Front. Math. China 2026, 21(1): 41–55
https://doi.org/10.3868/s140-DDD-026-0005-x

RESEARCH ARTICLE

Radicals of Morita context rings

Yao WANG[1], **Yanli REN**[2]

1 School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China
2 School of Mathematics and Information Technology, Nanjing Xiaozhuang University, Nanjing 211171, China

Abstract For a Morita context ring $T=\binom{RN}{MS}$, the structure of several radicals is given under certain conditions.

Keywords Morita context ring, radical
MR2010 16S80; 16U99

0 Introduction

Let R, S be two rings, ${}_RN_S, {}_SM_R$ be two bimodules, and $(\cdot,\cdot): N\otimes_S M\to R$ and $[\cdot,\cdot]: M\otimes_R N\to S$ be two bimodule homomorphisms satisfying the conditions of

$$(n,m)n'=n[m,n'],\ [m,n]m'=m\,(n,m')\,,\ \forall m,m'\in M,\ \forall n,n'\in N,$$

then the system $(R,S,M,N,(\cdot,\cdot),[\cdot,\cdot])$ is called a Morita context. Let

$$T=\begin{pmatrix} R & N\\ M & S\end{pmatrix}=\left\{\begin{pmatrix} r & n\\ m & s\end{pmatrix}\middle| r\in R, s\in S, n\in N, m\in M\right\}.$$

With respect to matrix addition, T is an Abel group. Define multiplication by

$$\begin{pmatrix} r & n\\ m & s\end{pmatrix}\begin{pmatrix} r' & n'\\ m' & s'\end{pmatrix}=\begin{pmatrix} rr'+(n,m') & rn'+ns'\\ mr'+sm' & [m,n']+ss'\end{pmatrix},$$

and then T becomes an associative ring, called a Morita context ring. The images of the two bimodule homomorphisms $(N,M)=NM$ and $[M,N]=MN$ refer to the ideals of R,S, respectively, called the trace ideals of this Morita

---

Translated from *Advances in Mathematics* (*China*), 2016, 45(2): 195–205
Corresponding author: Yao WANG, E-mail: wangyao@nuist.edu.cn; Yanli REN, E-mail: renyanlisx@163.com

context. If $A, B, V, W,$ are subsets of R, S, N, M, respectively, then $\begin{pmatrix} A & V \\ W & B \end{pmatrix}$ refers to the subset $\left\{ \begin{pmatrix} a & v \\ w & b \end{pmatrix} \middle| a \in A, b \in B, v \in V, w \in W \right\}$ of $\begin{pmatrix} R & N \\ M & S \end{pmatrix}$. When there is 1 in R, S and N, M are unital modules, if $I = \begin{pmatrix} A & V \\ W & B \end{pmatrix}$ is an ideal of T, then A, B are ideals of R, S, respectively, and $V, W,$ are sub-bimodules of N, M respectively.

Generally, let $R_1, R_2, \ldots, R_n$ be n rings $(n \geqslant 2)$, and M_{ij} be R_i-R_j- bimodules such that $M_{ii} = R_i, i, j = 1, 2, \ldots, n$. For any $i, j, k = 1,2, \ldots, n, i \neq j, k \neq j$, there exist R_i-R_j- bimodule homomorphisms

$$\varphi_{ikj} \colon M_{ik} \otimes_{R_k} M_{kj} \to M_{ij}.$$

For $i = j, k = j$, there are canonical isomorphisms

$$\varphi_{iij} \colon R_i \otimes_{R_i} M_{ij} \to M_{ij},$$

$$\varphi_{ijj} \colon M_{ij} \otimes_{R_j} R_j \to M_{ij}.$$

Define $ab = \varphi_{ikj}(a \otimes b)$, satisfying the associative law $(ab)c = a(bc)$ for all $a \in M_{ik}, b \in M_{kj}, c \in M_{jl}$ and all i, j, k, l. Let T be the set of all $n \times n$ matrices $\{(a_{ij}) \,|a_{ij} \in M_{ij}, \forall 1 \leqslant i, j \leqslant n\}$. With respect to the addition and multiplication of matrices, T becomes an associative ring, called a $n \times n$ Morita context ring.

For the $n \times n$ Morita context ring, let

$$R = R_1,$$

$$N = (M_{12}, M_{13}, \ldots, M_{1n}),$$

$$M = \begin{pmatrix} M_{21} \\ M_{31} \\ \vdots \\ M_{n1} \end{pmatrix},$$

$$S = \begin{pmatrix} R_2 & M_{23} & \ldots & M_{2n} \\ M_{32} & R_3 & \ldots & M_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n2} & M_{n3} & \ldots & R_n \end{pmatrix}.$$

Then S itself is a $(n-1) \times (n-1)$ Morita context ring. On the row and column multiplication of matrices, N is an R-S- bimodule and M indicates an S-R- bimodule. According to the multiplication that φ_{ikj} defines on T, there are bimodule homomorphisms $N \otimes_S M \to R$ and $M \otimes_R N \to S$, satisfying the associative law. Thus, $T \cong \left(\begin{smallmatrix} R N \\ M S \end{smallmatrix}\right)$. Next, it focuses on the discussion of 2×2 Morita context rings.

Various matrix rings play important roles in ring theory. Morita context rings generalize the concept of matrix rings over a given ring. As a main tool in the theory of module category equivalence, it has been widely used in algebraic branches. Every ring A with a non-trivial idempotent element e is isomorphic to the Morita context ring determined by the Morita context Morita context $(eAe,\ (1-e)A(1-e),\ (1-e)Ae,\ eA(1-e))$. The following shows examples of Morita contexts.

