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REVIEW ARTICLE

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Mutual Chern-Simons theory and its applications in condensed matter physics

Abstract In this paper, the mutual Chern-Simons (MCS) theory is introduced as a new kind of topological gauge theory in 2+1 dimensions. We use the MCS theory in gapped phase as an effective low energy theory to describe the Z_2 topological order of the Kitaev-Wen model. Our results show that the MCS theory can catch the key properties for the Z_2 topological order. On the other hand, we use the MCS theory as an effective model to deal with the doped Mott insulator. Based on the phase string theory, the *t-J* model reduces to a MCS theory for spinons and holons. The related physics in high T_c cuprates is discussed.

Keywords mutual Chern-Simons theory, topological order, doped Mott insulator

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1 Introduction

Gauge theory in 2+1 dimensions has been studied extensively since the discovery of the quantum fractional Hall effect [1–6]. People found that the Chern-Simons (CS) theory can be an effective theory to describe the ground states for the quantum fractional Hall (QFH) states. The Chern-Simons theory is a nontrivial gauge theory in 2+1 dimensions, of which a CS term is contained in its Lagrangian [3]:

$$\mathcal{L}_{CS} = \frac{1}{\theta}\varepsilon^{\mu\nu\lambda} A_{\mu}\partial_{\nu} A_{\lambda} \tag{1}$$

For the CS theory, there are several fundamental physics properties:

• *Massive character for the gauge fields*: The CS term when added to the Maxwell term, acts as the mass term to the gauge fields. Then the gauge field becomes massive. However, such mass term for the gauge field is gauge invariant which is very special to 2+1 dimensions.

• *Fractional statistics for the particles*: The CS theory describes the particles with fractional statistics. Let us minimally couple the Chern Simons theory to an external charge current J^{μ}. From the classical equation motion, one can see that each charge particle catches magnetic flux. And the element matter field obeys anionic commutation of statistics, which is shown in Fig. 1.

• *Symmetry*: In CS theories, the parity and time-reversal symmetries are explicitly broken.

• *Topological degeneracy*: The ground states have topological degeneracy on a torus.

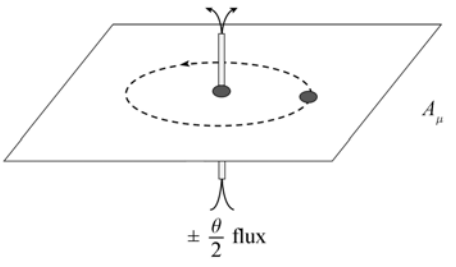

Fig. 1 Fractional statistics from Chern-Simons theory.

Because FQH states are described by their ground state wave functions, which are *complex* functions of infinite variables, it is not surprising that FQH states contain a new

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kind of orders that is described by the CS theory. It catches the key topological property for the FQH states. All the topological properties, including topological degeneracy, quantum numbers, and edge states, agree, indicating the equivalence between the FQH states and the CS theory [7].

The success for the application of the CS theory on FQH states encourages people to pursue new kinds of gauge theories for other quantum orders. In this paper we will introduce a new kind of gauge theory in 2+1 dimensions-mutual $U(1)\times U(1)$ Chern-Simons (MCS) theory. And we give two examples in condensed matter physics, of which the MCS theory is a low energy effective theory, the Z_2 topological order and the doped Mott insulator.

2 MCS theory and its applications in condensed matter physics

2.1 The fundamental physics properties for MCS theory

In this part we will introduce a new type of topological gauge theory in 2+1 dimensions-the mutual $U(1)\times U(1)$ Chern-Simons (MCS) theory [8, 9]. The Lagrangian for the MCS theory:

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4e_a^2}(f_{\mu\nu})^2 - \frac{1}{4e_A^2}(F_{\mu\nu})^2 + \mathcal{L}_{\text{MCS}} + \mathrm{i}a^{\mu} j_{\mu} + \mathrm{i}A^{\mu} J_{\mu} \tag{2}$$

where the currents are defined as $j_\mu = (j_i, \rho_a)$ and $J_\mu = (J_i, \rho_A)$. The mutual CS term is

$$\mathcal{L}_{\text{MCS}} = \frac{1}{\pi}\varepsilon_{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda \tag{3}$$

$f_{\mu\nu}$ is the gauge field strength for gauge field a_λ and $F_{\mu\nu}$ is the gauge field strength for gauge field A_μ. For the MCS theory, there are also several important physics consequences:

• *Massive character for the gauge fields*: The masses for both of the gauge fields come from the mutual CS term:

$$m_a \sim e_a^2, \quad m_A \sim e_A^2$$

In addition, such mass term for the gauge fields from MCS term is also gauge invariant.

