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RESEARCH ARTICLE

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Interaction and resonance phenomena of multi-soliton

Abstract As is well known, Korteweg-de Vries equation is a typical one which has planar solitary wave. By considering higher order transverse disturbance to planar solitary waves, we study a Kadomtsev-Petviashvili (KP) equation and find some interesting results. In this letter we investigate the three soliton interaction and their resonance phenomena of KP equation, and theoretically find that the maximum amplitude is 9 times of the initial interacting soliton for three same amplitude solitons. Three arbitrary amplitude soliton interaction of KP equation is also studied by numerical simulation, which can also results in resonance phenomena.

Keywords Kadomtsev-Petviashvili equation, soliton, resonance

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1 Introduction

In 1965, Zabusky and Kruskal [1] investigated the interaction of solitary waves and the recurrence of initial states. The term soliton was coined by Zabusky and Kruskal [1], who performed numerical studies of the KdV equation, and found particle like waves which retained their shapes and velocities after collisions. The interaction of two solitons emphasized the reality of the preservation of shapes and speeds and of the steady pulse like character of solitons. As a result of a great deal of work all over the world, the properties of solitons in (1+1) space-time manifold are understood in many cases quite well. It is found that the study of one-dimensional solitons is reaching a certain stage of maturity. However, the study of higher-dimension soliton effects is still in its infancy. Higher-dimension solitons exist in the sky as density waves in spiral galaxies, and the giant Red Spot in the atmosphere of Jupiter, they exist in the ocean as waves bombarding oil wells; they exist in much smaller natural and laboratory systems such as plasmas. The Kadomtsev-Petviashvili (KP) equation was one of the first model equations proposed for higher dimensional ocean pattern [2]. It was first proposed to study the transverse instability of the KdV solitary wave [3].

It was observed that when cylindrical or spherical solitons interact obliquely, a new nonlinear wave is formed during the interaction which moves ahead of the colliding solitons, both its velocity and position depend on the value of the polar angle and are, respectively, lager than those of colliding solitons. Ref. [4] have studied the two sam amplitude soliton resonance of the KP equation, and theoretically predict that the maximum amplitude is 4 times of the initial interacting soliton, which is agree with experiments. This phenomena has been reported in the fluid dynamics [5, 6], and in the plasma physics [7－12]. Similar problem of vector solitons have also been studied in many fields of modern physics [13－20].

It is well known that the KP equation is a approximation for complex physical systems which usually can be described by a set of partial differential equations and they can not be solved analytically. By using the perturbation methods [21－24] and other mathematical method with the help of the computer [25, 26], many authors have obtain the KP equation from many branches of physics [27－29]. Therefore, to solve the KP equation is one of the interesting problems nowadays. Since one and two arbitrary soliton solutions have been obtained analytically until now [4]. Here, in this paper, we want to study the three soliton solution analytically and find the characteristic of the interaction and the resonance phenomena among these three solitons.

The general form of KP equation can be written as follows:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+Bu\frac{\partial u}{\partial x}+C\frac{\partial^3 u}{\partial x^3}\right)+D\frac{\partial^2 u}{\partial u^2}=0 \qquad (1)$$

where B, C, D are all nonzero real constants. Eq. (1) is called

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KP-II equation when $CD > 0$ and it is called KP-I equation when $CD < 0$. In this paper, we only investigate the KP-II equation. By introducing the transformation $u = \frac{6}{B}C^{\frac{1}{3}}u'$, $x' = C^{-\frac{1}{3}}x$, $y' = \sqrt{\frac{3}{D}}C^{-\frac{1}{6}}y$ and dropping the primes, Eq. (1) can be rewritten as the so-called standard KP equation:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right)+3\frac{\partial^2 u}{\partial y^2}=0 \quad (2)$$

Eq. (2) is one of the integrable system. We will give out its multi-soliton solutions by using the Bäcklund transformation and discuss the three-soliton interactions.