(1) Let R be a ring, ${}_RM$ be a left R-module, and $M^* = \mathrm{Hom}(M,R)$, $E = \mathrm{End}_R M$, then (R, E, M^*, M) forms a Morita context.

(2) Let G be a finite group acting on R, $R^G = \{x \in R \mid x^g = x, \forall g \in G\}$ be the stable ring, and $R * G = \left\{\sum_{g\in G} r_g g \mid r_g \in R\right\}$ be the skew group ring. Among them, the multiplication is $rg \cdot sh = rs^{g^{-1}} gh, \forall g, h \in G, r, s \in R$. Define the multiplication as

$$x \cdot r = \sum_{g\in G} r_g r^{g^{-1}},\ r \cdot x = \sum_{g\in G} (rr_g)^g,\ \forall x = \sum_{g\in G} r_g g,\ \forall r \in R,$$

then R becomes (R^G, R^G)- bimodule as well as $(R * G, R * G)$- bimodule. $(R^G, R * G, R, R)$ forms a Morita context.

(3) Let H be a finite-dimensional Hopf algebra, and there exists a S-fixed integral $\int_l, 0 \neq t \in \int_l, \lambda \in H^*$ such that $th = \lambda(h)t$, $\forall h \in H$. For $a\sharp h \in A\sharp H, b \in A$, define

$$(a\sharp h)\cdot b = a(h\cdot b), b\cdot(a\sharp h) = \overline{S}h^{\lambda}\cdot(ab),$$

where $\overline{S}$ is the composition inverse of the antipode S. Then $(A^H, A\sharp H, {}_{A\sharp H}A_{A^H}, {}_{A^H}A_{A\sharp H})$, with respect to the maps $[\cdot,\cdot]\colon A\otimes_{A^H}A \to A\sharp H$, $[a,b] = atb$ and $(\cdot,\cdot)\colon A\otimes_{A\sharp H}A \to A^H, (a,b) = t\cdot(ab)$, form a Morita context.

(4) Let (Γ, M) be a weak Nobusawa Γ-ring. ΓM represents $\Gamma \otimes_{\mathrm{Z}} M$, and $M\Gamma$ is $\mathrm{M} \otimes_{\mathrm{Z}} \Gamma$. Then

$$\begin{pmatrix} \Gamma M & \Gamma \\ M & M\Gamma \end{pmatrix}$$

is a Morita context ring. Conversely, for any Morita pair $(Q, R, {}_RT_Q, {}_QS_R)$, there is Morita context Γ - ring (S, T).

There are many studies on Morita context rings. For example, Amitsur [1] discussed the relationship between the properties of T and those of R, S, N, M. Poole and Stewart [14] gave the canonical quotient ring of Morita context rings. Chen et al. [4] characterized the ideal lattice of Morita context rings. In particular, Jaegermann [11] explored the properties of the strong radical and hereditary radical of rings via Morita context. Nicholson and Watters [13] studied normal radicals and normal classes based on Morita context. In this paper, we promote the research in [11] and [13], and provide specific constructions of radicals for Morita context rings.

It is assumed that there is 1 for all rings discussed in this paper, and all modules are unital modules. Let I be an ideal of ring A. I is called a prime ideal of A if for any two ideals B, C of A, and $BC \subseteq I$ implies that $BC \subseteq I$ or $C \subseteq I$. A ring whose zero ideal is a prime ideal is called a prime ring. The intersection of all prime ideals of A, denoted by $P(A)$, is called the prime radical of A. I indicates a completely prime ideal of A. If for any two elements a, b of A, $ab \in I$ implies that $a \in I$ or $b \in I$. According to [17], the intersection of all completely prime ideals of A, denoted by $P_2(A)$, is called the generalized nilradical of A. The Jacobson radical $J(A)$ of ring A is the intersection of all maximal left ideals of A. The simple radical $S(A)$ of A is defined as the intersection of all maximal ideals of A. The Brown-McCoy radical $g(A)$ of A is the intersection of all maximal ideals I_α of A such that A/I_α is a simple ring. For other concepts of radicals not explained in this paper, refer to [17].

1 Generalized nilradical, strongly prime radical, and Brown-McCoy radical

Handelman and Lawrence [9] defined the concept of strongly prime rings. A finite subset F of A is called a right insulator of A. If for any $r \in A$, $Fr = 0$ implies $r = 0$. A ring A is called right strongly prime if every nonzero ideal of A contains a right insulator. The right strongly prime radical of A, denoted by $s_r(A)$, is defined as the intersection of all ideals of A such that A/I is right strongly prime, then such an ideal I is called a right strongly prime ideal of A. The left strongly prime radical (denoted by $s_l(A)$) and left strongly prime ideals of A are defined similarly.

Proposition 1.1 *If the trace ideals NM and MN of the Morita context $(R, S, M, N, (\cdot,\cdot), [\cdot,\cdot])$ are both nilpotent, then*
(1) *assume*

$$I = \begin{pmatrix} A & U \\ V & B \end{pmatrix}$$

is the prime ideal of

$$T = \begin{pmatrix} R & N \\ M & S \end{pmatrix},$$

then there must be $U = N, V = M$, and $A \supseteq NM, B \supseteq MN$ are prime ideals of R, S, respectively;
(2) *let I be a completely prime ideal of T. Then there must be $U = N, V = M$, and $A \supseteq NM, B \supseteq MN$ are completely prime ideals of R, S, respectively;*
(3) *let I be a right strongly prime ideal of T. Then there must be $U = N, V = M$, and $A \supseteq NM, B \supseteq MN$ are right strongly prime ideals of R, S, respectively.*

