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RESEARCH ARTICLE

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$O(n)$ tricriticality in two dimensions

Abstract We present exact results for several universal parameters of the tricritical $O(n)$ model in two dimensions. The results apply to the range $-2 \leqslant n \leqslant 3/2$, and include the central charge and three scaling dimensions, associated with temperature, magnetic field and the introduction of an interface. Since these results are based on an extrapolation of known relations between the $O(n)$ and the Potts model, they cannot be considered as rigorous. For this reason, we perform an accurate numerical analysis of the central charge and the critical exponents. This analysis, which is based on transfer-matrix calculations on the honeycomb lattice, is in a full and precise agreement with the theoretical predictions.

Keywords $O(n)$ model, tricriticality, critical exponents

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1 Introduction

The rapid development of the theory of critical phenomena and phase transitions has led to a considerable amount of exact results characterizing universality classes in two dimensions, including results for models with a continuously variable symmetry parameter, such as the random-cluster representation of the q-state Potts model and the loop representation of the $O(n)$ model. For a review of these results, see the review article by Nienhuis [1]. Conspicuously missing in this collection of exact results are, however, the universal parameters of the tricritical $O(n)$ loop model. Although numerical results are available [2], thus far, only the magnetic exponent [3] has been conjectured as a function of the central charge c, while the exact relation between c and the symmetry parameter n remains unknown. In the present article we bridge this gap by providing exact formulas for the central charge and the main scaling dimensions of the tricritical $O(n)$ model as a function of n. These results are, in part, based on assumptions but it is still reasonable to assume that they are exactly true, as we shall substantiate below.

The $O(n)$ model is originally defined as a system of n-component spins on a lattice. The $O(n)$ symmetry imposes full isotropy on the interactions acting between the spins. Thus, pair couplings must have the form $E_{ij} = \varepsilon(\boldsymbol{S}_i \cdot \boldsymbol{S}_j)$ where i and j represent two neighboring lattice sites, and ε is some arbitrary function that remains to be chosen. Graph expansion [4] of this model transforms the partition sum into a weighted sum of Eulerian graphs, in which n is no longer a discrete parameter, but instead assumes the role of a continuous variable parameter.

In this context, a remarkable possibility appears if one chooses the model on the honeycomb lattice, and the pair potential in the form $\varepsilon(p) \equiv -\log(1+xp)$, where x is a measure of the inverse temperature. Then the graph expansion reduces to a gas of non-intersecting and non-overlapping loops on the honeycomb lattice [5]. The partition sum of this loop model can be subjected to further mappings on the Kagomé 6-vertex model and the Coulomb gas, and thus opens the possibility to derive exact results for the honeycomb $O(n)$ model [6–9].

It is known that, in analogy with the Potts model, tricriticality can be induced in the $O(n)$ model when a sufficient number of vacancies is introduced. This was already confirmed for the case $n = 0$, which describes the collapse of a polymer at the so-called theta point, induced by

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polymer at the so-called theta point, induced by attractive interactions between the polymer segments [10, 11], and for the Ising case $n = 1$ [12, 13] where the existing results for the tricritical $q = 2$ Potts model are applicable [1]. For the general $O(n)$ loop model, the existence of tricritical points was revealed by transfer-matrix analyses [2, 15] for several values of n in the range $-2 \leqslant n \leqslant 2$ of n [2, 15]. Whereas this work yielded reasonably accurate values for some universal parameters, no exact formulas were found for these parameters as a function of n.

However, as noted recently by Janke and Schakel [3], the conformal classification of the magnetic exponent in terms of the Kac formula [16, 17] as $X_h = X_{m/2,\, m/2}$ (as explained below in Eq. (10) and the accompanying text), which is known for the two cases $n = 0$ and 1, is also applicable to other values of n. This classification is consistent with numerical data obtained for the tricritical $O(n)$ model. But the relation between n and the central charge c remains thus far unknown.