2 Theoretical investigation of the three-soliton resonance

2.1 Multi-soliton solutions of KP equation

The Bäcklund transformation is a well-known method for generating families of solutions [30—32]. In this letter, we try to find multi-soiton solutions of KP equation by using the Bäcklund transformation. Bäcklund transformation of equation (2) can be given by:

$$u = 2\frac{\partial^2}{\partial x^2}(\ln v) \quad (3)$$

Substituting Eq. (3) into Eq. (2) yields,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right)+3\frac{\partial^2 u}{\partial y^2}=$$

$$2\frac{\partial^2}{\partial x^2}\left(\frac{v(v_{xt}+v_{xxxx}+3v_{yy})-(v_x v_t+4v_x v_{xxx}-3v_{xx}^2+3v_y^2)}{v^2}\right) \quad (4)$$

Eq. (4) shows that Eq. (2) becomes true automatically if

$$v(v_{xt}+v_{xxxx}+3v_{yy})-(v_x v_t+4v_x v_{xxx}-3v_{xx}^2+3v_y^2)=0 \quad (5)$$

Eq. (5) is called τ-equation and can be solved by the ε-expansion method easily.

To obtain the single-soliton solution of Eq. (2), we write

$$v = 1+\mathrm{e}^{k_1 x+l_1 y-\omega_1 t+\delta_1} \quad (6)$$

where $\omega_1 = k_1^3+\frac{3l_1^2}{k_1}$, k_1 and l_1, are two arbitrary nonzero constants, δ_1 is also an arbitrary constant, then we have

$$u = \frac{k_1^2}{2}\mathrm{sech}^2\left(\frac{k_1 x+l_1 y-\omega_1 t+\delta_1}{2}\right) \quad (7)$$

Eq. (7) is one soliton solution of the KP equation. Its amplitude is $\frac{1}{2}k_1^2$.

To obtain the two-soliton solution, we write

$$v = 1+v_1+v_2+A_{12}v_1 v_2 \quad (8)$$

where

$$v_j = \mathrm{e}^{k_j x+l_j y-w_j t-\delta_j} \quad (9)$$

and

$$w_j = k_j^3+\frac{3l_j^2}{k_j} \quad (10)$$

where k_j, l_j, δ_j (j = 1, 2) are constants, and A_{12} is given by:

$$A_{12} = \frac{(k_1-k_2)(\omega_2-\omega_1)+(k_1-k_2)^4+3(l_1-l_2)^2}{(k_1+k_2)(\omega_1+\omega_2)-(k_1+k_2)^4-3(l_1+l_2)^2} \quad (11)$$

then we obtain the two-soliton solution for Eq. (2) as follows:

$$u = 2(v_1k_1^2+v_2k_2^2+v_1v_2(k_1-k_2)^2+A_{12}v_1v_2(k_1+k_2)^2$$
$$+A_{12}v_1^2v_2k_2^2+A_{12}v_1v_2^2k_1^2)/(1+v_1+v_2+A_{12}v_1v_2)^2 \quad (12)$$

In order to obtain the three-soliton solution, we write

$$v = 1+v_1+v_2+v_3+A_{12}v_1v_2+A_{13}v_1v_3+A_{23}v_2v_3$$
$$+A_{12}A_{23}A_{13}v_1v_2v_3 \quad (13)$$

where

$$v_j = \mathrm{e}^{k_j x+l_j y-w_j t-\delta_j} \quad (14)$$

and

$$\omega_j = k_j^3+\frac{3l_j^2}{k_j} \quad (15)$$

where k_j, l_j, δ_j (j = 1, 2, 3) are all constants, and A_{ij} is given by:

$$A_{ij} = \frac{(k_i-k_j)(\omega_j-\omega_i)+(k_i-k_j)^4+3(l_i-l_j)^2}{(k_i+k_j)(\omega_i+\omega_j)-(k_i+k_j)^4-3(l_i+l_j)^2} \quad (16)$$

In general, we can obtain the N-soliton solution of Eq. (2) analytically following this way. However, we only study three-soliton solutions in this paper. The N soliton solutions will be our future works.