Proof (1) Let $I = \begin{pmatrix} A & U \\ V & B \end{pmatrix}$ be a prime ideal of T. Take $J = \begin{pmatrix} NM & N \\ M & MN \end{pmatrix}$. It is verified that J is an ideal of T, and there are

$$J^3 \subseteq \begin{pmatrix} (NM)^3+(NM)^2 & (NM)^2N+(NM)N \\ (MN)^2M+(MN)M & (MN)^3+(MN)^2 \end{pmatrix},$$

$$J^5 \subseteq \begin{pmatrix} (NM)^5+(NM)^4+(NM)^3 & (NM)^4N+(NM)^3N+(NM)^2N \\ (MN)^4M+(MN)^3M+(MN)^2M & (MN)^5+(MN)^4+(MN)^3 \end{pmatrix},$$

$$J^7 \subseteq \begin{pmatrix} (NM)^7+(NM)^6+\cdots+(NM)^4 & (NM)^6N+(NM)^5N+\cdots+(NM)^3N \\ (MN)^6M+(MN)^5M+\cdots+(MN)^3M & (MN)^7+(MN)^6+\cdots+(MN)^4 \end{pmatrix},$$

$$\vdots$$

$$J^{2n-1} \subseteq \begin{pmatrix} (NM)^{2n-1}+\cdots+(NM)^n & (NM)^{2n-2}N+\cdots+(NM)^{n-1}N \\ (MN)^{2n-2}M+\cdots+(MN)^{n-1}M & (MN)^{2n-1}+\cdots+(MN)^n \end{pmatrix}, n\in\mathbb{N}.$$

Since NM and MN are nilpotent, there exists $m\in N$ such that $(NM)^m=0$, $(MN)^m=0$. It implies $J^{2m-1}\subseteq\begin{pmatrix}0 & N_1\\ M_1 & 0\end{pmatrix}$, where $M_1\subseteq M, N_1\subseteq N$. Then $(J^{2m-1})^{2m}=0$, so J is a nilpotent ideal of T. Therefore $J\subseteq I$, and hence $I=\begin{pmatrix}A_1 & N\\ M & B_1\end{pmatrix}$, where $A_1\supseteq NM, B_1\supseteq MN$. From $IT\subseteq I$ and $TI\subseteq I$, it follows that $A_1\lhd A, B_1\lhd B$.

It is assumed that $A_2\lhd R, A_3\lhd R$, and $A_2\supseteq NM, A_3\supseteq MN$ such that $A_2A_3\subseteq A_1$. Then $I_2=\begin{pmatrix}A_2 & N\\ M & B_1\end{pmatrix}$ and $I_3=\begin{pmatrix}A_3 & N\\ M & B_1\end{pmatrix}$ are ideals of T, and there is

$$I_2I_3\subseteq\begin{pmatrix}A_2A_3+NM & N\\ M & MN+B\end{pmatrix}\subseteq\begin{pmatrix}A_1 & N\\ M & B_1\end{pmatrix}=I.$$

Since I is prime, it follows that $I_2\subseteq I$ or $I_3\subseteq I$, hence $A_2\subseteq A_1$ or $A_3\subseteq A_1$. Therefore A_1 is a prime ideal of R. Similarly, B_1 is a prime ideal of S.

(2) Let I be any completely prime ideal of T. Then I is a prime ideal of T. By (1), $I=\begin{pmatrix}A & N\\ M & B\end{pmatrix}$, where $A\supseteq NM, B\supseteq MN$ are prime ideals of R, S, respectively.

For $a_1,a_2\in A, b_1,b_2\in B$, if $a_1a_2\in A, B_1B_2\in B$, then for any $v_1v_2\in N, w_1w_2\in M$, there is

$$\begin{pmatrix}a_1 & v_1\\ w_1 & b_1\end{pmatrix}\begin{pmatrix}a_2 & v_2\\ w_2 & b_2\end{pmatrix}=\begin{pmatrix}a_1a_2+v_1w_2 & v_3\\ w_3 & w_1v_2+b_2\end{pmatrix}\in\begin{pmatrix}A & N\\ M & B\end{pmatrix}=I.$$

Since I is completely prime, it follows that $\begin{pmatrix}a_1 & v_1\\ w_1 & b_1\end{pmatrix}\in I$ or $\begin{pmatrix}a_2 & v_2\\ w_2 & b_2\end{pmatrix}\in I$, hence $a_1\in A, b_1\in B$ or $a_2\in A, b_2\in B$, which shows that A, B are completely prime ideals of R, S, respectively.

(3) Let I_1 be any right strongly prime ideal of T, so I_1 is a prime ideal of T. By (1), $I_1=\begin{pmatrix}A_1 & N\\ M & B_1\end{pmatrix}$, where $A_1\supseteq NM, B_1\supseteq MN$ are prime ideals of R, S,

respectively. The following shows that A_1, B_1 are also right strongly prime ideals.

R/A_1 is a prime ring. Take any nonzero ideal A_2/A_1 of R/A_1. Then

$$A_2 \supsetneqq A_1, I_2 = \begin{pmatrix} A_2 & N \\ M & B_1 \end{pmatrix} \supsetneqq \begin{pmatrix} A_1 & N \\ M & B_1 \end{pmatrix}.$$

Since T/I_1 is right strongly prime and $0 \neq I_2/I_1 \lhd T/I_1, I_2/I_1 \cong A_2/A_1$ contain finite insulators

$$F = \begin{pmatrix} A_0/A_1 & 0 \\ 0 & 0 \end{pmatrix} \subseteq \begin{pmatrix} A_2/A_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where A_0/A_1 is a finite set, such that for any $\begin{pmatrix} \overline{a} & 0 \\ 0 & 0 \end{pmatrix} \in T/I_1 \cong A_0/A_1 \oplus B_0/B_1$, if

$$\begin{pmatrix} A_0/A_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{a} & 0 \\ 0 & 0 \end{pmatrix} = 0,$$

then there must be $\overline{a} = 0$. Equivalently, if $(A_0/A_1)\,\overline{a} = 0$, then $\overline{a} = 0$. It shows that $A_0/A_1 \subseteq A_2/A_1$ is a finite insulator for A_2/A_1, so R/A_1 is right strongly prime, and A_1 is a right strongly prime ideal. Similarly, B_1 is a right strongly prime ideal.