It is our purpose to provide the missing information, namely the relation between n and the central charge c. This will enable a proposal for the exact expressions for the central charge and three critical exponents of the tricritical $O(n)$ loop models as a function of n, which apply in the range $-2 \leqslant n \leqslant 3/2$. A less extensive report on this research has already appeared in Ref. [18]. The outline of the present paper is as follows. In the next section, Section 2, we introduce the model and its underlying theory. In Section 3, we perform the numerical analysis of the universal quantities predicted by the theory, and we conclude the paper with a short discussion in Section 4.

2 Model and theoretical background

As a representative of the supposed tricritical $O(n)$ universality class, we choose a generalized version of the $O(n)$ spin model on the honeycomb lattice studied by Domany *et al*. [5]. The generalization concerns the introduction of vacancies. This "dilution" is introduced by means of face variables that sit in the center of the elementary hexagons of the honeycomb lattice. These face variables have two possible states: vacant with weight v, or occupied with weight $1-v$. Furthermore, there is, as before, an n-component vector spin $\boldsymbol{S}_i$ on each vertex i, provided that it is surrounded by three occupied hexagons. The edges of a vacant face, i.e., a vacancy, cannot be visited by a loop. The one-spin weight distribution satisfies the $O(n)$ symmetry. For reasons of simplicity, it is normalized according to $\int \mathrm{d}\boldsymbol{S} = 1$ and $\int \mathrm{d}\boldsymbol{S}\boldsymbol{S}\cdot\boldsymbol{S} = n$. Therefore, the partition function given by [6, 7] generalizes to

$$Z = \sum_{\mathcal{L}} v^{N_v} (1-v)^{N-N_v} \int \prod_{i|\mathcal{L}} \mathrm{d}\boldsymbol{S}_i \prod_{\langle ij\rangle} (1 + w\boldsymbol{S}_i \cdot \boldsymbol{S}_j) \quad (1)$$

where the sum is on all allowed configurations of site and face variables, and $\mathcal{L}$ is a subset of the dual lattice and represents the occupied faces of the honeycomb lattice. The product over $i\,|\,\mathcal{L}$ includes all spins except those on the vertices of the vacant hexagons. N_v is the number of vacancies, N is the total number of faces (occupied or not), w is a parameter describing the spin-spin coupling, and $\langle ij\rangle$ represents all pairs of nearest-neighbor sites on the honeycomb lattice.

Since the reduced spin-spin interaction energy $(1+w\,\boldsymbol{S}_i \cdot \boldsymbol{S}_j)$ as implied by Eq. (1) retains the $O(n)$ symmetry, we expect that the universal properties are applicable not just to this model, but to a whole class of models with pair interactions of a similar nature.

As before, we apply an expansion [5] of the partition sum in powers of the coupling constant w, and thus obtain the loop representation of the model, but this time vacancies are included. The configurations of this loop gas are the graphs $\mathcal{G}$ consisting of any non-negative number of non-intersecting closed loops covering an arbitrary number of edges of the honeycomb lattice, while avoiding the edges of the vacant hexagons. The partition sum follows, completely analogous as in Ref. [5], as

$$Z = \sum_{\mathcal{L}} \sum_{\mathcal{G}|\mathcal{L}} v^{N_v} (1-v)^{N-N_v} w^{N_w} n^{N_l} \quad (2)$$

where N_w is the number of vertices visited by a loop, and N_l the number of closed loops. The first summation is over all possible configurations $\mathcal{L}$ of occupied faces. The second sum is over all graphs $\mathcal{G}$ allowed by the vacancy configuration $\mathcal{L}$. An example of a possible configuration is shown in Fig. 1.

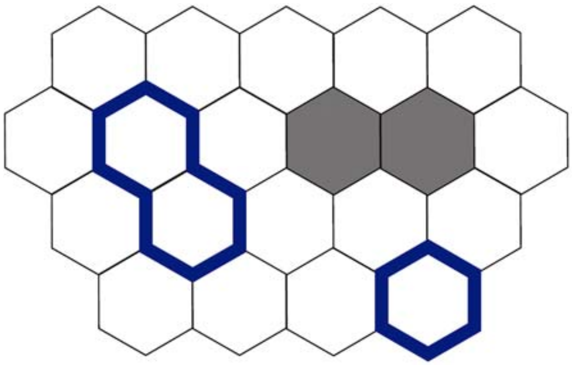

Fig. 1 A typical configuration of honeycomb $O(n)$ model with vacancies.

