

Propagation properties of beams generated by Gaussian mirror resonator in fractional Fourier transform plane

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Abstract Based on the definition of the fractional Fourier transform (FRFT) and irradiance moments in the cylindrical coordinate system, the propagation expressions and kurtosis parameter of beams generated by Gaussian mirror resonator passing through the ideal fractional Fourier transformation systems are obtained. The propagation properties and kurtosis parametric characteristic of the beams in the FRFT plane are analyzed in detail. Some numerical examples are given to illustrate the analytical results. The influences of the fractional order on the intensity distribution and the kurtosis parameter of the beams are also investigated. The results show that the intensity distribution and the kurtosis parameter of the beams in the FRFT plane are closely related to the fractional order and beam parameters.

Keywords Gaussian mirror resonator, fractional Fourier transform (FRFT), propagation property, kurtosis parameter

1 Introduction

Since the ordinary Fourier transform and related techniques find widespread use in physics and engineering, it is natural to expect the fractional Fourier transform (FRFT) to find many applications as well. The FRFT was first proposed as a new mathematical tool by Namias in 1980, and subsequently, its potential applications in optics were explored in 1993 by Ozaktas, Mendlovic and Lohmann [1–3]. Since then, it has become a research subject in optics and has been extensively applied in signal and image processing, in areas ranging from time/space-variant filtering, perspective projections, phase retrieval, image restoration, pattern recognition, tomography, data

compression, encryption, watermarking, and so forth [4,5]. Recently, much work has been done about the FRFT for different types of beam systems used frequently in modern optics [6–22].

On the other hand, unstable resonator with a variable reflectance mirror has been proposed [23,24] and successfully implemented in many gas and solid-state lasers [25] for many years. Mirror with a Gaussian radial variation of reflectivity presents many useful characteristics. For example, optical resonator with a Gaussian mirror offers advantages over standard unstable resonators of good mode discrimination, smooth output beam profile, and large mode volume. In addition, beams generated by Gaussian mirror resonator can be decomposed into a linear combination of the lowest order Gaussian beams, and their propagation properties have been investigated in detail [26–28].

In this paper, the FRFT is applied to study the transformation properties of beams generated by Gaussian mirror resonator. Based on the definition of FRFT and irradiance moments in the cylindrical coordinate system, analytical propagation expressions and kurtosis parameter for the FRFT of the beam are derived in the FRFT plane. The properties of the beam in the fractional Fourier plane and its dependence on the fractional order are studied in detail by using the derived formulas. Some typical numerical examples are given to illustrate the transformation characteristics of the beam in the FRFT plane. Finally, a simple conclusion is given.

2 FRFT in cylindrical coordinate system

The optical system for performing FRFT is depicted in Fig. 1. Assume a stationary quasi-monochromatic source field expressed by $E(x_1, y_1)$, the FRFT of $E(x_1, y_1)$ achieved by the optical system, as shown in Fig. 1, is given by [1–3]

$$E_p(x_2, y_2) = \frac{1}{i\lambda f \sin\phi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(x_1, y_1) \times \exp\left[-\frac{i\pi(x_1^2 + y_1^2 + x_2^2 + y_2^2)}{\lambda f \tan\phi}\right] \times \exp\left[\frac{2\pi i(x_1 x_2 + y_1 y_2)}{\lambda f \sin\phi}\right] dx_1 dy_1, \quad (1)$$

where f is focal length of the lens, λ is the optical wavelength, x_1, y_1 and x_2, y_2 are the rectangular coordinates in the input plane and the fractional Fourier plane, respectively. ϕ is given by

$$\phi = \frac{p\pi}{2}, \quad (2)$$

where p is called the fractional order of the Fourier transform and may take any arbitrary real number. When p takes a value of $4n + 1$, n being any integer, the FRFT reverts to a conventional Fourier transform. In Eq.(1), the factor $\frac{1}{i\lambda f \sin\phi}$ in front of the integral ensures energy conservation after the FRFT.

By setting

$$\begin{aligned} x_1 &= r_1 \cos\theta_1, \\ y_1 &= r_1 \sin\theta_1, \\ x_2 &= r_2 \cos\theta_2, \\ y_2 &= r_2 \sin\theta_2 \end{aligned}$$

in Eq. (1), we can get the expression of FRFT in the cylindrical coordinate system as follows:

$$E_p(r_2, \theta_2) = \frac{1}{i\lambda f \sin\phi} \int_0^{\infty} \int_0^{2\pi} E(r_1, \theta_1) \exp\left[-\frac{i\pi(r_1^2 + r_2^2)}{\lambda f \tan\phi}\right] \times \exp\left[\frac{2\pi i r_1 r_2 \cos(\theta_1 - \theta_2)}{\lambda f \sin\phi}\right] r_1 dr_1 d\theta_1, \quad (3)$$

where r_1, θ_1 and r_2, θ_2 are the radial and the azimuth angle coordinates in the input and FRFT planes, respectively.