Proposition 1.2 (1) *Let A_α, B_β be prime ideals of R, S, respectively. Then*

$$I_\alpha = \begin{pmatrix} A_\alpha & N \\ M & S \end{pmatrix}, J_\beta = \begin{pmatrix} R & N \\ M & B_\beta \end{pmatrix}$$

are prime ideals of T;
(2) *Let A_α, B_β be completely prime ideals of R, S, respectively. Then I_α, J_β are completely prime ideals of T;*
(3) *Let A_α, B_β be right strongly prime ideals of R, S, respectively.* Then I_α, J_β are right strongly prime ideals of T.

Proof (1) Let $I = \begin{pmatrix} A_1 & V_1 \\ W_1 & B_1 \end{pmatrix}, J = \begin{pmatrix} A_2 & V_2 \\ W_2 & B_2 \end{pmatrix}$ be two ideals of T such that $IJ \subseteq I_\alpha$. Then $A_1A_2 + V_1W_2 \subseteq A_\alpha$. Hence $A_1A_2 \subseteq A_\alpha$. Since A_1, A_2 are ideals of R, it follows that $A_1 \subseteq A_\alpha$ or $A_2 \subseteq A_\alpha$. Thus, $I \subseteq I_\alpha$ or $J \subseteq I_\alpha$. As a result, I_α is a prime ideal of T. Similarly, J_β is a prime ideal of T.
(2) The proof is similar to (1).
(3) By (1), $I_\alpha = \begin{pmatrix} A_\alpha & N \\ M & B \end{pmatrix}, J_\beta = \begin{pmatrix} A & N \\ M & B_\beta \end{pmatrix}$ are prime ideals of T. Take any nonzero ideal $I_1/I_\alpha = \begin{pmatrix} A_1/A_\alpha & 0 \\ 0 & 0 \end{pmatrix}$ of $T/I_\alpha \cong A/A_\alpha$. Then $A_1 \supsetneqq A_\alpha, A_1/A_\alpha$ is a nonzero ideal of A/A_α. Since A_α is a right strongly prime ideal of A, there exists a finite subset $A_0/A_\alpha \subseteq A_1/A_\alpha$ such that for any $\overline{a} \in A_0/A_\alpha$, if

$(A_0/A_\alpha)\,\overline{a}=0$, there must be $\overline{a}=0$. For any $\begin{pmatrix}\overline{a} & 0\\ 0 & 0\end{pmatrix}\in T/I_\alpha\cong A/A_\alpha$, if

$$\begin{pmatrix}A_0/A_\alpha & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\overline{a} & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}(A_0/A_\alpha)\,\overline{a} & 0\\ 0 & 0\end{pmatrix}=0.$$

There must be $\overline{a}=0$. Therefore, $\begin{pmatrix}(A_0/A_\alpha)\,\overline{a} & 0\\ 0 & 0\end{pmatrix}=0$ is a finite insulator for I_1/I_α, and I_α is a right strongly prime ideal of T. Similarly, J_β is a right strongly prime ideal of T.

Theorem 1.1 *If the trace ideals NM and MN of the Morita context $(R,S,M,N,(\cdot,\cdot),[\cdot,\cdot])$ are both nilpotent, then*
(1) *the generalized nilradical of T is*

$$P_2(T)=\begin{pmatrix}P_2(R) & N\\ M & P_2(S)\end{pmatrix};$$

(2) *the right strongly prime radical of T is*

$$s_r(T)=\begin{pmatrix}s_r(R) & N\\ M & s_r(S)\end{pmatrix}.$$

Proof (1) Let $Q=\begin{pmatrix}P_2(R) & N\\ M & P_2(S)\end{pmatrix}$. Since $T/Q\cong R/P_2(R)\oplus S/P_2(S)$, T/Q is a P_2- semisimple ring, then $Q\supseteq P_2(T)$.

On the other hand, by Proposition 1.1 (2), there is

$$\begin{aligned}P_2(T)&=\bigcap_\alpha P_\alpha\,(P_\alpha \text{ runs over all completely prime ideals of } T)\\ &\supseteq\bigcap_\alpha\begin{pmatrix}A_\alpha & N\\ M & B_\alpha\end{pmatrix}(A_\alpha\supseteq NM, B_\alpha\supseteq MN \text{ are completely prime ideals of } A,B)\\ &=\begin{pmatrix}\bigcap_\alpha A_\alpha & N\\ M & \bigcap_\alpha B_\alpha\end{pmatrix}\supseteq\begin{pmatrix}P_2(R) & N\\ M & P_2(S)\end{pmatrix}=Q,\end{aligned}$$

so $P_2(T)=Q$.
(2) The proof is similar to (1).

Proposition 1.3 *It is assumed that the trace ideals of the Morita context $(R,S,M,N,(\cdot,\cdot),[\cdot,\cdot])$ satisfy $NM\subseteq J(R), MN\subseteq J(S)$. Then*

$$I=\begin{pmatrix}A & U\\ V & B\end{pmatrix}\tag{1}$$

is a maximal left ideal of T if and only if

$$I=\begin{pmatrix}A_1 & N\\ M & S\end{pmatrix}\ \text{or}\ I=\begin{pmatrix}R & N\\ M & B_1\end{pmatrix},$$

where $A_1\supseteq NM, B_1\supseteq MN$ are maximal left ideals of R,S, respectively.

