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REVIEW ARTICLE

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Semiclassical states in loop quantum gravity: an introduction

Abstract Unifying general relativity and quantum mechanics is a great challenge left to us by Einstein. To face this challenge, considerable progress has been made in non-perturbative canonical (loop) quantum gravity during the past 20 years. The kinematical Hilbert space of the quantum theory is constructed rigorously. However, the semiclassical analysis of the theory is still a crucial and open issue. In this review, we first introduce our work on constructing a semiclassical weave state, using the $\hat{Q}[\omega]$ operator on the kinematical Hilbert space of loop quantum gravity. Then we give an introduction to the two different approaches currently investigated for constructing coherent states in the kinematical Hilbert space. The current status of semiclassical analysis in loop quantum gravity is then summarized.

Keywords loop quantum gravity, background independence, semiclassical states

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1 Introduction

The research on quantum gravity theory is rather active. Many quantization programs for gravity are being carried out (for a summary see, e.g., [1]). In these different kinds of approaches, the idea of loop quantum gravity is motivated by researchers in the community of general relativity. It follows closely the thoughts of general relativity, and hence it is a quantum theory born with background independence. For recent review in this field, we refer to [2, 3, 4]. Roughly speaking, loop quantum gravity is an attempt to construct a mathematically rigorous, non-perturbative, background independent quantum theory of four-dimensional, Lorentzian general relativity plus all known matter in the continuum. The project of loop quantum gravity inherits the basic idea of Einstein that gravity is fundamentally a space-time geometry. Here one believes in that the theory of quantum gravity is a quantum theory of space-time geometry with diffeomorphism invariance (this legacy is discussed comprehensively in Rovelli's book [5]).

To apply the background independent quantization technique, one casts general relativity into the Hamiltonian formalism of a diffeomorphism invariant Yang-Mills gauge field theory with a compact internal gauge group [6, 7]. The kinematical Hilbert space, $\mathcal{H}_{\text{kin}}$, of the quantum theory is then constructed rigorously. One can even solve the Gaussian and diffeomorphism constraints to arrive at a diffeomorphism invariant Hilbert space [8]. Certain geometrical operators corresponding to the measure of length [9], area [10, 11], volume [12] and the integrated norm of any smooth 1-form [13] are shown to have discrete spectra in the kinematical Hilbert space. The classical limit of the quantum theory is currently under investigation [14, 15, 16, 17]. Moreover, the fundamental discreteness in loop quantum gravity is crucially used to make much progress in symmetric models such as: loop quantum cosmology [18], quantum black hole [19] and static space-time [20]. However, despite the systematic efforts in constructing the Hamiltonian constraint operator [21, 22] and the master constraint operator [23, 24, 24], to accept the theory as a conceivable candidate for describing quantum space-time, we need to prove that its classical limit is general relativity(GR) or at least overlaps GR in the regime where GR is well tested. A weave state was first introduced in Ref.[14] to approximate the flat 3-geometry. Being one of the solutions in Ref.[26], it is regarded as a physical state of loop quantum gravity. However, this state is an eigenstate of the volume operator with vanishing eigenvalue. Moreover, as argued in Refs.[27,28], the classical correspondences of the solutions in Ref.[26] and their generalization [28] should all be degenerate metrics, which are not admitted in the traditional GR. There are also other kinematical weave states constructed at the unconstrained level [29, 30, 31, 32].

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One of the most remarkable physical results in loop quantum gravity is the evidence for a quantum discreteness of space at the Planck scale. This is due to the fact that certain geometric operators have discrete spectra. There is another geometrical operator $\hat{Q}[\omega]$ proposed in previous literatures [14], which corresponds to the integrated norm of any smooth 1-form ω_a on the 3-manifold. While the operators of area and volume had been shown to be well-defined self-adjoint operators on $\mathcal{H}_{\text{kin}}$, the general properties of $\hat{Q}[\omega]$ were still unclear. We even do not know if it is well-defined on $\mathcal{H}_{\text{kin}}$. The obstacles are due to two facts: first, the result of $\hat{Q}[\omega]$ operating on a cylindrical function will involve integrals over edges of the graph on which the function is defined, and hence it is not a cylindrical function any wore in general; second, the current effective regularization technique of smearing the triads in 2-dimension [11] could not be directly applied to the regularization of $\hat{Q}[\omega]$, whose classical expression involves the square of the triads while there is an integral over 3-dimensional manifold.

In this review, we will first introduce our work on the $\hat{Q}[\omega]$ operator [13] and a degenerate weave state [15].Since $\hat{Q}[\omega]$ operator is rather convenient for constructing certain weave states in the study of the classical approximation of the quantum theory [14], we first study the properties of $\hat{Q}[\omega]$ on $\mathcal{H}_{\text{kin}}$ in order to lay a foundation for its applications. To bypass the above mentioned obstacles, we will use a 3-dimensional smearing function for regularization. Then, instead of acting the regulated $\hat{Q}[\omega]$ on general cylindrical functions, we will operate it on spin network states, which form a complete orthonormal basis in $\mathcal{H}_{\text{kin}}$. It turns out that the operation gives a real discrete spectrum, which is in the same form as its eigenvalues on coloured loop states. Thus, $\hat{Q}[\omega]$ is a well-defined symmetric operator in $\mathcal{H}_{\text{kin}}$. A further discussion shows that it is also self-adjoint. By using the $\hat{Q}[\omega]$ operator, we then show that in loop quantum gravity, a quantum state based on the s-knot class of an infinite number of open curves can solve all quantum constraints and approximate a degenerate 3-geometry, from which a non-degenerate metric region can be evolved by the classical Ashtekar equations. Thus, a physical state of canonical quantum gravity is related to the familiar classical geometry.

Although we have gained great enlightenment from weave states, they are not enough for a full semiclassical analysis since weave states usually approximate only 3-geometries but not the connections. In order to check the classical limit of the Hamiltonian constraint operator in the kinematical Hilbert space, kinematical coherent states that can approximate both 3-geometries and the connection are currently being development. We will introduce two different approaches that appear in the literature so far. One leads to the so-called complexifier coherent states proposed by Thiemann *et al.* [33, 34, 35, 36]. The other is promoted by Varadarajan [37, 38, 39] and developed by Ashtekar *et al.* [17, 40].