The analytical expression of the beam generated by Gaussian mirror resonator at $z = 0$ in cylindrical coordinate system as follows [26–28]:

$$E(r, 0) = E_0 \exp\left(-\frac{r^2}{w_0^2} + ik\frac{r^2}{2R_0}\right) \times \left[1 - \kappa_0 \exp\left(-2\beta^2 \frac{r^2}{w_0^2}\right)\right]^{1/2}, \quad (4)$$

where E_0 represents the amplitude of a conventional Gaussian beam, w_0 is the beam waist, κ_0 is the on-axis reflectance of this mirror, β is a parameter given by w_0/w_c , w_c is the mirror spot size at which the reflectance is reduced to $1/e^2$ of its peak value, and R_0 is the wave-front curvature of the incident beam. For the simplicity, here, we set $E_0 = 1$. By using the binomial expansion method, Eq. (4) can be re-expressed as [26]

$$E(r, 0) = \sum_{m=0}^{\infty} E_m \exp\left(-\frac{r^2}{q_m}\right), \quad (5)$$

where

$$\begin{aligned} E_m &= \alpha_m E_0, \\ \alpha_0 &= 1, \\ \alpha_1 &= -\frac{\kappa_0}{2}, \end{aligned}$$

and

$$\alpha_m = \frac{(2m-3)(2m-5)\cdots(3)(1)}{m!} \left(\frac{\kappa_0}{2}\right)^m, \quad m \geq 2. \quad (6)$$

In Eq. (5), we introduce some parameters given by

$$\frac{1}{q_m} = \frac{1}{(w_0)_m^2} + i\frac{k}{2R_0}, \quad (7)$$

$$(w_0)_m = \frac{w_0}{(2m\beta^2 + 1)^{1/2}}, \quad m = 0, 1, 2, \dots \quad (8)$$

By substituting Eqs. (5)–(7) into Eq. (3) and applying the integral formula

$$\int_0^{\infty} \exp(-ax^2) J_0(bx) x dx = \frac{1}{2a} \exp\left(-\frac{b^2}{4a}\right), \quad (9)$$

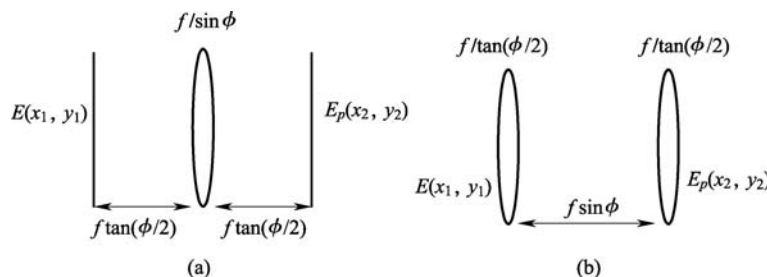


Fig. 1 Optical system for performing FRFT. (a) One-lens system; (b) two-lens system

with $J_0(\cdot)$ being the Bessel function of order zero, after integral calculations, an analytical expression for the optical field distribution of beams generated by Gaussian mirror resonator in the FRFT plane through the FRFT systems is obtained as follows:

$$E_p(r_2, \theta_2) = \frac{2\pi}{i\lambda f \sin\phi} \exp\left(-\frac{i\pi r_2^2}{\lambda f \tan\phi}\right) \times \sum_{m=0}^{\infty} \frac{E_m}{2a} \exp\left(-\frac{b^2}{4a}\right), \quad (10)$$

where

$$\begin{aligned} a &= \frac{1}{q_m} + \frac{i\pi}{\lambda f \tan\phi}, \\ b &= \frac{2\pi r_2}{\lambda f \sin\phi} = Br_2, \\ B &= \frac{2\pi}{\lambda f \sin\phi}. \end{aligned} \quad (11)$$

Here, we assume that beam waist w_0 is located in the plane of the Gaussian mirror (assumed here to be at $z=0$), namely, $R_0 \rightarrow \infty$. Under this assumption, Eq. (7) can be rewritten as

$$a = \frac{1}{(w_0)_m^2} + \frac{i\pi}{\lambda f \tan\phi}, \quad m = 0, 1, 2, \dots \quad (12)$$