A possible solution for the problem concerning the relation between n and the central charge c, may be suggested by the hypothesis that the generic *critical* $O(n)$ model corresponds with a *tricritical* $q = n^2$-state Potts model [6, 7]. Perhaps the solution can be found by bringing the tricritical Potts model into an even higher multicritical state.

It is known that this can be realized [19, 20] by the simultaneous introduction into the Potts model of vacancies and their dual counterparts. The latter interactions appear as four-spin couplings that "freeze" the four Potts variables into the same state. It is possible to map the random-cluster representation of this model on a loop model on the the surrounding lattice. The latter model appears to allow, for special choices of the interaction parameters, solutions of the Yang-Baxter equations [19]. There appears to be four branches of exact solutions parametrized by q. One of these branches could be interpreted in terms of tri-tricritical Potts transitions [19, 20]. The exact central charge and exponents of this model follow as a function of q from an alternative representation in terms of a Temperley-Lieb model [19]. These results are confirmed by subsequent numerical analyses [20].

It is now very tempting to identify the loop weight $\sqrt{q}$ of the equivalent loop model with the loop weight n of the model (2), and to assume that the universal properties of the q-state tri-tricritical Potts model are equivalent with those of the tricritical $O(n=\sqrt{q})$ loop model. The central charge derived in Ref. [19] is expressed in $n=\sqrt{q}$, determined by the following equations:

$$c=1-\frac{6}{m(m+1)},\quad 2\cos\frac{\pi}{m+1}=\Delta,\quad \Delta-\frac{1}{\Delta}=n \tag{3}$$

Furthermore, Ref. [19] yielded scaling dimensions of which we quote three as:

$$X_j=\frac{k_j^2-1}{2m(m+1)} \tag{4}$$

where we introduced an index j = 1, 2 or 3, and k_j is given by

$$\cos[k_j\pi/(m+1)]=\frac{\Delta_j}{2} \tag{5}$$

with

$$\begin{aligned}\Delta_1&=1/\Delta\\ \Delta_2&=-1/\Delta\\ \Delta_3&=-\Delta\end{aligned} \tag{6}$$

3 Numerical verification

In this section we provide a test of the validity of the relation proposed above between the tricritical $O(n)$ model and the tri-tricritical Potts model. This test is numerical in nature and employs transfer-matrix calculations for the honeycomb loop model with vacancies.

The geometry of our loop model is chosen as a model wrapped on a cylinder, oriented such that one of the lattice edge directions corresponds with that of the axis of the cylinder. We fix the unit of length such that the smaller diameter of the elementary hexagons equals 1. The weight factors appearing in the transfer matrix account for the change of the numbers of loops, vacancies, and vertices covered by a loop segment when a new layer of sites is appended to the cylinder. The largest eigenvalue of the transfer matrix determines the free energy density. The finite-size dependence of the latter quantity allows a numerical determination of the central charge [21] of the corresponding conformal field theory. The numerical analysis yielded three more eigenvalues λ_i. These correspond with the correlation lengths associated with three different three correlation functions. These results allow finite-size estimates $X_i(v, w, L)$ of the corresponding scaling dimensions X_i [22]. In this way, the temperature dimension X_t is associated with the second eigenvalue of the transfer matrix. The magnetic dimension X_h is associated with the largest eigenvalue of a modified transfer matrix describing a system with a single loop segment running in the length direction of the cylinder. The "interface" exponent X_m follows from the transfer matrix describing a system without such a single loop segment, but with a modified column of edges with bond weights of the opposite sign. For further details about the transfer-matrix technique, we refer the reader to Refs. [2, 15, 23].

Let the vicinity of the tricritical fixed point be parametrized by a leading relevant temperature-like field t_1, a subleading relevant temperature field t_2, and an irrelevant scaling field u. The corresponding renormalization exponents are y_{t_1}, y_{t_2} and y_u respectively, with $y_{t_1} > y_{t_2}$.