2 Kinematical Hilbert space of loop quantum gravity

Canonical gravity in the real Ashtekar formalism is defined over an oriented 3-manifold Σ. The basic variables are real $SU(2)$ connections, A_a^i, as the configuration and the densitized (weight 1) triads, E_j^b, corresponding to the conjugate momentum. We use $a, b, \cdots$ for spatial indices and $i, j, \cdots$ for internal $SU(2)$ indices. Given any graph γ, with the set of edges $E(\gamma) \equiv \{e_1, \cdots, e_n\}$ and the set of vertices $V(\gamma) \equiv \{\upsilon_1, \cdots, \upsilon_m\}$, embedded in Σ, the holonomy of the $SU(2)$ connection A_a^i along any edge e_I is an element of $SU(2)$ and can be expressed as:

$$A(e_I) := \mathcal{P}\exp\int_{e_I} \mathrm{d}s\dot{e}_I(s)A_a^i(e_I(s))\tau_i \quad (1)$$

where $\mathcal{P}$ denotes path ordering and τ_i are the $SU(2)$ generators in the fundamental representation. Given a function $f_n:[SU(2)]^n \to \mathbb{C}$, the cylindrical function is defined as:

$$f_\gamma(A) := f_n(A(e_1), \cdots, A(e_n)) \quad (2)$$

We denote $Cyl(\overline{A})$ as the set of cylindrical functions. Since any two cylindrical functions based on different graphs can always be viewed as being defined on the same graph that is just constructed as the union of the original ones, it is straightforward to define a scalar product for them by:

$$\langle f_\gamma \mid g_\gamma \rangle := \int_{[SU(2)]^n} \mathrm{d}\mu_\gamma \overline{f_n(A(e_1), \cdots, A(e_n))} g_n(A(e_1), \cdots, A(e_n)) \quad (3)$$

where $\mathrm{d}\mu_\gamma$ is the Haar measure of $[SU(2)]^n$, which is naturally induced by that of $SU(2)$. The Hilbert space, $\mathcal{H}_{\text{kin}} \equiv L^2(\overline{A}, \mathrm{d}\mu)$, is obtained by completing the space of all finite linear combinations of cylindrical functions in the norm induced by the quadratic form Eq.(3) on a cylindrical function.

We now introduce the spin network basis in $\mathcal{H}_{\text{kin}}$. To each e_I we assign a non-trivial irreducible spin j_I representation of $SU(2)$. This is called a colouring of the edge. Next, consider a vertex υ_α, say a K-valent one, i.e., there are K edges $e_1, \cdots, e_K$ meeting at υ_α. Let $\mathcal{H}_{j1}, \cdots, \mathcal{H}_{jK}$ be the Hilbert spaces of the representations, $j_1, \cdots, j_K$, associated to the K edges. Consider the tensor product of these spaces $\mathcal{H}_{\upsilon_\alpha} = \mathcal{H}_{j1} \otimes \cdots \otimes \mathcal{H}_{jK}$, and fix, once and for all, an orthonormal basis, N_α, in $\mathcal{H}_{\upsilon_\alpha}$. This is called a colouring of the vertex. A (non-gauge invariant) spin network, S, is then defined as the embedded graph whose edges and vertices have been coloured. The (non-gauge invariant) spin network state, $\Psi_S(A)$, based on S is defined as:

$$\Psi_S(A) = \bigotimes_{e_I \in E(\gamma)} jI(A(eI)) \cdot \bigotimes_{v_\alpha \in V(\gamma)} N_\alpha \quad (4)$$

where $j_I(A(e_I))$ is the representation matrix of the holonomy $A(e_I)$ in the spin j_I representation associated to the edge e_I, and the holonomy matrices are constructed with the vector N_α at each vertex υ_α where the edges meet. By varying the graph, the colours of the edges, and the colours of the vertices, we obtain a family of spin network states. It turns out that these states form a complete orthonormal basis in the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$ [11].

Since a $SU(2)$ gauge transformation acts on a spin network state simply by $SU(2)$ transforming the colouring of the vertices N_α, it is easy to recover the gauge invariant spin network states by colouring each vertex with a $SU(2)$ invariant basis. These states form a complete orthonormal basis in the $SU(2)$ gauge invariant Hilbert space $\mathcal{H}_G$ [41, 11].

3 $\hat{Q}$ operator and degenerate weave state

The operator $\hat{Q}[\omega]$ is constructed to represent the classical quantity [14]

$$Q[\omega] = \int \mathrm{d}^3x \sqrt{E_i^a(x)\omega_a(x)E^{bi}(x)\omega_b(x)} \quad (5)$$

where ω_a is any smooth 1-form on Σ that makes the integral meaningful, and the integral is well-defined since the integrand is a density of weight 1. If we know $Q[\omega]$ for all smooth ω_a, the triad E_i^a can be reconstructed up to local $SU(2)$ gauge transformations. Hence, the collection of $Q[\omega]$ provides a good coordinates system on the space of the triads fields.

Since E_i^a represents the conjugate momentum of the configuration variable A_a^i, the formal expression of the corresponding momentum operator would be some functional derivative with respect to A_a^i, i.e.,

$$\hat{E}_i^a(x) = -\mathrm{i}G\hbar \frac{\delta}{\delta A_a^i(x)} \quad (6)$$

This is an operator-valued distribution rather than a genuine operator, hence it has to be integrated against smearing functions in order to be well-defined. Let $f_\epsilon(x,y)$ be a 1-parameter family of fields on Σ which tends to $\delta(x,y)$ as ϵ tends to zero, such that $f_\epsilon(x,y)$ is a density of weight 1 in x and a function in y. We then define the smeared version of $E_i^a(x)\omega_a(x)$ as:

$$[E_i\omega]_f(x) := \int \mathrm{d}^3y f_\epsilon(x,y)E_i^a(y)\omega_a(y) \quad (7)$$

Hence, $[E_i\omega]_f(x)$ tends to $E_i^a(x)\omega_a(x)$ as ϵ tends to zero. Then $Q[\omega]$ can be regulated as:

$$Q[\omega] = \lim_{\epsilon \to 0} \int \mathrm{d}^3x([E_i\omega]_f(x)[E^i\omega]_f(x))^{\frac{1}{2}} \quad (8)$$