(2) I is a maximal ideal of T if and only if $I=\begin{pmatrix} A_1 & N \\ M & S \end{pmatrix}$ or $I=\begin{pmatrix} R & N \\ M & B_1 \end{pmatrix}$, where $A_1 \supseteq NM, B_1 \supseteq MN$ are maximal ideals of R, S, respectively.

Proof (1) Necessity: Let $I=\begin{pmatrix} A_1 & Y \\ X & B_1 \end{pmatrix}$ be a maximal left ideal of $T=\begin{pmatrix} R & N \\ M & S \end{pmatrix}$. Then there is

$$RA_1+NX\subseteq A_1, RY+NB_1\subseteq Y, MA_1+SX\subseteq X, MY+SB_1\subseteq B_1,$$

which implies that A_1, B_1 are left ideals of R, S, respectively. Consider two cases:
(i) If $A_1 \neq R$, then since R has an identity, there exists a maximal left ideal $A_2 \supseteq A_1$ of R. According to the assumption

$$NM\subseteq J(R)=\bigcap_{\alpha} J_\alpha \,(J_\alpha \text{ runs over all maximal left ideals of } R)$$

there is $NM\subseteq A_2$, and there is a left ideal $\begin{pmatrix} A_2 & N \\ M & S \end{pmatrix} \supseteq \begin{pmatrix} A_1 & Y \\ X & B_1 \end{pmatrix}$ of T. According to the maximality of I, there must be

$$A_2=A_1, Y=N, X=M, B_1=S,$$

namely $I=\begin{pmatrix} A_2 & N \\ M & S \end{pmatrix}$.
(ii) If $A_1=R$, then according to $MA_1+SX\subseteq X$, there is $MR\subseteq X$. Since there is 1 for R and $M\subseteq X$, then $X=M$. Moreover, it is known from $MN\subseteq J(S)$ that $\begin{pmatrix} R & N \\ M & B_1 \end{pmatrix}\left(\supseteq \begin{pmatrix} R & Y \\ M & B_1 \end{pmatrix}\right)$ is a left ideal of T. According to the maximality of I, there is $Y=N$ and $I=\begin{pmatrix} R & N \\ M & B_1 \end{pmatrix}$. Let B_2 be a maximal left ideal of S and $B_2\supseteq B_1$. Then $B_2\supseteq J(S)\supseteq MN$. It is verified that $J=\begin{pmatrix} R & N \\ M & B_2 \end{pmatrix}\supseteq I$ is a left ideal of T. By the maximality of I, there is $B_2=B_1$.

The sufficiency is straightforward.
(2) The proof is similar to (1).

Theorem 1.2 *It is assumed that the trace ideals of the Morita context* $(R, S, M, N, (\cdot,\cdot), [\cdot,\cdot])$ *satisfy* $NM\subseteq J(R), MN\subseteq J(S)$. *Then*
(1) *the simple radical of* T *is*

$$S(T)=\begin{pmatrix} S(R) & N \\ M & S(S) \end{pmatrix};$$

(2) *the Brown-McCoy radical of* T *is*

$$g(T)=\begin{pmatrix} g(R) & N \\ M & g(S) \end{pmatrix}.$$

Proof It can be obtained from the definitions of the simple radical and the Brown-McCoy radical and Proposition 1.3.

Corollary 1.1 *If the trace ideals* NM *and* MN *of the Morita context* $(R, S, M, N, (\cdot,\cdot), [\cdot,\cdot])$ *are both nilpotent, then*
(1) T *is a semiprime ring if and only if* R, S *are semiprime rings and* $N = 0, M = 0$;
(2) *if* T *is a prime ring, then* R, S *are prime rings and* $N = 0, M = 0$.

Proof (1) T is a semiprime ring if and only if $P(T) = 0$. By Theorem 1.1 (1), it holds if and only if $N = 0, M = 0$ and R, S are semiprime rings.
(2) If T is a prime ring, then the zero ideal of T is prime. By Proposition 1.1 (1), it follows that $N = 0, M = 0$ and R, S are prime rings.

The converse of Corollary 1.1 (2) is not true. For example,

$$I = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}, J = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}$$

are both ideals of $T = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}$ and $IJ = 0$, but I, J are nonzero. Thus, the zero ideal is not prime in T.

A ring R is called a left quasi-pseudo ring [15] if every maximal left ideal of R is an ideal of R.

Corollary 1.2 *Suppose the trace ideals of the Morita context* $(R, S, M, N, (\cdot,\cdot), [\cdot,\cdot])$ *satisfy* $NM \subseteq J(R), MN \subseteq J(S)$. *Then* T *is a left quasi-pseudo ring if and only if* R, S *are left quasi-pseudo rings.*

Proof It is assumed that R, S are left quasi-pseudo rings. According to Proposition 1.3 (1), every maximal left ideal of T is an ideal of T, so T is a left quasi-pseudo ring. Conversely, it is supposed that T is a left quasi-pseudo ring. Then for any maximal left ideals A of R and B of S, $R + NM, S + MN$ are left ideals of R, S respectively. Based on the maximality of A, B, there is $NM \subseteq A, MN \subseteq B$. Then by Proposition 1.3 (1), $I = \begin{pmatrix} A & N \\ M & S \end{pmatrix}$ and $J = \begin{pmatrix} R & N \\ M & B \end{pmatrix}$ are maximal left ideals of $T = \begin{pmatrix} R & N \\ M & S \end{pmatrix}$, hence they are ideals of T, and A, B are ideals of R, S respectively. R, S are left quasi-pseudo rings.