2.2 Investigation of the three-soliton resonance

For simplicity, we first consider a simple case on which we assume $k_1 = k_2 = k_3 = k$ and $l_1 = -l_2 = l$, $l_3 = 0$. In this case, we have $\omega_1 = \omega_2 = \omega = k^3+\frac{3l^2}{k}$ and $\omega_3 = k^3$. Then we find that

$$A_{12} = \frac{l^2}{-k^4+l^2}, \quad A_{13} = A_{23} = \frac{l^2}{-4k^4+l^2} \quad (17)$$

In order to obtain the maximum value of u at resonance angle, we let

$$\frac{\partial u}{\partial x} = 0 \quad (18)$$

$$\frac{\partial u}{\partial y} = 0 \quad (19)$$

$$\frac{\partial u}{\partial t}=0 \tag{20}$$

We then find that at the point

$$x=\frac{k^3}{6l^2}\ln\frac{A_{12}}{A_{13}}+\frac{1}{k}\ln\frac{1}{A_{13}} \tag{21}$$

$$y=0 \tag{22}$$

$$t=\frac{k}{6l^2}\ln\frac{A_{12}}{A_{13}} \tag{23}$$

the maximum value of u is obtained, which is given by:

$$u_{\max}=-\frac{7}{2}k^2+\frac{2l^2}{k^2}-\frac{2l^2}{k^2}\sqrt{1-\frac{4k^4}{l^2}}\sqrt{1-\frac{k^4}{l^2}} \tag{24}$$

From Eq. (24) we find that when $l\rightarrow 2k^2$, the maximum value of $u_{\max}$ is reached as follows:

$$u_{\max}=\frac{9}{2}k^2 \tag{25}$$

It seems that $u_{\max}$ is equal to 9 times that of initial soliton amplitude $\frac{k^2}{2}$. We, therefore, note that the maximum amplitude for three same amplitude soliton during interaction is 9 time that of one initial soliton amplitude. This is also agree with our numerical results.

Figure 1 shows the resonance phenomena of three same amplitude solitons when $l\approx 2k^2$, where we take

$$\begin{pmatrix}k_1\\k_2\\k_3\\l_1\\l_2\\l_3\end{pmatrix}=\begin{pmatrix}0.2\\0.2\\0.2\\0.08+10^{-16}\\-(0.08+10^{-16})\\0\end{pmatrix}$$

Both theoretical and numerical results indicate that the amplitude of the fourth wave is about 0.18, which is 9 times of the initial soliton amplitude. Figure 2 and Fig. 3 show the interaction of three solitons where we let

$$\begin{pmatrix}k_1\\k_2\\k_3\\l_1\\l_2\\l_3\end{pmatrix}=\begin{pmatrix}0.2\\0.2\\0.2\\0.0804\\-0.0804\\0\end{pmatrix} \text{ and } \begin{pmatrix}k_1\\k_2\\k_3\\l_1\\l_2\\l_3\end{pmatrix}=\begin{pmatrix}0.2\\0.2\\0.2\\0.1\\-0.1\\0\end{pmatrix}$$

respectively. Figure 2 indicates that the maximum amplitude of the wave is about 0.155, which is less than 9 times of the initial interacting solitons. Figure 3 shows that the maximum amplitude of the wave is about 0.085, which is much less than 9 times of the initial interacting solitons. It shows that if $l\neq 2k^2$, the maximum amplitude is less than 9 times of the initial interacting solitons. We also find that the maximum amplitude of the wave tends to 0.06 when l is large enough. It suggests that the resonance does not take place when $l\neq 2k^2$.

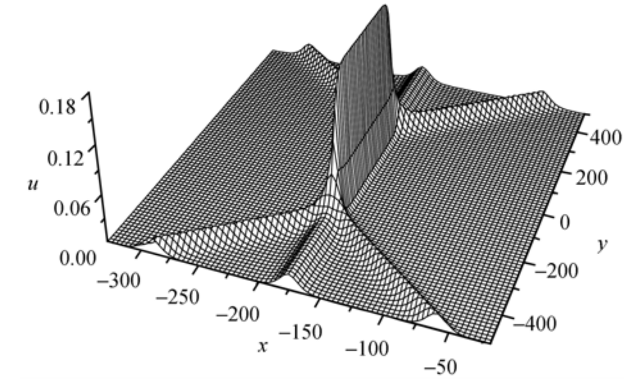

Fig. 1 The profile of u for three same amplitude soliton interaction, where $k_1=k_2=k_3=k=0.2$, $l_1=-l_2\approx 0.08$, $l_3=0$.