To go over to the quantum theory, we simply replace E_i^a by $\hat{E}_i^a$ and obtain

$$\hat{Q}[\omega] = \lim_{\epsilon \to 0} \int \mathrm{d}^3x \ ([\hat{E}_i\omega]_f(x) \ [\hat{E}^i\omega]_f(x))^{\frac{1}{2}} \quad (9)$$

In Ref.[?], we show that the action of $\hat{Q}[\omega]$ on spin network states yields

$$\hat{Q}[\omega] \circ \Psi_S(A) = l_p^2 \sum_{I=1}^{n} \left[\int_{e_I} \mathrm{d}t \, | \dot{e}_I^a(t)\omega_a(e_I(t)) | \sqrt{j_I(j_I+1)} \right] \cdot \Psi_S(A) \quad (10)$$

Therefore, spin network states are also eigenvectors of $\hat{Q}[\omega]$ The complete spectrum of $\hat{Q}[\omega]$ with respect to the spin network basis in the Hilbert space $\mathcal{H}_{\text{kin}}$ is obtained. In contrast to the volume operator, which acts only on vertices [12], $\hat{Q}[\omega]$ acts only on edges of spin networks. Hence, the spin network states based on a same graph with same colouring of the edges are all degenerate with respect to this operator. As a result, the action of $\hat{Q}[\omega]$ on the gauge invariant spin network states gives the same result as Eq.(10). In this sense, the spectrum of $\hat{Q}[\omega]$ respects the physically relevant states in $\mathcal{H}_{\text{kin}}$.

We have shown that $\hat{Q}[\omega]$ is diagonalized in the spin network basis with real eigenvalues, hence it is a well-defined symmetric operator in the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$. Moreover, it is obvious from Eq.(9) that the expression of $\hat{Q}[\omega]$ is purely real, and hence it commutes with the complex conjugation. Therefore, it follows from Von Neumann's theorem [42] that $\hat{Q}[\omega]$ admits self-adjoint extensions on $\mathcal{H}_{\text{kin}}$. The same reasons lead to that $\hat{Q}[\omega]$ is also self-adjoint on the gauge invariant Hilbert space $\mathcal{H}_G$.

The Ashtekar theory admits a generalization of GR to involve degenerate metrics, since the inverse of the triads is not necessary for the whole formalism. The geometry that we want to approximate is a degenerate "flat" 3-metric, h_{ab}, of rank 2 on R^3. The metric is "flat" in the sense that there exists a foliation $R^3 = R^2 \times R$ such that the induced 2-metric, q_{ab}, of h_{ab} on R^2 is the flat Euclidean metric. Let $\{X, Y, Z\}$ be the Cartesian coordinates on R_3 compatible with the decomposition $R^3 = R^2 \times R$ and $\left(\frac{\partial}{\partial Z}\right)^a$ be the degenerate vector field of h_{ab}. Thus the line element of h_{ab} reads

$$\mathrm{d}s^2 = \mathrm{d}X^2 + \mathrm{d}Y^2 \quad (11)$$

The weave states that approximate classical 3-metrics were first constructed as the eigenstates of geometrical operators such as $\hat{Q}[\omega]$ and the operators of area and volume [14, 29, 31]. The corresponding eigenvalues are required to agree with the classical values of the geometrical quantities at large scales. The successful construction of $\mathcal{H}_{\rm kin}$ prompts us now to approximate a classical geometry by the expectation values of the geometrical operators.

For the operator $\hat{Q}[\omega]$, one can define the following: A quantum state Ψ is said to approximate a classical metric on Σ at scales larger than a macroscopic length scale L accessible by current measurement if, for all ω_a on Σ,

$$\text{(i)}\ \langle Q\rangle := \langle\Psi\,|\,\hat{Q}[\omega]\,|\,\hat{\Psi}\rangle = Q[\omega] + O\left(\frac{\delta}{L}\right) \tag{12}$$

$$\text{(ii)}\ \Delta_Q := (\langle Q^2\rangle - \langle Q\rangle^2)^{\frac{1}{2}} \ll Q[\omega] \tag{13}$$

where δ is a fixed length chosen as $l_{Pl} < \delta \ll L$ .Such a state Ψ is said to be semiclassical with respect to the observables $\hat{Q}[\omega]$. However, this definition may face obstruction when it is used for non-compact Σ, such as R^3. As argued in Ref.[32], the weave states that describe the geometries on R^3 have to be based on graphs of an infinite number of curves, while the states in $\mathcal{H}_{\rm kin}$ constructed so far are based on graphs of finite collections of curves. We now think of a way to overcome the obstruction to a certain extent. Suppose there is a cover $\{C_i\}$, consisting of 3-dimensional regions C_i, of a non-compact Σ, such that for any C_i, a weave state W_Σ based on a graph Γ (may consist of an infinite number of curves) can always be expressed as:

$$W_\Sigma = W_{C_i} W_{\Sigma - C_i} \tag{14}$$

where the cylindrical functions W_{C_i} and $W_{\Sigma-C_i}$ are based, respectively, on the subgraphs of Γ restricted to C_i and $\Sigma - C_i$, and the subgraphs of the regions C_i all consist of finite numbers of curves. Then we can define that W_Σ approximates a classical metric on Σ if all W_{C_i} approximate, according to Eqs. (12) and (13), the metrics restricted to C_i. Note that this definition is valid for all of the weave states and their 3-metrics presented so far.