The axial irradiance can be obtained from Eq. (10) by making $r_2 = 0$ as

$$I_p(0, \theta_2) = |E_p(0, \theta_2)|^2. \quad (13)$$

$$K = \frac{\sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} \frac{6! E_m E_{m'}^* a a^* \sqrt{\frac{\pi a a^*}{a + a^*}} \sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} \frac{E_m E_{m'}^* \sqrt{\frac{\pi a a^*}{a + a^*}}}{2 B a a^*}}{\left(\sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} \frac{E_m E_{m'}^* \sqrt{\frac{\pi a a^*}{a + a^*}}}{B^3 (a + a^*)} \right)^2}, \quad (17)$$

where $E_{m'} = \alpha_m E_0$, ‘*’ is the complex conjugate. Equation (17) describes the evolution of kurtosis parameter of the intensity distribution for beams generated by Gaussian mirror resonator in the fractional plane.

3 Properties in FRFT plane

In this section, we study the properties of beams generated by Gaussian mirror resonator in the FRFT plane by using the formulas derived in the above section. The series in Eq. (10) is fast convergent, thus usually, the use of ten terms of the series is sufficient to achieve satisfactory numerical results.

By using Eq. (10), we calculate the normalized intensity of the beams generated by Gaussian mirror resonator in the

Further, to characterize the behavior of a light beam passing through these two types of FRFT systems in a quantitative way, the kurtosis parameter, K , which is defined as [29]

$$K = \frac{\langle r^4 \rangle}{\langle r^2 \rangle^2}, \quad (14)$$

where the second- and fourth-order moment are defined as

$$\langle r^n \rangle = \frac{\int_{-\infty}^{+\infty} r^n |E(r)|^2 dr}{\int_{-\infty}^{+\infty} |E(r)|^2 dr}, \quad n = 2, 4. \quad (15)$$

The kurtosis parameter K written in terms of the higher-order irradiance moments can be used to describe the degree of sharpness (or flatness) of any laser beams. The irradiance profile of beams is classified as leptokurtic, mesokurtic, or platykurtic depending on K being larger, equal or less than 3, which is the kurtosis value of the fundamental Gaussian beam for the one-dimensional case. Nevertheless, the kurtosis parameters of Gaussian beams, flattened Gaussian beams, elegant and standard Hermite-Gaussian beams, etc., have been derived [29,30].

Substituting Eq. (10) into Eqs. (14) and (15), and using the integral formula

$$\int_0^{\infty} x^{2n} \exp(-\alpha x^2) dx = \frac{(2n-1)!!}{2^{n+1} \alpha^n} \sqrt{\frac{\pi}{\alpha}}, \quad \text{Re } \alpha > 0, \quad (16)$$

we obtain

fractional plane through the FRFT system with different fractional order p as depicted in Fig. 2. The parameters used in the calculation are $w_0 = 1.5$ mm, $f = 200$ mm, and $\lambda = 632.8$ nm. In Fig. 2, we can find that the fractional order has a strong influence on the intensity distribution in the fractional plane. When $0 < p < 1$, with the increasing of fractional order, the peak value of the intensity distribution becomes larger. Moreover, the beam width becomes much smaller, which means the spot size decreases. Namely, the field distribution becomes more and more convergent. When $1 < p < 2$, with the increasing of fractional order, the peak value of the intensity distribution becomes smaller, the beam width and spot size increases gradually, which means that the field distribution becomes more and more divergent. Figure 2(c) corresponds to the case of Fourier transformation.

Figure 3 shows the effect of different focal lengths and different beam widths on the on-axis intensity distribution. In Fig. 3, we find that the on-axis intensity distribution changes periodically with the fractional orders when the beam generated by Gaussian mirror resonator passing through the two types of optical system, and the fundamental period is 2. Figure 3(a) denotes the variation of the normalized on-axis intensity distribution with the

fractional order p for different focal lengths f ; the dotted curve is the result of $f=200$ mm; and the solid curve represents $f=2000$ mm. Figure 3(b) explores the variation of the normalized on-axis intensity distribution with the fractional order p for different beam widths, in which the dotted curve is the result of $w_0 = 1.5$ mm, and the solid curve is $w_0 = 0.5$ mm.

