Supplementary Material for

Programmable Synthetic Temporal Lattices for High-Fidelity Chaotic Synchronization

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Supplementary Contents:

I. Experimental setup of nonlinear temporal coupled rings.

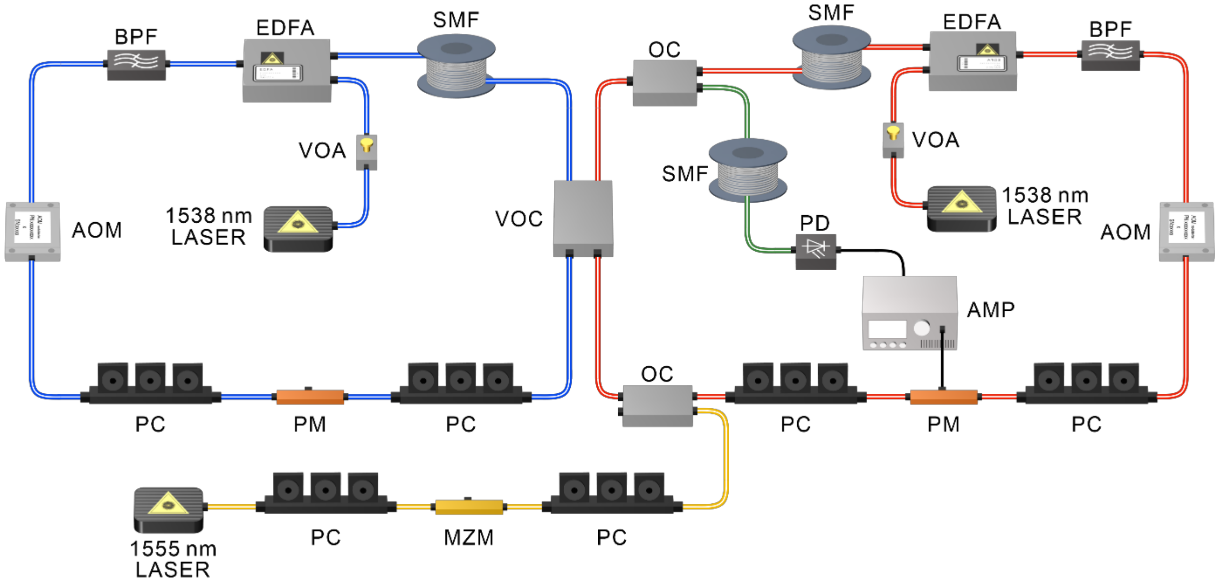

Fig. S1. Experimental setup. The blue and red loops represent the long and short loops and the green and black circuits denote the feedback circuit. All optical and electric components are as follows: variable optical coupler (VOC), optical coupler (OC), photodetector (PD), amplifier (AMP), phase modulator (PM), erbium-doped fiber amplifiers (EDFA), polarization controller (PC), Mach-Zehnder modulator (MZM), optical coupler (OC), variable optical coupler (VOC), single mode fiber (SMF), erbium-doped fiber amplifiers (EDFA), band-pass filter (BPF), acousto-optic modulator (AOM), phase modulator (PM), photodiode (PD), arbitrary waveform generator (AWG), variable optical attenuator (VOA), amplifier (AMP).