We now construct a weave state that approximates the above given degenerate metric h_{ab}. The basic idea is to consider a family of an infinite number of non-intersecting open curves, $\{\gamma_i\}$, instead of closed loops on R^3. All of the γ_i are required to be the integral curves of the degenerate vector field of h_{ab}, and hence match the Z-coordinate curves exactly. This kind of curve was called "large loops" in Refs.[29, 30]. Using the induced 2-metric q_{ab} on a 2-surface $Z = const.$, we fix the intersections of γ_i and the surface as the lattice sites of a square lattice on R^2 with lattice spacing λ . As mentioned in Ref.[32], a way of dealing with states based on curves of infinite length is to consider a compacti.cation of Σ [43]. Thus γ_i may also be regarded as a closed loop on $\bar{R}^3$,where $\bar{R}^3 := R^3 \cup \infty$ is the one-point compactification of R^3. Following Ref.[32], we define the desired "quasi-coherent" state, $W_{\{\}}$, based on $\{\gamma_i\}$ as:

$$W_{\{\}} := \lim_{n\to\infty} \prod_{i=1}^{n} \psi_i \tag{15}$$

where

$$\psi_i := \eta \exp(\beta T_r[\rho_1(H[\gamma_i] - e)]) \tag{16}$$

here, β is an arbitrary constant, e is the identity in $SU(2)$, and η is a normalization factor.

To see if $W_{\{\}}$ weaves the classical geometry determined by h_{ab}, let us consider a cover $\{\mathcal{O}_m\}$ of R^3, where $\mathcal{O}_m$ denotes the 3-dimensional region $\{(X,Y,Z)|X^2+Y^2<m^2, m\in N\}$. Here, N is the collection of natural numbers. Let n be the number of curves γ_i in region $\mathcal{O}_m$, it is obvious from Eqs. (15) and (16) that, for any $\mathcal{O}_m$,

$$W_{\{\}} = W_n W_{\{\}-n} \tag{17}$$

where W_n and $W_{\{\}-n}$ are based, respectively, on the graphs $\{\gamma_i \subset \mathcal{O}_m\}$ and $\{\gamma_i \subset (R^3 - \mathcal{O}_m)\}$, which are the subgraphs of $\{\gamma_i\}$ restricted, respectively, to $\mathcal{O}_m$ and ($R^3 - \mathcal{O}_m$), and

$$W_n = \prod_{i=1}^{n} \psi_i \tag{18}$$

The remaining task is to prove that W_n approximates the geometry of h_{ab} on $\mathcal{O}_m$. Calculations in Ref.[15] show that $W_{\{\}}$ approximates the degenerate metric h_{ab} on R^3 at scales larger than L.

The "quasi-coherent" feature of the weave $W_{\{\}}$ can be seen from its construction of Eqs. (15), (16), and (18). The function W_n takes on its maximum value when $H[\gamma_i] = e$ and hence, as $n\to\infty$, the function W_n becomes increasingly peaked around the connections A_a^i, which gives a trivial holonomy along all curves γ_i. Note that the function $Q[\omega]$ carries suffcient information about the 3-metric. If we know $Q[\omega]$ for every smooth 1-form ω_a, the metric is known completely. Using the area operator [14, 11], it is not diffcult to check that the weave $W_{\{\}}$ will reproduce as well the correct values of the areas of any 2-surfaces measured by h_{ab} in R^3. Since the curves γ_i are non-intersecting, $W_{\{\}}$ will give a zero expectation value of the volume operator [12] for any 3-dimensional regions. This is the right result because h_{ab} is degenerate.

The other interesting result of Ref.[15] is that a non-degenerate space-time region can be evolved by the classical Ashtekar equations from the degenerate 3-metric woven above. In other words, $W_{\{\}}$ has approximated the degenerate 3-metric from which a non-degenerate metric region can be evolved by the classical Ashtekar equations.

Another approving point is that the $SU(2)$ connection in the solution gives a trivial holonomy along all curves on R^3, and hence is one of the connections peaked around by the same weave state.

The weave state based on "large loops" was first proposed in Ref.[44] to approximate a flat 3-metric. But, further investigations show that this kind of weave cannot give the "correct eigenvalue" for the area operator [29, 30]. However, our construction shows that "large loops" are well suited to weave degenerate metrics without any problem for the area operator, because the presence of preferred directions of the curves just respects the feature of degenerate metrics. Our example in Ref.[15] shows that the degeneracy of 3-metrics is not preserved by Ashtekar equations, although it is concluded in Ref.[45] that the "degeneracy type of triads" is locally preserved by the evolution. Moreover, in contrast to the solution in Ref.[46] where the causality has to be broken in order to evolve a degenerate metric from non-degenerate initial data, the causal structure, which may be degenerate [47], of the whole space-time can still be well without any breaking in the inverse evolution. In this sense the non-degenerate region in the example is causally evolved from the degenerate initial data.

It is straight for ward to see that the weave state $W_{\{\}}$ solves the quantum Hamiltonian constraint. A common point to all different regularization procedures in loop quantum gravity is that the Hamiltonian constraint operator acts only on the nodes of spin networks [21, 28]. From the definition (15), it is obvious that $W_{\{\}}$ can be expanded by spin network basis as:

$$W_{\{\}} = \sum_{\{P_i\}} c\{P_i\}\Psi_{\{P_i\}} \tag{19}$$

where, $\Psi_{\{P_i\}} = \lim_{n\to\infty}\prod_{i=1}^{n}\Psi_{Pi}[\gamma_i]$ is based on the spin network $\{P_i\}$ which is obtained by colouring P_i to every γ_i, and the sum is over all possible choices of colouring of γ_i. Since the graph $\{\gamma_i\}$ consists of non-intersecting curves, $\Psi_{\{P_i\}}$ and hence $W_{\{\}}$ are annihilated by the Hamiltonian constraint operator. In fact, $W_{\{\}}$ can be viewed as a special kind of combinatorial solution in Ref.[28].