• *Mutual statistics for the particles*: From the equation motions for a_λ and A_λ,

$$\begin{aligned} -\frac{1}{2e_a}(\partial_\mu f_{\mu\nu}) + \frac{1}{\pi}\varepsilon^{\mu\nu\lambda} F_{\mu\nu} &= -\mathrm{i}\, j_\mu \\ -\frac{1}{2e_A^2}(\partial_\mu F_{\mu\nu}) + \frac{1}{\pi}\varepsilon^{\mu\nu\lambda} f_{\mu\nu} &= -\mathrm{i}\, J_\mu \end{aligned} \tag{4}$$

we find that a $U(1)$ charge for gauge field A_μ induces flux of gauge field a_μ. As a result, the $U(1)$ charge for gauge field A_μ and the $U(1)$ charge for gauge field a_μ have a semionic mutual statistics. That is, moving an A_μ-charge around an a_μ-charge generate a phase π. Figure 2 shows the semionic mutual statistics due to the MCS term.

• *Topological degeneracy*: The ground states have topological degeneracy on a torus. On an even-by-even lattice with periodic boundary condition (a torus), there are 4 ground states with almost the same energy.

• *Symmetry*: The physical symmetries, which include parity and time-reversal, are precisely preserved in a 2+1-dimensional MCS theory.

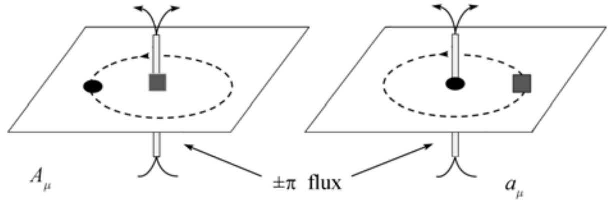

Fig. 2 Mutual semion statistics from mutual Chern-Simons theory.

2.2 MCS theory for Z_2 topological order

Let us firstly consider the Kitaev-Wen model:

$$H_{\text{exact}} = g\sum_i \hat{F}_{\breve{i}}, \quad \hat{F}_{\breve{i}} = \sigma^y_{\breve{i}} \sigma^x_{\breve{i}+\breve{x}} \sigma^y_{\breve{i}+\breve{x}+\breve{y}} \sigma^x_{\breve{i}+\breve{y}} \tag{5}$$

where $\sigma^{x,y,z}$ are the Pauli matrices and $\breve{i} = (i_x, i_y)$ labels the site of a square lattice [10, 11, 7]. The Kitaev-Wen model is an exact solved model in 2+1D. It was found that all the excitations above the ground state are gapped and the ground state contains a non-trivial Z_2 topological order.

It is known that such Z_2 topological order contains Z_2 vortex and Z_2 charge excitations, and the Z_2 vortex and the Z_2 charge have semionic mutual statistics between them. Thus, we will identify the A_μ-charge as the Z_2 charge and the a_μ-charge as the Z_2 vortex. Figure 2 shows such nontrivial topological relationship between Z_2 charge and Z_2 vortex.

Such a topological state without gapless excitations can be described by a MCS theory. It catches the key topological properties for the Z_2 topological order of the Kitaev-Wen model. We reach the conclusion by comparing the topological properties of the MCS theory with those of the Z_2 state. All the topological properties, including topological degeneracy, quantum numbers, and edge states, agree, indicating the equivalence between the Z_2 topological state on lattice and the MCS theory [9].