In this section, we discuss the construction of synthetic temporal lattice and the experimental setup in detail. The long (blue) and short (red) loops are connected via a variable optical coupler (VOC) at the center. The periodic optical pulses with identical amplitudes are injected into the short loop via an optical coupler (OC). These pulses are then split by the VOC and propagate through both the long and short loops. Due to the length difference between the loops, the pulses take different times to reach the VOC, causing a single pulse to gradually turn into a pulse sequence. We define the interference of light pulses arriving simultaneously at the VOC as a lattice site n, and each complete circulation of a pulse through a loop as a longitudinal step m. This allows us to map the pulse sequences into a synthetic temporal lattice, as shown in Fig. 1(b). Here, the red (blue) lines represent that the pulse travels through the short (long) loop, corresponding to a hop from step m and position n to step $m + 1$ and position $n - 1$ ($n + 1$). The experimental setup is shown in Fig. S1, the incident optical pulses are generated by pulse generation module consisting of a 1555nm laser, a Mach-Zehnder modulator (MZM) and two polarization controllers (PCs). The generated incident pulses are coupled into the short loop through an optical coupler (OC), and then splitted by a variable optical coupler (VOC) into two parts, which enter the short loop and the long loop respectively. The lengths of these two loops are ~ 5km on average, corresponding to a mean pulse travel time ~ 25 ms. It is worth noting that due to the 300m length difference between the two loops, the pulses entering the two loops at the same time have a time difference of 150μs after optical pulses circulation, which causes a single incident pulse to gradually evolve into a pulse sequence. Erbium-doped fiber amplifiers (EDFAs) are used to compensate for the pulse propagation and insertion losses. They are driven by two lasers with 1538 nm wavelength to clamp the gain of the devices and overcome the transient response. The variable optical attenuators (VOAs) provide optical power regulation to prevent overload conditions at the EDFA receiver input. After passing EDFAs, the driving signal are blocked by the band-pass filter (BPFs). All PCs are used to control the polarization state of light pulses and compensate for polarization-dependent losses in optical fibers. Acoustic optical modulators (AOMs) are used to absorb the pulse sequence after four circulations. To achieve opto-electronic feedback, we first couple a portion of the optical pulse from the short loop into the feedback circuit through an OC. To ensure that the outgoing light of the previous period can precisely feedback to the second step of pulse evolution in the next period, the optical pulse coupled into the feedback circuit must first pass through a single mode fiber (SMF) of ~10 km in length for delay. Subsequently, the delayed optical signal was converted into an electrical signal by the photodiode (PD). The electrical signal is amplified by an amplifier (AMP) and then fed to the phase modulator (PM) in the short loop. It is worth noting that, although pulses in each step of one period are coupled out, PM is controlled to only receive the feedback signal from the fourth step of each period to ensure the accurate operation of the feedback.

II. Derivation of transmittance T with different N and N_q.

In this section, we derive the expression for the transmittance T under values different N and N_q. Starting from the pulse evolution equation Eq. (1) in the temporal lattice described in the main text, we extend it to a truncated temporal lattice, which can be expressed as

$$\begin{pmatrix} u_{q+1} \\ v_{q+1} \end{pmatrix} = \begin{pmatrix} -\sin\beta & i\cos\beta \\ i\cos\beta & -\sin\beta \end{pmatrix} \begin{pmatrix} u_q \\ v_q \end{pmatrix} = M \begin{pmatrix} u_q \\ v_q \end{pmatrix}, \tag{S2-1}$$

where each iteration corresponds to two discrete time steps m and M represents the iteration matrix. Assuming that the incident pulse undergoes N iterations, and phase modulation is applied at $N = N_q$, then the pulse amplitude in the long and short loops after N iterations can be expressed as

$$\begin{pmatrix} u_N \\ v_N \end{pmatrix} = M^{N-N_q} \begin{pmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{pmatrix} M^{N_q} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}. \tag{S2-2}$$

We know that the formula for the Nth power of a trigonometric function matrix is given by

$$\begin{pmatrix} \cos\alpha & i\sin\alpha \\ i\sin\alpha & \cos\alpha \end{pmatrix}^N = \begin{pmatrix} \cos(N\alpha) & i\sin(N\alpha) \\ i\sin(N\alpha) & \cos(N\alpha) \end{pmatrix}. \tag{S2-3}$$

Therefore, by setting $\beta = \alpha - \pi/2$, the iterative matrix M can be expressed as

$$M = \begin{pmatrix} \cos\alpha & i\sin\alpha \\ i\sin\alpha & \cos\alpha \end{pmatrix}. \tag{S2-4}$$