To get the state solving the diffeomorphism constraint, we use the loop representation [26, 41] and define the spin network state $\Phi_{K\{P_i\}}$ on non-intersecting coloured curves $\alpha_{p'}$ by:

$$\Phi_{K\{P_i\}}[\alpha_{P'}] := \begin{cases} 1, & \text{if } \alpha_{P'} \in K(\{P_i\}) \\ 0, & \text{otherwise} \end{cases} \tag{20}$$

where the s-knot $K(\{P_i\})$ is the equivalence class of the embedded spin networks $\{P_i\}$ under the action of the diffeomorphism group, $Diff(R^3)$, on R^3, i.e., $\{P_i\},\{P^j\}\in K$, if there exists a $\phi \in Diff(R^3)$, such that $\{P'_j\} = \phi \cdot \{P_i\}$. Replacing the spin network basis $\Psi_{\{P_i\}}$ in Eq.(19) by the diffeomorphism-invariant knot states $\Phi_{K\{P_i\}}$, we obtain the corresponding quantum state:

$$w_{K\{\}} = \sum_{\{P_i\}} c_{\{p_i\}}\Phi_{K\{p_i\}} \tag{21}$$

which solves all the quantum constraints. Hence, $W_{K\{\}}$ should be a physical state of loop quantum gravity for R^3, although it does not belong to the Hilbert space constructed currently for the states based on graphs of a finite number of curves. Since the spin network $\{P_i\}$ corresponds to a rank 2 degenerate flat metric h_{ab}, the equivalence class $K(\{P_i\})$ of spin networks should correspond to the equivalence class of all metrics related to h_{ab} by a spatial diffeomorphism. Thus it is natural to interpret $W_{K\{\}}$ as representing the rank 2 degenerate flat 3-geometry at large scales. Moreover, this degenerate 3-geometry can be related to some locally non-degenerate geometry by classical Ashtekar's equations, and hence it plays the role of a bridge between a physical state in canonical quantum gravity and the familiar classical geometry.

4 Complexifier coherent state

The complexifier approach is somehow motivated by the coherent state construction for compact Lie groups [48].One begins with a positive function C(complexifier)on the classical phase space and arrives at a "coherent state" ψ_m, which more possibly belongs to the space Cyl^* dual to $Cyl(\overline{\mathcal{A}})$, rather than $\mathcal{H}_{\text{kin}}$. However, one may consider the so-called "cut-off state" of ψ_m with respect to a finite graph as a graph-dependent coherent state in $\mathcal{H}_{\text{kin}}$ [1]. By construction, the coherent state ψ_m is an eigenstate of an annihilation operator coming also from the complexifier C and hence has desired semiclassical properties [34, 35]. We now sketch the basic idea of its construction. Let $\mathcal{M}$ be the phase space of a dynamical system. The semiclassical states in the Hilbert space $\mathcal{H}$ of the corresponding quantum theory are defined in GNS terminology as follows.

Definition 4.1 *Given a class of observables $\mathcal{S}$ which comprises a subalgebra in the space $\mathcal{L}(\mathcal{H})$ of linear operators on the Hilbert space, a family of (pure) states $\{\omega_m\}_{m\in M}$ are said to be semiclassical with respect to $\mathcal{S}$ if and only if:*

(1) *The observables in $\mathcal{S}$ should have correct semiclassical limit on semiclassical states, i.e.,*

$$\lim_{\hbar\to 0}\left|\frac{\omega_m(\hat{a}) - a(m)}{a(m)}\right| = 0$$

and

(2) *the fluctuations should be small, i.e.,*

$$\lim_{\hbar\to 0}\left|\frac{\omega_m(\hat{a}^2) - \omega_m(\hat{a})^2}{\omega_m(\hat{a})^2}\right| = 0$$

for all $\hat{a} \in \mathcal{S}$.

Kinematical coherent states $\{\Psi_m\}_{m\in M}$ are semiclassical states that in addition satisfy the annihilation operator property [33, 1], namely there exists certain non-self-adjoint operator $\hat{z}=\hat{a}+\mathrm{i}\lambda\hat{b}$ with $\hat{a}$, $\hat{b}\in\mathcal{S}$ and certain squeezing parameter λ, such that

$$\hat{z}\Psi_m = z(m)\Psi_m \tag{22}$$

Moreover, the set of $\{\Psi_m\}_{m\in M}$ should be dense in $\mathcal{H}$. Note that Eq.(22) implies that the minimal uncertainty relation is saturated for the pair of elements ($\hat{a},\hat{b}$), i.e.,

$$\Psi_m([\hat{a}-\Psi_m(\hat{a})]^2)=\Psi_m([\hat{b}-\Psi_m(\hat{b})]^2)=\frac{1}{2}\left|\Psi_m([\hat{a},\hat{b}])\right| \tag{23}$$

Note also that coherent states are usually required to satisfy the additional peakedness property, namely for any $m\in\mathcal{M}$ the overlap function $\left|\langle\Psi_m,\Psi_{m'}\rangle\right|$ is concentrated in a phase volume $\frac{1}{2}\left|\Psi_m([\hat{q},\hat{p}])\right|$, where $\hat{q}$ is a configuration operator and $\hat{p}$ a momentum operator. So the central stuff in the construction is to define a suitable "annihilation operator" $\hat{z}$ in analogy with the simplest case of harmonic oscillator. A powerful tool named as "complexifier" is introduced in Ref.[33] to define a meaningful $\hat{z}$ operator which can give rise to kinematical coherent states for a general quantum system.

Definition 4.2 *Given a phase space $\mathcal{M}=T^*C$ for some dynamical system with configuration coordinates q and momentum coordinates p, a complexifier, C, is a positive smooth function on $\mathcal{M}$, such that*

(1) *$C/\hbar$ is dimensionless;*

(2) $\lim\limits_{\|p\|\to\infty}\frac{|C(m)|}{\|p\|}=\infty$ *for some suitable norm on the space of the momentum*;

(3) *Certain complex coordinates* ($z(m)$, $\bar{z}(m)$) *of $\mathcal{M}$ can be constructed from C.*

Given a well-defined complexifier C on phase space $\mathcal{M}$, the program for constructing coherent states associated with C can be carried out as the following.

• *Complex polarization*

The condition (3) in Definition 4.2 implies that the complex coordinate $z(m)$ of M can be constructed via

$$z(m):=\sum_{n=0}^{\infty}\frac{\mathrm{i}^n}{n!}\{q,C\}_{(n)}(m) \tag{24}$$

where the multiple Poisson bracket is inductively defined by $\{q,C\}_{(0)}=q$, $\{q,C\}_{(n)}=\{\{q,C\}_{(n-1)},C\}$. One will see that $z(m)$ can be regarded as the classical version of an annihilation operator.