Because Z_2 vortex is boson, one can describe it by massive relativistic complex fields (ϕ_1, ϕ_2). The Lagrangian of them is written as

$$\mathcal{L}_\phi = (\partial_\mu - a_\mu)\phi_1^\dagger(\partial_\mu + a_\mu)\phi_1 + m_\phi^2 \phi_1^\dagger \phi_1 \tag{6}$$

$$+(\partial_\mu - A_\mu)\phi_2^\dagger(\partial_\mu + A_\mu)\phi_2 + m_\phi^2 \phi_2^\dagger \phi_2 \tag{7}$$

where A_μ and a_μ are the gauge fields coupled to Z_2 vortexes and m_ϕ is mass for them. Because of the mutual statistics between the Z_2 vortexes, one needs to add a mutual Chern-Simons term:

$$\mathcal{L}_{\mathrm{MCS}} = \frac{\mathrm{i}}{\pi}\varepsilon_{\mu\nu\lambda} A_{\mu}\partial_{\nu} a_{\lambda} \tag{8}$$

To obtain the right MCS theory, we note that a Z_2 vortex for the Kitaev-Wen model exist only on the even plaquettes [11, 12, 9]. The vortex on the odd plaquettes is actually a Z_2 charge. So under a translation by one lattice spacing, a Z_2 vortex is changed into a Z_2 charge! So in the MCS theory that describes the topological state, a_μ and A_μ must exchange under the translation by one lattice spacing. The above discussion motivates us to define the MCS theory by considering special translation symmetry. Let $\hat{T}_i$ ($i = x, y$) be the translations by one lattice spacing in the x and y directions, respectively. The gauge fields transform as follows:

$$T_i^{-1}\hat{A}_j T_i = \hat{a}_j, \quad T_i^{-1}\hat{a}_j T_i = \hat{A}_j, \quad i = x, y$$

This twist boundary condition means that $\hat{A}_\mu$ and $\hat{a}_\mu$ can be viewed as a single gauge field on a lattice whose size is doubled in the x-direction. As a result the ground-state degeneracy on even-by-odd, odd-by-even and odd-by-odd is reduced to 2. The ground-state degeneracy on even-by-even is reduced to 4. Table 1 shows the topological degeneracy for both Z_2 topological order and the MCS theory. We also used the MCS theory to calculate the crystal momenta of the ground states. The results agree with the above prediction.

In summary, the Z_2 topological order of the Kitaev-Wen model can be well described by the MCS theory in the gapped phase.

Table 1 The Z_2 topological order of the Kitaev-Wen model.

	Even by even	Even by odd	Odd by even	Odd by odd
MCS theory	4	2	2	2
Kitaev-Wen model	4	2	2	2

2.3 Mutual Chern-Simons theory for doped Mott insulator

Next, we use the MCS theory to describe the doped Mott insulator. The MCS theory in studying the t-J model is given as an effective theory from the phase string description. To characterize such a Hilbert space restriction for the t-J model, a spin-charge separation description, namely, by introducing spinless "holon" of charge $+e$ and neutral spin-1/2 "spinon" as the essential building blocks of the restricted Hilbert space, has become an effective and useful way [13, 14] based on a distinctive decomposition of the electron operator:

$$c_{i\sigma} = h_i^{\dagger} b_{i\sigma} \mathrm{e}^{\mathrm{i}\hat{\Theta}_{i\sigma}} \tag{10}$$

which is known as the bosonization or phase string decomposition [15, 16] because holon and spinon operators, $h_i^{\dagger}$ and $b_{i\sigma}$, are both bosonic, with the fermionic commutations relations of the electron operator being restored by the phase string operator, $\mathrm{e}^{\mathrm{i}\hat{\Theta}_{i\sigma}} = (-\sigma)^i\,\mathrm{e}^{\mathrm{i}\frac{1}{2}(\Phi_i^b - \sigma\Phi_i^h)}$. Here, internal gauge invariance appears as $U(1) \times U(1)$: $h_i \to \mathrm{e}^{\mathrm{i}\phi_i^h} h_i$ and $\Phi_i^b \to \Phi_i^b + 2\phi_i^h$; $b_{i\sigma} \to \mathrm{e}^{\mathrm{i}\sigma\chi_i} b_{i\sigma}$ and $\Phi_i^h \to \Phi_i^h + \chi_i$. Consequently there exist a pair of $U(1)\times U(1)$ gauge fields coupling to the holon and spinon fields, respectively, in the resulting gauge theory, called the phase string model, derived based on the decomposition (10) and the bosonic resonating-valence- bond (RVB) mean-field saddle-point.