The tricritical point is numerically approximated by solving the unknowns v and w in the two equations:

$$X_i(v,w,L)=X_i(v,w,L-1)=X_i(v,w,L-2) \tag{7}$$

where the functions X_i $(i = h, t, m)$ are provided by the transfer-matrix algorithm. Expansion of the finite-size-scaling function at the tricritical point indicates that the solution $v(L)$ of Eq. (7) converges to the tricritical value $v^{(\mathrm{tri})}$ of v according to

$$v(L)=v^{(\mathrm{tri})}+aL^{y_u-y_{t_2}}+\cdots \tag{8}$$

where a is an, in principle, unknown amplitude. Similarly $w(v, L)$ converges to its tricritical value $w^{(\mathrm{tri})}$. Furthermore, the values $X_i(L)$ taken at the solutions of Eq. (7) are found to converge to the tricritical scaling dimension X_i as :

$$X_i(L)=X_i+buL^{y_u}+\cdots \tag{9}$$

where b is another unknown amplitude. This procedure to locate the tricritical points and to estimate the tricritical exponents from Eq. (9) was performed both for $X_i = X_h$ and for $X_i = X_m$. These calculations were performed along the same lines as in Ref. [2], but here we use larger finite sizes up to L = 14, and moreover we include several values for $-2 \leqslant n < 0$. The results for the tricritical points are listed in Table 1, together with the estimated error margins and are shown in Fig. 2. In general they agree well with those reported in Ref. [2], but in some cases, we found that the apparent finite-size convergence was less rapid than suggested by the smaller range of system sizes used in Ref. [2], so that the estimated error margins had to be enlarged. The new analyses using X_h and X_m generated consistent results and thus provides a consistency check for the numerical uncertainties.

Table 1 Tricritical points determined from the scaling equations for the magnetic and the interface correlation lengths. The estimated numerical uncertainty in the last decimal place is given between parentheses.

n	v	w
−2.0	0.350 3 (1)	0.815 6 (1)
−1.75	0.364 9 (1)	0.833 0 (1)
−1.50	0.380 814 (1)	0.852 082 (1)
−1.25	0.397 935 2 (1)	0.872 640 4 (1)
−1.00	0.416 356 8 (3)	0.894 926 8 (1)
−0.75	0.436 008 8 (1)	0.918 961 7 (2)
−0.50	0.456 683 4 (2)	0.944 610 0 (2)
−0.25	0.478 147 5 (2)	0.971 742 8 (2)
0	1/2	1
0.25	0.521 805 (1)	1.028 950 (1)
0.50	0.543 13 (1)	1.058 12 (1)
0.75	0.563 61 (2)	1.087 08 (2)
1.00	0.583 0 (1)	1.115 5 (1)
1.25	0.601 0 (1)	1.142 9 (1)
1.50	0.617 5 (1)	1.168 8 (1)
1.75	0.632 1 (1)	1.192 8 (1)
2.00	0.645 2 (1)	1.214 5 (1)

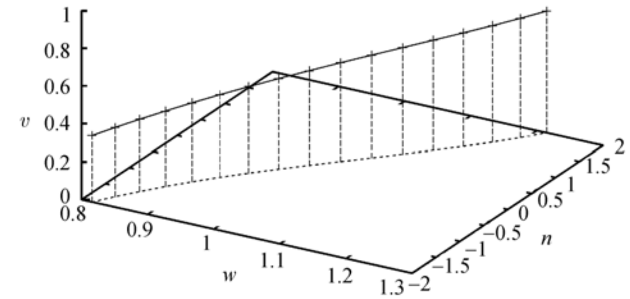

Fig.2 Tricritical line of the $O(n)$ model as a function of n. The data points show the numerical data. The curve and its projection on the w-n plane are added to guide the eye.

Furthermore, we used the second eigenvalue of the transfer matrix at the tricritical points thus calculated, in order to obtain finite-size estimates of X_t. These data were extrapolated; the results are, together with the central charge and the two other exponents, listed in Table 2. The estimated error margins are added between parentheses.