Figure 4 explains the evolutions of kurtosis parameter K

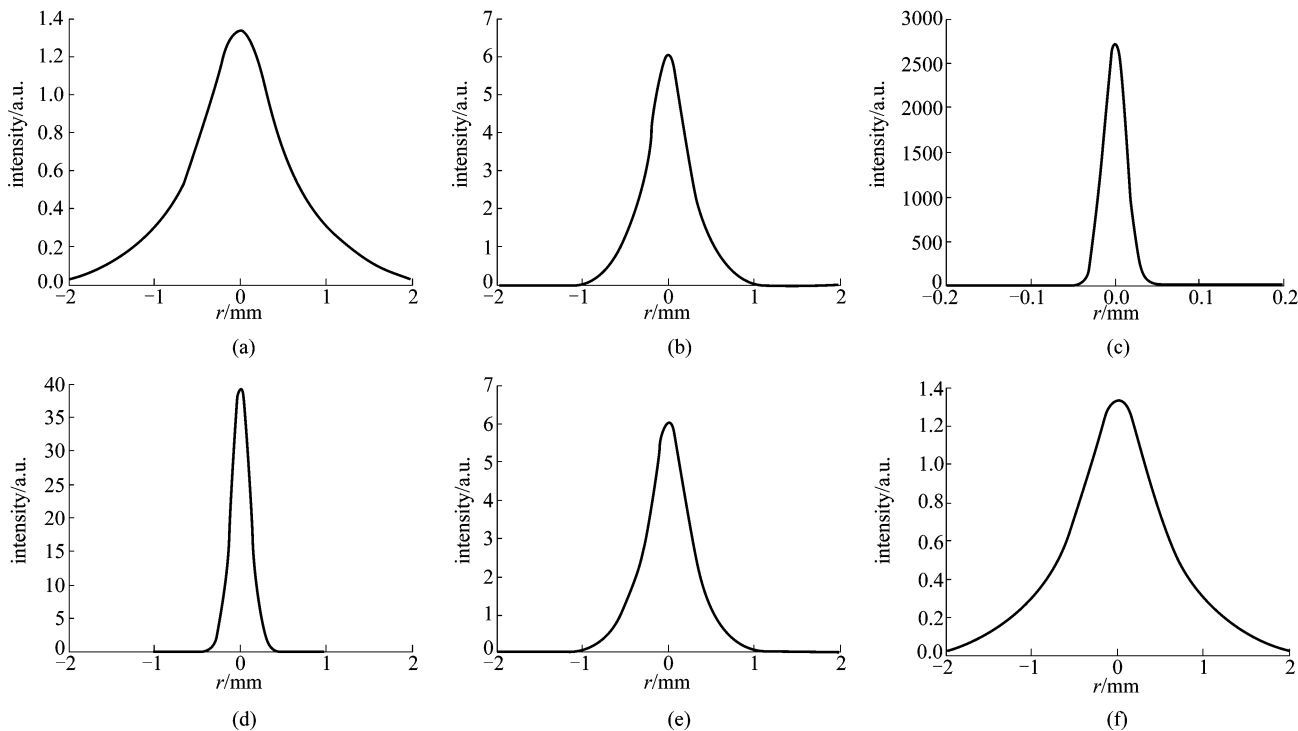


Fig. 2 Intensity distribution of beams generated by Gaussian mirror resonator in fractional plane after going through FRFT systems with different fractional order p . (a) $p = 0.1$; (b) $p = 0.7$; (c) $p = 1$; (d) $p = 1.1$; (e) $p = 1.3$; (f) $p = 1.9$

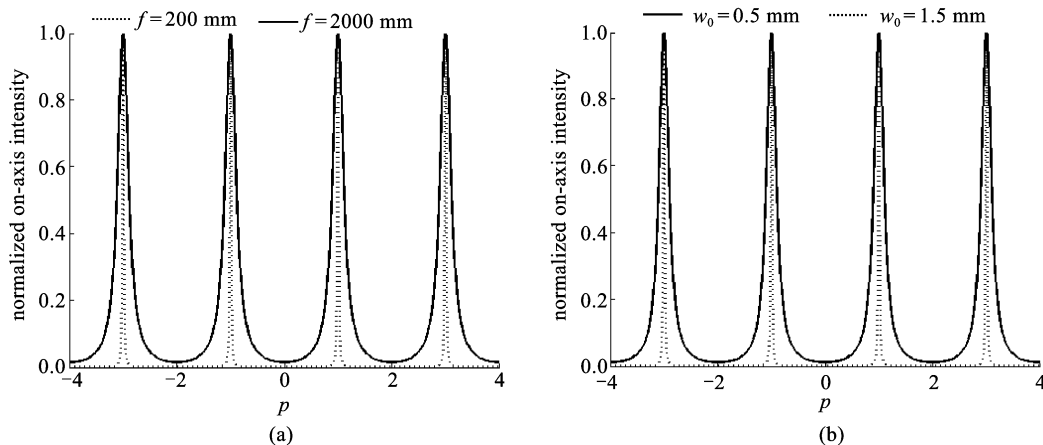


Fig. 3 Normalized on-axis intensity with different beam waists and focal lengths in fractional plane with varying fractional order p ($\kappa_0 = 0.7, \beta = 1.5$). (a) $w_0 = 1.5$ mm; (b) $f = 200$ mm