The low-energy effective theory of the continuum versions for the t-J model from phase string decomposition can be written as [8, 17, 18]:

$$\mathcal{L}_{\mathrm{eff}} = \mathcal{L}_h + \mathcal{L}_s + \mathcal{L}_{\mathrm{MCS}} \tag{11}$$

with

$$\mathcal{L}_h = h^{\dagger}[\partial_0 - \mathrm{i}(A_0^s + eA_0^e)]h + h^{\dagger}\frac{(-\mathrm{i}\partial_\alpha - A_\alpha^s - eA_\alpha^e)^2}{2m_h}h$$

$$\mathcal{L}_s = \frac{1}{2g}\left[|(\partial_\mu - \mathrm{i}\sigma A_\mu^h)z_\sigma|^2 + m_s^2 |z_\sigma|^2\right]$$

$$\mathcal{L}_{\mathrm{MCS}} = \frac{\mathrm{i}}{\pi}\mathrm{e}^{\mu\nu\lambda} A_\mu^s \partial_\nu A_\lambda^h$$

g is the coupling constant. A_μ^e is the vector potential of the external electromagnetic field, and $-e$ is the electron electric charge. Note that the chemical potential μ in $\mathcal{L}_h$ has been absorbed into $\mathrm{i}A_0^s$ for simplicity. The mutual Chern-Simons therm here $\mathcal{L}_{\mathrm{MCS}}$ denotes the nontrivial quantum entanglemant between holon and spinon in a doped Mott insulator.

However, the MCS theory to describe the t-J model is different from that to describe the Kitaev-Wen model. For the effective Lagrangian here is an MCS theory in the Higgs phase; while for the Z_2 topological order in the gapped phase.

At low doping, the spinon condensation $\langle z \rangle \neq 0$ leads to a spin antiferomagnetic (AF) order and forces a "confinement" on the holon part, making holons self-localized to ensure the AF long range order. On the other hand, at a higher doping, the condensation of bosonic holons $\langle h \rangle \neq 0$ forces a "confinement" on the spinon part, resulting in a superconducting (SC) phase coherence. Two phases are characterized by dual Meissner effects and dual flux quantization conditions, accompanied by a dual confinement, which are the direct consequences of the mutual-Chern-Simons gauge fields interacting with two matter fields when one of them experiences Bose condensation. The global phase diagram is shown in Fig. 3.

Firstly in the AF phase, the spinon condensation may be viewed as a two-component "superfluidity". By introducing a unit vector $\tilde{\boldsymbol{n}}$ defined by $\tilde{\boldsymbol{n}} = \tilde{z}^{\dagger}\boldsymbol{\sigma}\tilde{z}$, $\tilde{z} \equiv (z_\uparrow, z_\downarrow^*)^{\mathrm{T}}$, the low-energy effective Lagrangian in this phase reduces to

$$\mathcal{L}_{\mathrm{eff}} = \frac{1}{8\tilde{g}}(\nabla\tilde{\boldsymbol{n}})^2 + \frac{\tilde{g}}{2\pi^2}(\boldsymbol{E}^s)^2 + \mathrm{i}A_0^s \mathcal{K}_0^s + \mathcal{L}_h \tag{12}$$

where $\boldsymbol{E}^s = -\nabla A_0^s$, $\mathcal{K}_0^s \equiv \frac{1}{4\pi}\varepsilon_{0\nu\lambda}\tilde{\boldsymbol{n}}\cdot\partial^\nu\tilde{\boldsymbol{n}}\times\partial^\lambda\tilde{\boldsymbol{n}}$ and $\frac{1}{\tilde{g}} \equiv \frac{1}{g} - \frac{1}{g_c} > 0$ (here $g_c = \frac{4\pi}{\Lambda}$ with Λ denoting a cutoff parameter in the regularization). The dual Meissner effect means that a holon is an "alien" object in the spinon condensate, and the dual flux quantization condition means that a meron (vortex) is produced in the spinon condensate to which a holon must be confined to, just like a spinon is confined to a magnetic vortex core in the above-mentioned SC state. As a result, only the "neutral" object of a holon-meron composite, not the holon itself, appears in the low-energy physical spectrum, which has a dipolar spin configuration at long distance, co-existing with the AF long range order in a dilute hole concentration regime [19].