In our experiment configuration, the incident light enters the temporal coupled loops through the short loop, which corresponds to the initial condition $u_0 = 1$, $v_0 = 0$. By substituting these conditions into the evolution equation Eq. (S2-2), we obtain the pulse amplitude after N iterations as

$$\begin{pmatrix} u_N \\ v_N \end{pmatrix} = \begin{pmatrix} \cos\left[\left(N-N_q\right)\alpha\right] & i\sin\left[\left(N-N_q\right)\alpha\right] \\ i\sin\left[\left(N-N_q\right)\alpha\right] & \cos\left[\left(N-N_q\right)\alpha\right] \end{pmatrix} \begin{pmatrix} \cos\left(N_q\alpha\right)e^{i\phi} & i\sin\left(N_q\alpha\right) \\ i\sin\left(N_q\alpha\right) & \cos\left(N_q\alpha\right) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = M_N \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{S2-5}$$

We define the output intensity of the system as the optical pulse intensity in the short loop after the N^{th} iteration. Consequently, the transmittance T_p of the system after N iterations is defined as the ratio of the output intensity to the incident intensity, which can be described as

$$\begin{aligned} T_p &= \left|M_N(1,1)\right|^2 \\ &= \cos^2\left[\left(N-N_q\right)\alpha\right]\cos^2\left(N_q\alpha\right) + \sin^2\left[\left(N-N_q\right)\alpha\right]\sin^2\left(N_q\alpha\right) \\ &\quad -\frac{1}{2}\sin\left[2\left(N-N_q\right)\alpha\right]\sin\left(2N_q\alpha\right)\cos\phi_p \\ &= \cos^2\left(N\alpha\right) - \frac{1}{2}\sin\left[2\left(N-N_q\right)\alpha\right]\sin\left(2N_q\alpha\right)\left(\cos\phi_p - 1\right). \end{aligned} \tag{S2-6}$$

By substituting $\beta = \alpha - \pi/2$ into the above expression, we can also obtain the expression for transmittance T in terms of β, which can be written as

$$T_p = \cos^2\left(N\beta + \frac{N}{2}\pi\right) - \frac{1}{2}\sin\left[2\left(N - N_q\right)\beta + \left(N - N_q\right)\pi\right]\sin\left(2N_q\beta + N_q\pi\right)\left(\cos\phi_p - 1\right). \qquad \text{(S2-7)}$$

III. Details of decoding process of the sequential encoding method

In this section, we present the detailed decoding process of the individual encoding method. When using the sequential encoding method, the message loading onto the chaotic carrier causes errors between the synchronized chaotic signal and the original chaotic signal. When dealing with regular message signal of simple harmonic wave, we first directly demodulate the encoded signal I_p' with the synchronized signal R_p. At this point, the decoded message can be described as $D_p = \log(I_p'/R_p)$. After normalization, we can obtain the results shown in Fig. S2.

To further refine the existing decoded message signal, we consider using a digital low-pass filter during the decoding process to filter the signal that has just been demodulated. Based on the algorithm proposed by Gustafsso, this filter performs zero-phase digital filtering by processing the input data both forward and backward. By setting a suitable order and cutoff frequency $\omega_n = 0.4$, we can remove the high-frequency chaotic noise signals in the decoded message, as shown in Fig. S2. It can be seen that the decoded message signal has been significantly improved after filtering, but there are still some errors and it cannot achieve complete and precise decoding.

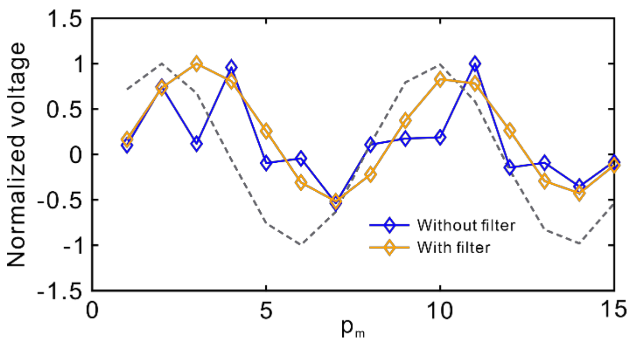

Fig. S2: Decoded message signal using sequential encoding method for h = 0.2 and Ω = 0.8. The blue and yellow marks represent the decoded message without filter and with filter (ω_n = 0.4), respectively. The dashed curve represents the original message signal.