A ring R is called periodic [8], if for every element x of R, there exist two distinct positive integers m, n such that $x^m = x^n$. A ring R is called N-nil [16], if for any $x \in R$, there exist positive integers n, k depending on x such that $(nx)^k = 0$. A ring R is called a p-ring [16] if for any $a \in R$, there exists a polynomial $f(x) = k_n x^n + k_{n-1} x^{n-1} + \cdots + k_1 x$ with zero constant term such that $f(a) = 0$. Periodicity, N-nil property and the p-ring property are both radical properties. The periodic radical $P_r(R)$ of ring R is the sum of all periodic ideals of R. The

N-nil radical $K_N(R)$ of ring R is the sum of all N-nil ideals of R. The p-radical $p_f(R)$ of ring R is the sum of all p-ideals of R.

Theorem 1.3 *It is assumed that the trace ideals of the Morita context* $(R, S, M, N, (\cdot,\cdot), [\cdot,\cdot])$ *satisfy* $NM = 0, MN = 0$. *Then*
(1) *the nil radical of* T *is*

$$K(T) = \begin{pmatrix} K(R) & N \\ M & K(S) \end{pmatrix};$$

(2) *the* p*-radical of* T *is*

$$p_f(T) = \begin{pmatrix} p_f(R) & N \\ M & p_f(S) \end{pmatrix};$$

(3) *the* N*-nil radical of* T *is*

$$K_N(T) = \begin{pmatrix} K_N(R) & N \\ M & K_N(S) \end{pmatrix};$$

(4) *the periodic radical of* T *is*

$$P_r(T) = \begin{pmatrix} P_r(R) & N \\ M & P_r(S) \end{pmatrix}.$$

Proof (1) According to [16, Theorem 2.1], T is a nil ring if and only if R, S are nil rings. Thus, as the largest nil ideal of T, $K(T) = \begin{pmatrix} K(R) & N \\ M & K(S) \end{pmatrix}$.
(2)–(3) can be obtained from [16, Theorems 2.4–2.5].
(4) The proof is similar to (1).

2 Generalized Prime Radical, Singular Radical, and Behrens Radical

A ring R is called a 2-prime ring [3], if the prime radical $P(R)$ of R equals the set $\mathrm{Nil}(R)$ of all nilpotent elements of R, namely $P(R) = \mathrm{Nil}(R)$. In [3], Birkenmeier et al. systematically studied 2-prime rings and proved that one of its sub-classes $\mathfrak{R}^2 = R$ every prime ideal of R is completely prime} forms an Amitsur-Kurosh root, called the generalized prime radical. The generalized prime radical $\mathfrak{P}_c(R)$ of a ring R is the sum of all ideals I such that $I \in \mathfrak{R}^2$.

Theorem 2.1 *It is assumed that the trace ideals of the Morita context* $(R, S, M, N, (\cdot,\cdot), [\cdot,\cdot])$ *satisfy* $NM = 0, MN = 0$. *Then the generalized prime radical of* T *is*

$$\mathfrak{P}_c(T) = \begin{pmatrix} \mathfrak{P}_c(R) & N \\ M & \mathfrak{P}_c(S) \end{pmatrix}.$$

Proof Let $\mathfrak{P} = \begin{pmatrix} \mathfrak{P}_c(R) & N \\ M & \mathfrak{P}_c(S) \end{pmatrix}$. Since $T/\mathfrak{P} \cong R/\mathfrak{P}_c(R) \oplus S/\mathfrak{P}_c(S)$ and $T/\mathfrak{P}$ is $\mathfrak{P}_c-$ semisimple, then $\mathfrak{P} \supseteq \mathfrak{P}_c(T)$. On the other hand, by Proposition 1.1 (1), for the ideal $P_1 = \begin{pmatrix} \mathfrak{P}_c(R) & N \\ M & 0 \end{pmatrix}$ of T, every prime ideal of $\mathfrak{P}_1$ has the form $I = \begin{pmatrix} A_1 & N \\ M & 0 \end{pmatrix}$, where A_1 is a prime ideal of $\mathfrak{P}_c(R)$. However, the ring $\mathfrak{P}_c(R) \in R^2$, so A_1 is a completely prime ideal and I is a completely prime ideal of $\mathfrak{P}_1$. It proves that $\mathfrak{P}_1$ is a R^2- ring and $\mathfrak{P}_1 \subseteq \mathfrak{P}_c(T)$. Similarly, $\mathfrak{P}_2 = \begin{pmatrix} 0 & N \\ M & \mathfrak{P}_c(S) \end{pmatrix}$ is also a R^2-ring, so $\mathfrak{P}_2 \subseteq \mathfrak{P}_c(T)$ and $(\mathfrak{P}_1 + \mathfrak{P}_2) \subseteq \mathfrak{P}_c(T)$, namely $\mathfrak{P} \subseteq \mathfrak{P}_c(T)$. Therefore, $\mathfrak{P}_c(T) = \begin{pmatrix} \mathfrak{P}_c(R) & N \\ M & \mathfrak{P}_c(S) \end{pmatrix}$.

In [6], Ferrero and Puczyłowski introduced an important radical, namely the (right) singular radical $\varphi = \{R | R$ cannot be homomorphically mapped onto a nonzero semiprime non-singular ring$\}$. It is the upper radical determined by the class of semiprime non-singular rings, and $\varphi \supseteq P$. A radical R is called an N-radical [5] if R contains all nilpotent rings and is a left hereditary and strong radical. The prime radical P, the locally nilpotent radical L, the Jacobson radical J, and the singular radical φ are all N-radicals.