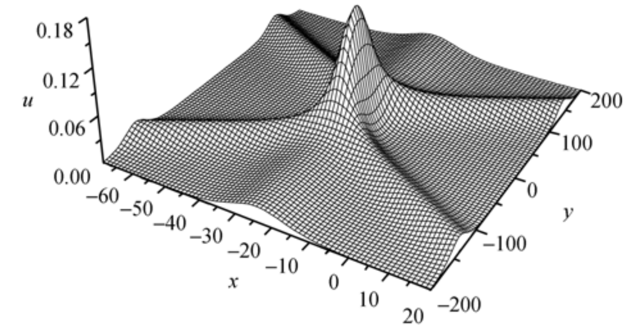

Fig. 2 The profile of u for three same amplitude soliton interaction, where $k_1=k_2=k_3=k=0.2$, $l_1=-l_2=0.0804$, $l_3=0$.

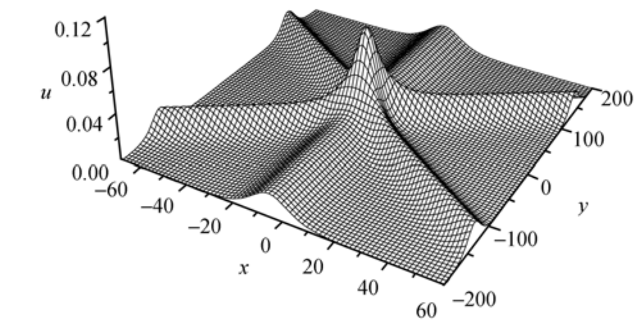

Fig. 3 The profile of u for three same amplitude soliton interaction, where $k_1=k_2=k_3=k=0.2$, $l_1=-l_2=0.1$, $l_3=0$.

Now we study a more general case where $l_1=-l_2=l$, $l_3=0$, but $k_1=k_2\neq k_3$. We assume that there are three solitons. We try to find whether the similar resonance phenomena takes place for this case. If it take place the question is how it takes place. We will study the three soliton solution of KP equation numerically and theoretically. By using the similar procedure as above for the three same amplitude soliton interaction, we found theoretically that when $l=k_1^2+k_1k_3=k_2^2+k_2k_3$, then the maximum amplitude during interaction between three solitons is reached. The maximum amplitude is

$$u_{\max} = \frac{1}{2}(k_1 + k_2 + k_3)^2 \tag{26}$$

Figures 4 and 5 are the solution of the KP equation for

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.3 \\ 0.4 \\ 0.21+10^{-16} \\ -(0.21+10^{-16}) \\ 0 \end{pmatrix} \text{and} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.6 \\ 0.4 \\ 0.6+10^{-16} \\ -(0.6+10^{-16}) \\ 0 \end{pmatrix}$$

respectively.

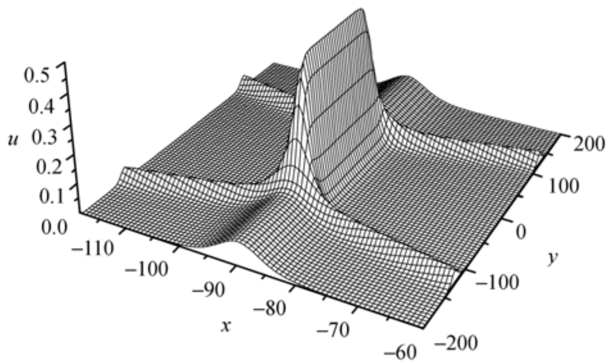

Fig. 4 The profile of u for three-soliton interaction, where $k_1 = k_2 = 0.3$, $k_3 = 0.4$, $l_1 = -l_2 \approx 0.21$, $l_3 = 0$.