• *Defining annihilation operator*

After the quantization procedure, a Hilbert space $H=L^2(C,\mathrm{d}\mu)$ with a suitable measure $\mathrm{d}\mu$ on a suitable configuration space C can be constructed. It is reasonable to assume that C can be defined as a positive self-adjoint operator $\hat{C}$ on $\mathcal{H}$. Then a corresponding operator $\hat{z}$ can be defined by transforming the Poisson brackets in Eq.(24) into commutators, i.e.,

$$\hat{z}:=\sum_{n=0}^{\infty}\frac{\mathrm{i}^n}{n!}\frac{1}{(\mathrm{i}\hbar)^n}[\hat{q},\hat{C}]_{(n)}=\mathrm{e}^{-\hat{C}/\hbar}\hat{q}\mathrm{e}^{\hat{C}/\hbar} \tag{25}$$

which is called as an annihilation operator.

• *Constructing coherent states*

Let $\delta_{q'}(q)$ be the δ-distribution on C with respect to the measure $\mathrm{d}\mu$. Since $\hat{C}$ is assumed to be positive and self-adjoint, the conditions (1) and (2) in Definition 4.2 imply that $\mathrm{e}^{-\hat{C}/\hbar}$ is a well-defined "smoothening operator". So it is quite possible that the heat kernel evolution of the δ-distribution, $\mathrm{e}^{-\hat{C}/\hbar}\delta_{q'}(q)$, is a square integrable function in H, which is even analytic. Then one may analytically extend the variable q' in $\mathrm{e}^{-\hat{C}/\hbar}\delta_{q'}(q)$ to complex values $z(m)$ and obtain a class of states ψ'_m as

$$\psi'_m(q):=[\mathrm{e}^{-\hat{C}/\hbar}\delta_{q'}(q)]_{q'\to z(m)} \tag{26}$$

such that one has

$$\begin{aligned}\hat{z}\psi'_m(q)&:=[\mathrm{e}^{-\hat{C}/\hbar}\hat{q}\delta_{q'}(q)]_{q'\to z(m)}=[q'\mathrm{e}^{-\hat{C}/\hbar}\delta_{q'}(q)]_{q'\to z(m)}\\&=z(m)\psi'_m(q)\end{aligned} \tag{27}$$

Hence ψ'_m is automatically an eigenstate of the annihilation operator $\hat{z}$. So it is natural to define coherent states $\psi_m(q)$ by normalizing $\psi'_m(q)$.

One may check that all the coherent state properties usually required are likely to be satisfied by the above complexfier coherent states $\psi_m(q)$ [1]. As a simple example, in the case of one-dimensional harmonic oscillator with Hamiltonian $H=\frac{1}{2}\left(\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2\right)$, one may choose the complexifier $C=p^2/(2m\omega)$. It is straightforward to check that the coherent state constructed by the above procedure coincides with the usual harmonic oscillator coherent state up to a phase [1]. So the complexifier coherent state can be considered as a suitable generalization of the concept of usual harmonic oscillator coherent state.

The complexifer approach can be used to construct kinematical coherent states in loop quantum gravity. Given a suitable complexifier C, for each analytic path $e\subset\Sigma$ one can define

$$A^C(e):=\sum_{n=0}^{\infty}\frac{\mathrm{i}^n}{n!}\{A(e),C\}_{(n)} \tag{28}$$

where $A(e)\in SU(2)$ is assigned to e. As the complexifier C is assumed to give rise to a positive selfadjoint operator $\hat{C}$ on the kinematical Hilbert space $\mathcal{H}_{\mathrm{kin}}$, one further supposes that $\hat{C}/\hbar T_s=\tau\lambda_s T_s$, where τ is a so-called classicality parameter, $\{T_s(A)\}_s$ form a system of basis in $\mathcal{H}_{\mathrm{kin}}$ and are analytic in $A \in \overline{A}$ .Moreover, the δ -distribution on the quantum configuration space $\overline{A}$ can be formally expressed as $\delta_{A'}(A) = \sum_s T_s(A')\overline{T_s(A)}$. Thus by applying Eq.(26) one obtains coherent states

$$\psi'_{A^C}(A) = (\mathrm{e}^{-\hat{C}/\hbar})\delta_{A'}(A)|_{A' \to A^C} = \sum_s \mathrm{e}^{-\tau\lambda_s} T_s(A^C)\overline{T_s(A)} \quad (29)$$

However, since there is an uncountable infinite number of terms in the expression (29), the norm of $\psi'_{A^C}(A)$ would in general be divergent. So $\psi'_{A^C}(A)$ is generally not an element of $\mathcal{H}_{\text{in}}$ but rather a distribution on a dense subset of $\mathcal{H}_{\text{kin}}$. In order to test the semiclassical limit of quantum geometric operators on $\mathcal{H}_{\text{kin}}$, one may further consider the "cut-offstate" of $\psi'_{A^C}(A)$ with respect to a finite graph γ as a graph-dependent coherent state in $\mathcal{H}_{\text{kin}}$ [1]. So the key input in the construction is to choose a suitable complexifer. There are vast possibilities of choices. For example, a candidate complexifier *C* is considered in Ref.[2] such that the corresponding operator acts on cylindrical functions f_γ by

$$(\hat{C}/\hbar)f_\gamma = \frac{1}{2}\left(\sum_{e \in E(\gamma)} l(e)\hat{J}_e^2\right)f_\gamma \quad (30)$$

where $\hat{J}_e^2$ is the Casimir operator associated to the edge *e*, the positive numbers *l*(*e*) satisfy $l(e \circ e') = l(e) + l(e')$ and $l(e^{-1}) = l(e)$. Then it can be shown from Eq.(28) that $A^C(e)$ is an element of *SL*(2,C). So the classical interpretation of the annihilation operators is simply the generalized complex *SU*(2) connections. It has been shown in Refs. [34] and [35] that the "cut-off state" of the corresponding coherent state,

$$\psi_{A^C,\lambda}(A) = \psi'_{A^C,\gamma}(A) / \left\|\psi'_{A^C,\gamma}(A)\right\| \quad (31)$$

has desired semiclassical properties, where

$$\psi'_{A^C,\gamma}(A) := \sum_{s,\gamma(s)=\gamma} \mathrm{e}^{-\frac{1}{2}\sum_{e \in E(\gamma(s))} l(e) j_e(j_e+1)} T_s(A^C)\overline{T_s(A)} \quad (32)$$

But unfortunately, these cut-off coherent states cannot be directly used to test the semiclassical limit of the Hamiltonian constraint operator $\hat{\mathcal{H}}(N)$, since $\hat{\mathcal{H}}(N)$ is graph-changing so that its expectation values with respect to these cut-off states are always zero! So further work in this approach is expected in order to overcome the difficulty. Anyway, the complexifier approach provides a clean construction mechanism and manageable calculation method for semiclassical analysis in loop quantum gravity.