Lemma 2.1 [5, Theorem 1] *Let $\Re$ be an N-radical and R be the sum of two subrings R_1, R_2. If R_1 is nilpotent and $R_2 \in \Re$, then $R \in \Re$.*

Theorem 2.2 *Let $\Re$ be any N-radical. It is supposed that the trace ideals of the Morita context $(R, S, M, N, (\cdot,\cdot), [\cdot,\cdot])$ satisfy $NM \subseteq \Re(R), MN \subseteq \Re(S)$. Then the $\Re$-radical of the Morita context ring $T = \binom{RN}{MS}$ is*

$$\Re(T) = \begin{pmatrix} \Re(R) & N \\ M & \Re(S) \end{pmatrix}.$$

Proof Let

$$T' = \left[\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \Re(R) & 0 \\ 0 & \Re(S) \end{pmatrix}\right] + \begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix} = T_1 + T_2,$$

where

$$T_1 = \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \Re(R) & 0 \\ 0 & \Re(S) \end{pmatrix} = T_3 + T_4.$$

Since $T_3 = \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}$ is a nilpotent ring and $T_4 = \begin{pmatrix} \Re(R) & 0 \\ 0 & \Re(S) \end{pmatrix} \cong \Re(R) \oplus \Re(S)$ is a $\Re$-radical ring, according to Lemma 2.1, T_1 is a $\Re$-radical ring and $T_2 = \begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix}$ is nilpotent. Based on Lemma 2.1, $T' = T_1 + T_2$ is a $\Re$-radical ring. However, T' is an ideal of T, so $T' \subseteq \Re(T)$. On the other hand,

$T/T' \cong R/\Re(R) \oplus S/\Re(S)$ is $\Re$- semisimple, so $T \subseteq \Re(T')$. Therefore, $\Re(T) = \begin{pmatrix} \Re(R) & N \\ M & \Re(S) \end{pmatrix}$.

Corollary 2.1 *It is supposed that the trace ideals of the Morita context* $(R, S, M, N, (\cdot,\cdot), [\cdot,\cdot])$ *satisfy* $NM \subseteq \varphi(R), MN \subseteq \varphi(S)$. *Then*
(1) *the singular radiccal of* T *is*

$$\varphi(T) = \begin{pmatrix} \varphi(R) & N \\ M & \varphi(S) \end{pmatrix};$$

(2) T *is a semiprime singular ring* $\Leftrightarrow$ R *and* S *are semiprime singular rings.*

Proof (1) can be obtained from Theorem 2.2.
(2) According to [9, Theorem 1.14],

T is semiprime singular $\Leftrightarrow T$ is φ-semisimple $\Leftrightarrow \varphi(T) = 0$

$$\Leftrightarrow \varphi(R) = 0, \varphi(S) = 0, N = 0, M = 0$$
$$\Leftrightarrow R, S \text{ are semiprime singular rings and } N = 0, M = 0.$$

The proof is complete.

Corollary 2.2 (1) It is supposed that the trace ideals of the Morita context $(R, S, M, N, (\cdot,\cdot), [\cdot,\cdot])$ satisfy $NM \subseteq P(R), MN \subseteq P(S)$. Then the prime radical of T is

$$P(T) = \begin{pmatrix} P(R) & N \\ M & P(S) \end{pmatrix}.$$

(2) It is assumed that the trace ideals NM and MN of the Morita context $(R, S, M, N, (\cdot,\cdot), [\cdot,\cdot])$ are nilpotent. Then T is a 2-prime ring if and only if R, S are 2-prime rings.
(3) It is assumed that the trace ideals of the Morita context $(R, S, M, N, (\cdot,\cdot), [\cdot,\cdot])$ satisfy $NM \subseteq L(R), MN \subseteq L(S)$. Then the locally nilpotent radical of T is

$$L(T) = \begin{pmatrix} L(R) & N \\ M & L(S) \end{pmatrix}.$$

Proof (1) and (3) can be obtained from Theorem 2.2.
(2) According to [3, Proposition 2.1], T is a 2-prime ring if and only if $P(T) = P_2(T)$. By (1) and Theorem 1.1, it holds if and only if $P(R) = P_2(R)$ and $P(S) = P_2(S)$, namely if and only if R, S are 2-prime rings.

A ring R is called a Jacobson ring if every prime ideal of R is an intersection of maximal left ideals of R. Equivalently, $J(R) = P(R)$.

Corollary 2.3 *It is supposed that the trace ideals of the Morita context*

$(R, S, M, N, (\cdot,\cdot), [\cdot,\cdot])$ *satisfy* $NM \subseteq J(R), MN \subseteq J(S)$. *Then*
(1) *the Jacbson radical of* T *is*

$$J(T) = \begin{pmatrix} J(R) & N \\ M & J(S) \end{pmatrix};$$

(2) T *is a Jacobson ring if and only if* R, S *are Jacobson rings.*
Beidar et al. [2] introduced an Amitsur-Kurosh root $\mathfrak{P}_e$, called the Behrens radical, and proved that a ring R was a Behrens radical ring if and only if every left ideal of R was a Brown-McCoy radical ring (see [2, Proposition 3.1]). Moreover, $J < \mathfrak{P}_e < g$.

Theorem 2.3 *It is assumed that the trace ideals of the Morita context* $(R, S, M, N, (\cdot,\cdot), [\cdot,\cdot])$ *satisfy* $NM \subseteq J(R), MN \subseteq J(S)$. *Then the Behrens radical of* T *is*

$$\mathfrak{P}_e(T) = \begin{pmatrix} \mathfrak{P}_e(R) & N \\ M & \mathfrak{P}_e(S) \end{pmatrix}.$$

Proof Let $T_0 = \begin{pmatrix} \mathfrak{P}_e(R) & N \\ M & \mathfrak{P}_e(S) \end{pmatrix}$. Since $T/T_0 \cong R/\mathfrak{P}_e(R) \oplus S/\mathfrak{P}_e(S)$ is $\mathfrak{P}_e-$semisimple, then $\mathfrak{P}_e(T) \supseteq T_0$. For the ideal T_0 of T, any ideal of T_0 has the form $L = \begin{pmatrix} I & N_1 \\ M_1 & J \end{pmatrix}$, where I, J are left ideals of $\mathfrak{P}_e(R), \mathfrak{P}_e(S)$ respectively. By Theorem 1.2 (2), L is a Brown-McCoy radical ring $\Leftrightarrow$ I, J are Brown-McCoy radical rings. It shows that every left ideal of T_0 is a Brown-McCoy radical ring. Therefore, according to [2, Proposition 3.1], T_0 is a Behrens radical ring and $T_0 \subseteq \mathfrak{P}_e(T)$.