A comparison of the numerical results for the central charge with Eq. (3), as given in Table 2, appears to be in good agreement with the exact classification of the tricritical $O(n)$ model proposed above. Our numerical results for X_t match X_2 in Eq. (4), those for X_m agree with X_1. Using the value of the central charge and m as a function n, we confirm that the numerical results for the magnetic scaling dimension agree with the entry ($i = m / 2, j = m / 2$) in the Kac formula

Table 2 Transfer-matrix results for the central charge and three tricritical exponents. Estimated error margins in the last decimal place are added between parentheses. The numerical results are indicated by "(num)". For comparison, we include theoretical values obtained from Eqs. (3), (4), and (10). For $n < -3/2$, the temperature exponent X_t becomes complex.

n	c (num)	c (exact)	X_m (num)	X_m (exact)
−2.0	−0.991 4 (2)	−0.991 559 9	−0.202 (1)	−0.201 799 0
−1.75	−0.910 8 (2)	−0.910 998 6	−0.176 5 (2)	−0.176 972 3
−1.50	−0.819 6 (2)	−0.819 736 5	−0.151 66 (3)	−0.151 644 7
−1.25	−0.716 4 (1)	−0.716 455 6	−0.125 96 (3)	−0.125 930 1
−1.00	−0.600 0 (1)	−6/10	−0.100 01 (2)	−1/10
−0.75	−0.469 62 (1)	−0.469 619 5	−0.074 10 (1)	−0.074 095 5
−0.50	−0.325 28 (1)	−0.325 282 9	−0.048 53 (1)	−0.048 531 9
−0.25	−0.167 99 (1)	−0.167 995 3	−0.023 691 (1)	−0.023 691 7
0	0	0	0	0
0.25	0.175 26 (1)	0.175 263 0	0.022 111 (1)	0.022 111 0
0.50	0.353 48 (1)	0.353 479 2	0.042 24 (1)	0.042 235 7
0.75	0.529 94 (1)	0.529 948 9	0.059 99 (1)	0.060 00 04
1.00	0.700 00 (1)	7/10	0.074 9 (1)	3/40
1.25	0.860 (1)	0.858 976 9	0.086 7 (2)	0.086 505 2
1.50	1.001 (2)	1	0.094 (5)	0.088 019 2
1.75	1.04 (4)		0.098 (5)	
2.00	1.05 (2)		0.10 (1)	
n	c (num)	c (exact)	X_m (num)	X_m (exact)
−2.0	–	–	−0.094 (1)	−0.095 162 7
−1.75	–	–	−0.087 (1)	−0.087 643 1
−1.50	0.709 (1)	0.709 784 7	−0.079 2 (1)	−0.079 090 9
−1.25	0.481 7 (2)	0.481 473 9	−0.069 4 (1)	−0.069 365 3
−1.00	0.400 0 (2)	2/5	−0.058 4 (1)	−7/120
−0.75	0.344 5 (2)	0.344 668 1	−0.045 93 (3)	−0.045 889 5
−0.50	0.303 90 (2)	0.303 930 9	−0.031 99 (1)	−0.031 982 8
−0.25	0.273 220 (1)	0.273 219 9	−0.016 645 (1)	−0.016 643 5
0.00	1/4	1/4	0	0
0.25	0.232 500 (1)	0.232 495 7	0.017 731 (1)	0.017 729 52
0.50	0.219 3 (1)	0.219 238 6	0.036 28 (1)	0.036 276 58
0.75	0.209 0 (2)	0.208 874 1	0.055 39 (1)	0.055 397 46
1.00	0.200 0 (1)	1/5	0.075 00 (2)	3/40
1.25	0.193 (1)	0.190 680 0	0.095 0 (2)	0.095 497 14
1.50	0.180 (5)	0.168 449 9	0.12 (1)	1/8
1.75	0.183 (10)		0.13 (1)	
2.00	0.184 (10)		0.15 (2)	

$$X_{i,j} = \frac{[i(m+1) - jm]^2 - 1}{2m(m+1)} \tag{10}$$

The $O(n)$ model with $n > 0$ appears to correspond with branch 1 as defined in Ref. [19], and those for $n < 0$ with branch 2. The numerical results and theoretical values of the central charge and the three scaling dimensions are shown as a function of n in Figs. (3) and (4).