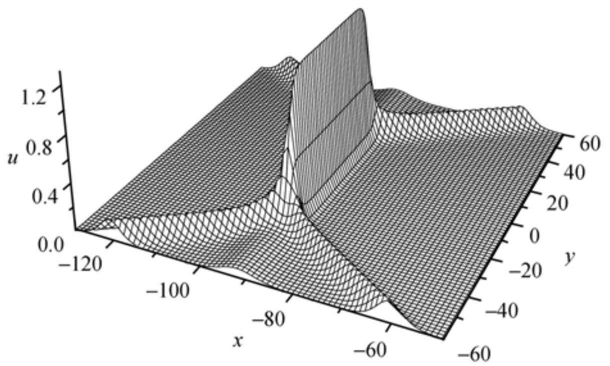

Fig. 5 The profile of u for three-soliton interaction, where $k_1 = k_2 = 0.6$, $k_3 = 0.4$, $l_1 = -l_2 \approx 0.6$, $l_3 = 0$.

Figures 4 and 5 show that the maximum amplitudes of the waves are 0.499 999 98 and 1.279 999 96, respectively, which are nearly same as the corresponding analytical results (0.5 and 1.28) given by Eq. (26) respectively.

If $k_1 = k_2 = k_3$, Eq. (26) is just same as Eq. (25). This is the same as that obtained for special case of three same amplitude soliton interaction.

Figure 6 is the solution of KP equation for

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.6 \\ 0.4 \\ 0.8 \\ -0.8 \\ 0 \end{pmatrix}$$

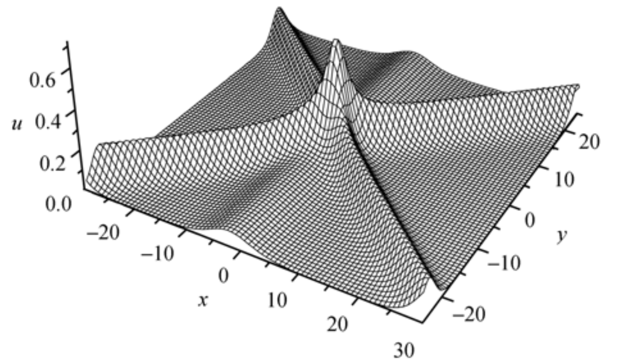

Fig. 6 The profile of u for three-soliton interaction, where $k_1 = k_2 = 0.6$, $k_3 = 0.4$, $l_1 = -l_2 = 0.8$, $l_3 = 0$.

In this case, we find numerically that the maximum amplitude of the wave is 0.568 962 0, which is less than 1.28. It suggests that the resonance does not take place when $l \neq k_1^2 + k_1 k_3$.

We can see from the figures presented above that during interaction between three solitons, there are phase shifts after the interations. At the resonance points, the phase shifts tends to infinity. We can see this results as the following procedures. We assume that there are three solitons, which have the arbitrary amplitudes and the arbitrary propagation directions. Assuming that $k_1 x + l_1 y - \omega_1 t \to +\infty$ and $k_2 x + l_2 y - \omega_2 t \to +\infty$, but $k_3 x + l_3 y = \omega_3 t$ is a finite value, we then obtain that $u \sim \frac{1}{2} k_3^2 \operatorname{sech}^2 \left[\frac{1}{2}(k_3 x + l_3 y - \omega_3 t + \delta_{p3}) \right]$ where $\delta_{p3} = \ln A_{13} + \ln A_{23}$. Similarly, letting $k_1 x + l_1 y - \omega_1 t \to \infty$ and $k_2 x + l_2 y - \omega_2 t \to -\infty$, but $k_3 x + l_3 y - \omega_3 t$ is a finite value, we then obtain that $u \sim \frac{1}{2} k_3^2 \operatorname{sech}^2 \left[\frac{1}{2}(k_3 x + l_3 y - \omega_3 t) \right]$. We can see that the amplitude and width of the third soliton after interacting is the same as the initial interaction soliton. We also find that there are a phase shift of $\frac{1}{2}\delta_{p3}$. As $\ln A_{13} + \ln A_{23}$ increases, then the phase shift increases. In a similar way, we can investigate the first and the second soliton. For the special case $k_1 = k_2 = k_3 = k$, $l_1 = -l_2 = l$, $l_3 = 0$, we find that δ_{p3} tends to ∞ as $l \to 2k^2$. This is completely agreement with the numerical results.