5 Laplacian coherent state

As we have seen, loop quantum gravity is based on quantum geometry, where the fundamental excitations are 1-dimensional polymer-like. On the other hand, low energy physics is based on quantum field theories that are constructed on a flat space-time continuum. The fundamental excitations of these fields are 3-dimensional, typically representing wavy undulations on the background Minkowskian geometry. The core strategy in this approach is then to relate the polymer excitations of quantum geometry to Fock states used in low energy physics and to locate Minkowski Fock states in the background independent framework. Since quantum Maxwell field can be constructed in both Fock representation and polymer-like representation, one first gains insights from the comparison between the two representations, then generalizes the method to quantum geometry. A "Laplacian operator" can be defined on $\mathcal{H}_{\text{kin}}$ [49, 17], from which one may define a candidate coherent state Φ_0, also in ***Cyl****, corresponding to the Minkowski space-time. To calculate the expectation values of kinematical operators, one considers the so-called "shadow state" of Φ_0, which is the restriction of Φ_0 to a given finite graph. However, the construction of shadow states is subtly different from that of cut-off states.

We will only describe the simple case of Maxwell field to illustrate the ideas of construction [37, 38, 3]. Following the quantum geometry strategy, the quantum configuration space $\overline{\boldsymbol{A}}$ for the polymer representation of the *U*(1) gauge theory can be similarly constructed. A generalized connection $\overline{\boldsymbol{A}} \in \boldsymbol{A}$ assigns each oriented analytic edge in Σ an element of *U*(1). The space $\overline{\boldsymbol{A}}$ carries a diffeomorphism and gauge invariant measure μ_0 induced by the Haar measure on *U*(1), which give rise to the Hilbert space, $H_0 := L^2(\overline{\boldsymbol{A}}, \mathrm{d}\mu_0)$, of polymer states. The basic operators are holonomy operators $\overline{\boldsymbol{A}}(e)$ labeled by 1-dimensional edges *e*, which act on cylindrical functions by multiplication, and smeared electric field operators $\hat{E}(g)$ for suitable test 1-forms *g* on Σ, which are self-adjoint. Note that, since the gauge group *U*(1) is Abelian, it is more convenient to smear the electric fields in 3 dimensions [3]. The eigenstates of $\hat{E}(g)$, so-called flux network states $\mathcal{N}_{\alpha,\vec{n}}$, provide an orthonormal basis in $\mathcal{H}_0$, which are defined for any finite graph α with *N* edges as:

$$N_{\alpha,\vec{n}}(\boldsymbol{A}) := [\boldsymbol{A}(e_1)]^{n_1}[\boldsymbol{A}(e_2)]^{n_2} \cdots [\boldsymbol{A}(e_N)]^{n_N} \quad (33)$$

where $\vec{n} \equiv (n_1, \cdots, n_N)$ assigns an integer n_I to each edge e_I. The action of $\hat{E}(g)$ on the flux network states reads

$$\hat{E}(g)\mathcal{N}_{\alpha,\vec{n}} = -\hbar\left(\sum_I n_I \int_{e_I} g\right) = \mathcal{N}_{\alpha,\vec{n}} \quad (34)$$

In this polymer-like representation, cylindrical functions are finite linear combinations of flux network states and span a dense subspace of $\mathcal{H}_0$. Denote ***Cyl*** the set of cylindrical functions and ***Cyl**** its algebraic dual. One then has a triplet $\boldsymbol{Cyl} \subset \mathcal{H}_0 \subset \boldsymbol{Cyl}^*$ in analogy with the case of loop quantum gravity.

The Schrödinger or Fock representation of the Maxwell field, on the other hand, depends on the Minkowski background metric. Here the Hilbert space is given by $\mathcal{H}_F = L^2(S', d\mu_F)$, where S' is the appropriate space of tempered distributions on Σ and μ_F is the Gaussian measure. The basic operators are connections $\hat{A}(f)$ smeared in 3 dimensions with suitable vector densities f and smeared electric fields $\hat{E}(g)$. But $\hat{A}(e)$ fail to be well defined. To resolve this tension between the two representations, one proceeds as follows. Let $\vec{x}$ be the Cartesian coordinates of a point in $\Sigma = \mathbf{R}^3$. Introduce a test function by using the Euclidean background metric on $\mathbf{R}^3$,

$$f_r(\vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}} r^3} \exp(-|\vec{x}|^2 / 2r^2) \quad (35)$$

which approximates the Dirac delta function for small r. The Gaussian smeared form factor for an edge e is defined as

$$X^a_{(e,r)}(\vec{x}) := \int_e ds\, f_r(\vec{e}(s) - \vec{x}) \dot{e}^a \quad (36)$$

Then one can define a smeared holonomy for e by

$$\boldsymbol{A}_{(r)}(e) := \exp\left[-i \int_{R^3} X^a_{(e,r)}(\vec{x}) A a(\vec{x})\right] \quad (37)$$

where $A_a(\vec{x})$ is the $U(1)$ connection 1-form of the Maxwell field on Σ. Similarly one can define Gaussian smeared electric fields by

$$E_{(r)}(g) := \int_{R^3} g_a(\vec{x}) \int_{R^3} f_r(\vec{y} - \vec{x}) E^a(\vec{y}) \quad (38)$$

In this way one obtains two Poission brackets algebras. One is formed by smeared holonomies and electric fields with

$$\{\boldsymbol{A}_{(r)}(e), \boldsymbol{A}_{(r)}(e')\} = 0 = \{E(g), E(g')\} \quad (39)$$

$$\{\boldsymbol{A}_{(r)}(e), E(g)\} = -i\left(\int_{R^3} X^a_{(e,r)} g_a\right) \boldsymbol{A}_{(r)}(e)$$