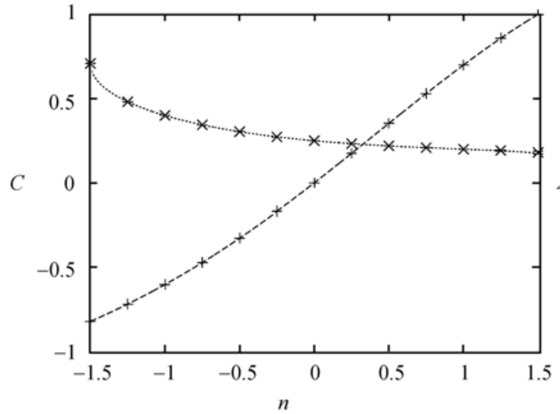

Fig. 3 Central charge (+) and temperature dimension (×) of the tricritical $O(n)$ model (×) of the tricritical $O(n)$ model vs n. The numerical results are indicated by the data points, and the theoretical predictions by the two curves.

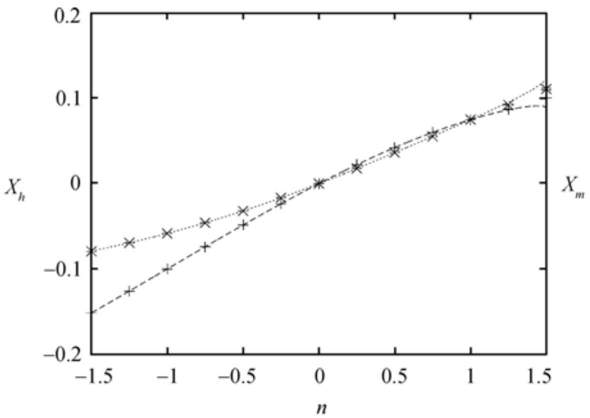

Fig. 4 Scaling dimensions X_h (×) and X_m (+) of the tricritical $O(n)$ model vs n. The data points represent the numerical results, the curves the theoretical predictions.

4 Discussion

It appears that the formulas for X_1 and X_2, as given by Eq. (4), do not correspond with entries in the Kac table, at least not with pairs of indices that are linear in m. This explains the apparent difficulty to conjecture the exact values of X_m and X_t from numerical data alone, even if supplemented by data for c. This problem did not apply to X_h whose classification in terms of the Kac table was already given above.

It is noteworthy that, unlike the critical branch, which extends to $q = 4$, the Potts tri-tricritical branch ends at $q = 9/4$. For $q > 9/4$ the exact solution is no longer critical, and the transition probably turns first-order [19, 20]. The correspondence $q = n^2$ thus yields the result that the tricritical $O(n)$ branch ends at $n = 3/2$, possibly with a discontinuous transition for $n > 3/2$. At first sight, the numerical results for $3/2 < n \leqslant 2$ are not suggestive of a discontinuous transition, and allow at most a weak discontinuity. But at the same time it is clear from Tables 1 and 2 that the estimated errors are increasing with n for $n \gtrsim 3/2$, as a result of deteriorating finite-size convergence. This is suggestive of the possibility that an operator becomes marginal at $n = 3/2$, in line with $c = 1$ (see Table 2). This is reminiscent of the $q > 4$ Potts model, where the marginal operator leads to anomalously slow finite-size convergence, which obscures the weak first-order character in a range of q near 4.

The scenario sketched above indicates that the critical and tricritical $O(n)$ branches are *not connected*. Furthermore, it is not suggestive of a relation between the tricritical $O(n)$ model and the critical Potts model, such as was recently quoted [3].

The results presented above apply to the non-intersecting loop model. Loop intersections are irrelevant in the critical $O(n)$ model, but they are relevant in the low-temperature phase [25]. While the possible relevance of such intersections could modify the universal behavior, this appears not to be the case for the $n = 1$ tricritical $O(n)$ loop model, since its exponents are known to agree with those of the corresponding spin model, i.e., the tricritical Blume-Capel model.

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