From Ref. [4], we can see that when resonance phenomena take place between two same amplitude solitons, the maximum amplitude is 4 times of the initial interacting soliton. In this paper, we find that when resonance take place among the three same amplitude solitons, the maximum amplitude is 9 times of the initial interacting soliton. So we can guess that when resonance take place among the N same amplitude solitons, the maximum amplitude is N^2 times of the initial interacting soliton. However, this should be proved. It is one of our future works. The resonance phenomena of the KP equation among the multi-soliton solutions can be useful to many branches of physics, such as: in the plasmas physics, for shallow water waves, for nonlinear optics, matter waves etc.

References

1. Zabusky N. J. and Kruskal M. D., Phys. Rev. Lett., 1965, 15: 240
2. Kadomtesv B. B. and V. I. Petviashvili., Sov. Phys. Doklady., 1970, 15: 539
3. Tsikis E. K., Raychaudhuri S., Gabl E. F., and Lonngren K. E., Plasma Phys. Controlled Fusion, 1985, 27: 419
4. Duan W. S., Shi Y. R., and Hong X. R., Phys. Lett. A, 2004, 323: 89
5. Miles J. W., J. Fluid. Mech., 1977, 79: 171
6. Maxworthy T., J. Fluid. Mech., 1980, 96: 47
7. Lonngren K. E., Ion acoustic soliton experiments in a plasma, Opt. Quantum. Electron., 1998, 30: 615
8. Nakamura Y., Bailung H., and Lonngren K. E., Phys. Plasmas, 1999, 6: 3466
9. Tsukabayashi I. and Nakamura Y., Resonant interaction of cylindrical ion-acoustic solitons, Phys. Lett. A, 1981, 85: 151
10. Tsukabayashi I., Nakamura Y., kako F., and Lonngren K. E., Phys. Fluid, 1983, 26: 790
11. Nakamura Y., bailung H., and Lonngren K. E., J. Plasma. Phys., 1999, 6: 3466
12. Ze F., Hershkowitz N., Chen C., and Lonngren K. E., Phys. Rev. Lett., 1979, 42: 1747
13. Kengne E., Chui S. T., and Liu M., Phys. Rev. E, 2006, 74: 036614
14. Li L., Malomed B. A., Mihalache D., and Liu W. M., Phys. Rev. E, 2006, 73: 066610
15. Kengne E. and Liu W. M., Phys. Rev. E, 2006, 73: 026603
16. Xie Z.W., Cao Z.X., Kats E. I., and Liu W. M., Phys. Rev. A, 2005, 71: 025601
17. Li D., He P. B., Li L., Liang J. Q., and Liu W. M., Phys. Rev. A, 2005, 71: 053611
18. Li L., Li Z. D., Malomed B. A., Mihalache D., and Liu W. M., Phys. Rev. A, 2005, 72: 033611
19. Wu B., Liu J., and Niu Q., Phys. Rev. Lett., 2002, 88: 034101
20. Liu J., Zhang C., Raizen M. G., and Niu Q., Phys. Rev. A, 2006, 73: 013601
21. He J. H., International Journal of Nonlinear Sciences and Numerical Simulation, 2005, 6: 207
22. He J. H., Computer Methods in applied Mechanics and Engineering, 1999, 178: 3—4, 257—262
23. Lin M. M. and Duan W. S., Phys. Plasmas, 2004, 11: 5710
24. Lin M. M. and Duan W. S., Chaos, Solitons, Fractals, 2005, 23: 929—937
25. Shi Y. R., Guo P., Lv K. P., and Duan W. S., Acta Physica Sinica, 2004, 53: 3265
26. Shi Y. R., Lv K. P., Duan W. S., Hong X. R., Zhao J. B., and Yang H. J., Acta Physica Sinica, 2003, 52: 267
27. Duan W. S., Wan G. X., Wang X. Y., and Lin M. M., Phys. Plasmas, 2004, 11: 4408
28. Duan W. S., Phys. Lett. A, 2003, 317: 275
29. Duan W. S. and Shi Y. R., Chaos, Solitons, Fractals, 2003, 18: 321
30. Adler M. and Moser J., Comm. Math. Phys., 1978, 61: 1
31. Ehlers F. and Knorrer H., Comment. Math. Helv., 1982, 57: 1
32. Svinolupov S. I., Sokolov V., and Yamilov R., Soviet. Math. Dokl., 1983, 28: 165