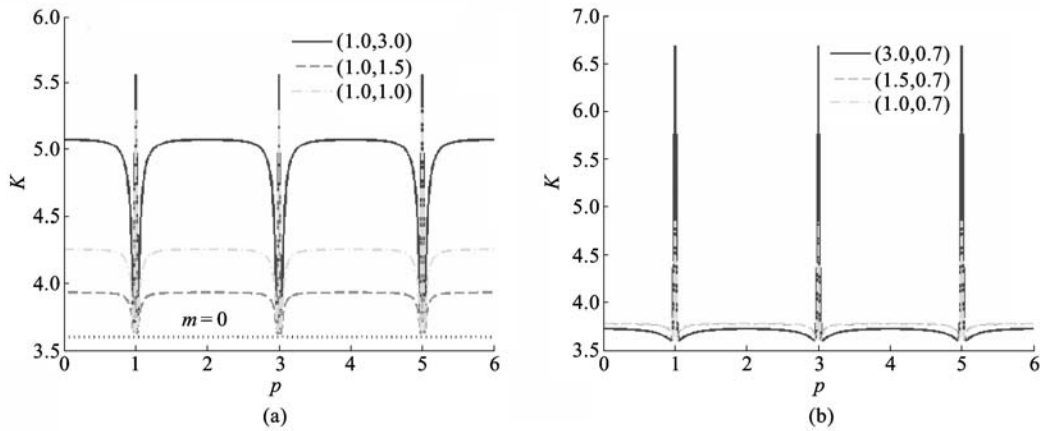


Fig. 4 Variations of kurtosis parameter for different beams (β, κ_0) generated by Gaussian mirror resonator versus fractional orders passage through ideal FRFT systems ($w_0 = 1.5$ mm, $f = 200$ mm). (a) $\beta = 1.0$; (b) $\kappa_0 = 0.7$

of beams generated by Gaussian mirror resonator with different values of κ_0 and β passing through the ideal fractional Fourier transformation systems. In Fig. 4, we can find that the evolution of the kurtosis along with the fractional factor is periodic, and the period is 2. That is to say, the sharpness (or flatness) of the beam passing through the ideal fractional Fourier transformation systems of order p is the same as order $p + 2n$, where $n = \pm 1, \pm 2, \dots$. When the fractional factor takes the value $p = 2n + 1$, where $n = \pm 1, \pm 2, \dots$, the kurtosis parameter takes the maximum value that means the flatness of the beam generated by Gaussian mirror resonator is the poorest. Further, the kurtosis parameter of the beam with larger κ_0 changes sharply. In addition, we can find that the beam parameters κ_0 and β have a much stronger influence on the maximum value of the kurtosis parameter for the beam generated by Gaussian mirror resonator, but they have a weaker influence on the minimum value of the kurtosis parameter. Moreover, in Fig. 4(a), we also can see that the kurtosis parameter is kept unchanged in the special case when $m = 0$, which corresponds to a fundamental Gaussian beam.

4 Conclusion

In conclusion, we have studied the propagation properties of beams generated by Gaussian mirror resonator through an FRFT optical system. Based on the definition of FRFT in the cylindrical coordinate system, analytical formulas of the FRFT for the beams are derived. By using the derived formulas, the intensity properties of the beams generated by Gaussian mirror resonator in the fractional Fourier plane and their dependence on the fractional order are studied in detail. The results show that the intensity distribution properties and kurtosis parameter of beams generated by Gaussian mirror resonator in the FRFT are

closely related to the fractional order and beam parameters. The evolution of the intensity distributions and kurtosis parameter with the fractional order are periodic, and the fundamental period is 2. The FRFT optical system provides a convenient way for controlling the properties of the beams by properly choosing the fractional order of the FRFT optical system. Moreover, the method presented in this paper is useful in the design of optical systems for beam shaping, which manipulate the intensity distribution and the wave-front of a laser beam.

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