The other is formed by unsmeared holonomies and Gaussian smeared electric fields with

$$\{\boldsymbol{A}(e), \boldsymbol{A}(e')\} = 0 = \{E_{(r)}(g), E_{(r)}(g')\} \quad (40)$$

$$\{\boldsymbol{A}(e), E_{(r)}(g)\} = -i\left(\int_{R^3} X^a_{(e,r)} g_a\right) \boldsymbol{A}(e)$$

Obviously, there is an isomorphism between them,

$$I_r : (\boldsymbol{A}_{(r)}(e), E(g)) \to (\boldsymbol{A}(e), E_{(r)}(g)) \quad (41)$$

Using the isomorphism I_r, one can pass back and forth between the polymer and the Fock representations. Specifically, the image of the Fock vacuum can be shown to be the following element of ***Cyl**** [37][38],

$$(V| = \sum_{\alpha,\vec{n}} \exp\left[-\frac{\hbar}{2} \sum_{IJ} G_{IJ} n_I n_J\right] (N_{\alpha,\vec{n}}| \quad (42)$$

where $(\mathcal{N}_{\alpha,\vec{n}}| \in$ ***Cyl**** maps the flux network function $|N_{\alpha,\vec{n}}\rangle$ to one and every other flux network functions to zero. While the states $(\mathcal{N}_{\alpha,\vec{n}}|$ do not have any knowledge of the underlying Minkowskian geometry, this information is coded in the matrix G_{IJ} associated with the edges of the graph α, given by [3]

$$G_{IJ} = \int_{e_I} dt \dot{e}^a_I(t) \int_{e_J} dt' \dot{e}^b_J(t') \cdot \int d^3x \delta_{ab}(\vec{x}) \left[f_r(\vec{x} - \vec{e}_I(t)) |\Delta|^{-\frac{1}{2}} f(\vec{x}, \vec{e}_J(t')) \right] \quad (43)$$

where δ_{ab} is the flat Euclidean metric and Δ its Laplacian. Therefore, one can single out the Fock vacuum state directly in the polymer representation by invoking Poincaré invariance without any reference to the Fock space. Similarly, one can directly locate in ***Cyl**** all coherent states as the eigenstates of the exponentiated annihilation operators. Since ***Cyl**** does not have an inner product, one uses the notion of shadow states to do semiclassical analysis in the polymer representation. From Eq.(42), the action of the Fock vacuum $(V|$ on $\mathcal{N}_{\alpha,\vec{n}}$ reads

$$(V|\mathcal{N}_{\alpha,\vec{n}}\rangle = \int_{\bar{\mathcal{A}}_\alpha} d\mu^0_\alpha \bar{V}_\alpha \mathcal{N}_{\alpha,\vec{n}} \quad (44)$$

where the state V_α is in the Hilbert space $\mathcal{H}_\alpha$ for the graph α and given by

$$V_\alpha(\boldsymbol{A}) = \sum_{\vec{n}} \exp\left[-\frac{\hbar}{2} \sum_{IJ} G_{IJ} n_I n_J\right] N_{\alpha,\vec{n}}(\boldsymbol{A}) \quad (45)$$

Thus for any cylindrical functions φ_α associated with α,

$$(V|\varphi_\alpha\rangle = \langle V_\alpha | \varphi_\alpha\rangle \quad (46)$$

where the inner product in the right hand side is taken in $\mathcal{H}_\alpha$. Hence $V_\alpha(\boldsymbol{A})$ are referred to as "shadows" of $(V|$ on the graphs α. The set of all shadows captures the full information in $(V|$. By analyzing shadows on sufficiently refined graphs, one can introduce criteria to test if a given element of ***Cyl**** represents a semiclassical state [3]. It turns out that the state $(V|$ does satisfy this criterion and hence can be regarded as semiclassical in the polymer representation.

The mathematical and conceptual tools gained from simple models like the Maxwell fields are currently being used to construct semiclassical states of quantum geometry. A candidate kinematical coherent state corresponding to the Minkowski space-time has been proposed by Ashtekar and Lewandowki in the light of a "Laplacian operator" [17, 3]. However, the detailed structure of this coherent state is yet to be analyzed and there is no a priori guarantee that it is indeed a semiclassical state.

6 Concluding remark

Looking for semiclassical states is one of the crucial open issues of current research in loop quantum gravity. We have shown by the degenerate weave state how a physical state can be related to a familiar classical geometry. However, weave states are not suitable to analyze the classical limit of the Hamiltonian constraint operator in the kinematical Hilbert space. Thus, two different approaches in constructing kinematical coherent state are now under investigation. One may find comparisons of the two approaches from both sides [16, 3]. It turns out that the Varadarajan's Laplacian coherent state for polymer Maxwell field can also be derived from Thiemann's complexifier method. However, one cannot find a complexifier to get the coherent state proposed by Ashtekar *et al*. for loop quantum gravity. Both approaches have their own virtues and need further developments. The complexifier approach provides a clean construction mechanism and manageable calculation method, while the Laplacian operator approach is related closely with the well-known Fock vacuum state. We expect that a judicious combination of the two approaches may lead to significant progress in the construction of semiclassical states in loop quantum gravity.

Although powerful tools have been developed to construct semiclassical states, the analysis of the classical limits of the Hamiltonian constraint operator and the corresponding constraints algebra is yet to be carried out. Furthermore, to do semiclassical analysis of the master constraint operator, one still needs diffeomorphism invariant coherent states in the diffeomorphism invariant Hilbert space $\mathcal{H}_{Diff}$ (see Refs. [16, 50] for recent progress in this aspect). Moreover, a crucial question of the semiclassical analysis is whether there are enough physical semiclassical states in certain unknown physical Hilbert space of loop quantum gravity, which may correspond to all classical solutions of the Einstein equation. This is the final theoretical criterion for any candidate theory of quantum gravity with general relativity as its classical limit. The physical semiclassical states are also relevant, if one wishes to use the full theory rather than symmetric models to analyze cosmology and black holes. In the matter coupled to gravity content, one would like to check whether the coupled quantum system approaches quantum field theory in curved space-time in suitable semiclassical limit. This issue is being studied at the kinematical level [51, 52].

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