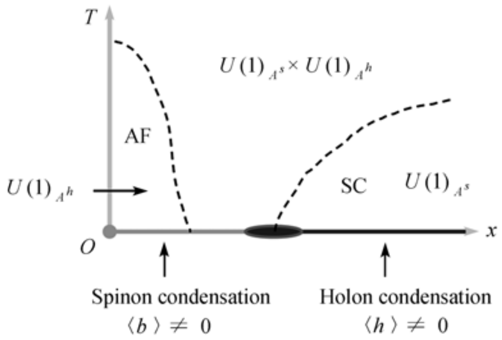

Fig. 3 The global phase diagram.

Now let us consider the Higgs phase with the Bose condensation of holons, $\langle h\rangle \neq 0$, whose ground state is an SC one [20] with the Meissner effect and charge $2e$ minimal flux quantization as shown below. The resulting effective Lagrangian takes the following form:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_s + \frac{m_h}{2\pi^2\rho_h}\cdot|\boldsymbol{E}^h|^2 - \mathrm{i}A_0^h\mathcal{Q}^h \tag{13}$$

in which $\boldsymbol{E}^h = \partial_0\boldsymbol{A}^h - \nabla A_0^h$ and $\mathcal{Q}^h \equiv \frac{1}{\pi}\varepsilon^{0\nu\lambda}\partial_\nu(\partial_\lambda\phi_h - A_\lambda^e)$. The Meissner effect and $hc/2e$ flux quantization in SC order are similar to the predictions by a conventional superconductivity theory, and the spinons are found to be confined such that to drop out of the physical spectrum. Only integer spin excitations, as composed of confined spinon pairs, are allowed in the bulk state. But as a unique prediction, a single spinon (an $S=1/2$ moment) does appear in the center of a magnetic vortex core. It forecasts that the spin fractionalization will occur in the pseudogap phase, as the latter may be viewed as the proliferation of the vortex core state above the superconducting transition T_c [20, 21].

The properties for the MCS theory in the Higgs phases for the *t-J* model are summarized in Table 2.

In summary, the MCS theory in the Higgs phase catches some key properties for the doped Mott insular related to high T_c cuprates.

Table 2 The properties for the MCS theory in the Higgs phases for the *t-J* model.

	AF	SC
Bose condensation	$\langle z\rangle \neq 0$	$\langle h\rangle \neq 0$
Coulomb gauge field	A_0^s	A_0^h
"Charged" particle of Coulomb gauge field	holon	spinon
External source of Coulomb gauge field	meron	magnetic flux
"Charge neutral" object	Holon-meron pair	(a) spinon pair (b) magnetic flux+a spinon
Dual flux quantization	$\lvert\mathcal{K}_0^s\rvert = \frac{1}{2}$	$\lvert\Phi_{\min}^e\rvert = \frac{hc}{2e}$
Dual Meissner effect	holon confinement	(a) spinon confinement (b) spinon bound to magnetic flux

3 conclusions

In this paper, we studied a new class of nontrivial 2+1- dimensional gauge field structure-the mutual-Chern-Simons theory. The Lagrangian of such a mutual-Chern-Simons theory is derived as an effective low-energy description for the Z_2 topological order and the phase-string model for doped Mott insulators. This effective theory retains the full symmetries of parity and time-reversal, in contrast to the conventional Chern-Simons theories where the two symmetries are usually broken. For the Z_2 topological order, the MCS theory is in a gapped phase, of which all excitations are massive; for the phase-string model of the doped Mott insulators, the MCS theory is in two Higgs phases, of which the dual Meissner effect is the most important character. Furthermore, the MCS theory can describe other interesting many-particle systems, such as the deconfined quantum critical point between Néel order and valence-band-solid state. This will be discussed elsewhere.

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