A ring R is called strongly regular, if for every a of R, there exists $t \in R$ such that $a = a^2 t$. The class of strongly regular rings is a radical class. The strongly regular radical $G(R)$ of R is the only largest strongly regular ideal of R [7].

Theorem 2.4 *For any Morita context* $(R, S, M, N, (\cdot,\cdot), [\cdot,\cdot])$, *there is*

$$G(T) = \begin{pmatrix} G(R) & 0 \\ 0 & G(S) \end{pmatrix}.$$

Proof According to [18, Theorem 1.3], $T_\alpha = \begin{pmatrix} A_\alpha & V_\alpha \\ W_\alpha & B_\alpha \end{pmatrix}$ is a G-ideal of T if and only if

$$G(A_\alpha) = A_\alpha, G(B_\alpha) = B_\alpha, V_\alpha = 0, W_\alpha = 0.$$

Therefore

$$\begin{aligned} G(T) &= \sum_{\alpha} T_\alpha \, (T_\alpha \text{ runs over all } G\text{-ideals of } T) \\ &= \sum_{\alpha} \begin{pmatrix} A_\alpha & 0 \\ 0 & B_\alpha \end{pmatrix} (A_\alpha, B_\alpha \text{ run over all } G\text{-ideals of } R \text{ and } S \text{ respectively}) \\ &= \begin{pmatrix} G(R) & 0 \\ 0 & G(S) \end{pmatrix}. \end{aligned}$$

The proof is completed.

Corollary 2.4 *It is assumed that the trace ideals of the Morita context* $(R,S,M,N,(\cdot,\cdot),[\cdot,\cdot])$ *satisfy* $NM=0, MN=0$. *Then* T *is* R*-semisimple if and only if* R,S *are* R*-semisimple and* $N=0, M=0$, *where* R *represents all the radicals mentioned above.*

A ring R is called an NI-ring [10] if its nil radical $K(R)=\mathrm{Nil}(R)$. A ring R is called a J-ring [12] if $J(R)=\mathrm{Nil}(R)$.

Corollary 2.5 *It is supposed that the trace ideals of the Morita context* $(R,S,M,N,(\cdot,\cdot),[\cdot,\cdot])$ *satisfy* $NM=0, MN=0$. *Then*
(1) T is an NI-ring if and only if R,S are NI-rings.
(2) T *is a J-ring if and only if* R,S *are* J*-rings.*

Proof It firstly requires to proof

$$Nil(T) = \begin{pmatrix} Nil(R) & N \\ M & Nil(S) \end{pmatrix}.$$

It is known that $C = \begin{pmatrix} Nil(R) & N \\ M & Nil(S) \end{pmatrix}$ is a subset of nilpotent elements of T, so $C \subseteq \mathrm{Nil}(T)$. From $\left(\begin{smallmatrix} a & v \\ w & b \end{smallmatrix}\right)^n = 0$, there are $a^n=0$, $b^n=0$ and $\begin{pmatrix} a & v \\ w & b \end{pmatrix} \in C$. As a result, $C \supseteq Nil(T)$.
(1) T is an NI-ring $\Leftrightarrow K(T)=Nil(T) \Leftrightarrow K(R)=Nil(R), K(S)=Nil(S) \Leftrightarrow R,S$ are NI-rings.
(2) The proof is similar to (1).

Example 2.1 (1) Let R,S be two rings, and ${}_RN_S$ be a bimodule. The upper triangular matrix ring $\begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ and the zero product extension ring $\begin{pmatrix} R & N \\ 0 & S \end{pmatrix}$ are Morita context rings with trace ideals equal to 0;
(2) let V be an n-dimensional vector space over the field K,

$$R = \Lambda(V) = K \oplus \Lambda^1(V) \oplus \Lambda^2(V) \oplus \cdots \oplus \Lambda^n$$

be the exterior algebra of V, and $m = \Lambda^1(V) \oplus \Lambda^2(V) \oplus \cdots \oplus \Lambda^n(V)$. Then m is an ideal of R, and $m^{n+1}=0$, $J(R)=m$. The trace ideals Rm and mR of the Morita context (R,m,R,R) are nilpotent;
(3) let $A_{x\times y}$ be the set of x-by- y matrices over the ring A. Then the Morita

context ring

$$\begin{pmatrix} R_{n\times n} & J(R)_{n\times m} \\ R_{m\times n} & R_{m\times m} \end{pmatrix}$$

satisfies $J(R)_{n\times m}R_{m\times n} \subseteq J\left(R_{n\times n}\right)$ and $R_{m\times n}J(R)_{n\times m} \subseteq J\left(R_{m\times m}\right)$.

References

1. Amitsur, S.A., Rings of quotients and Morita contexts, J. Algebra, 1971, 17(2): 273–298
2. Beidar, K.I., Fong, Y. and Puczy^lowski, E.R., Polynomial rings over nil rings cannot be homomorphically mapped onto rings with nonzero idempotents, J. Algebra, 2001, 238(1): 389–399
3. Birkenmeier, G.F., Heatherly, H.E. and Lee, E.K., Completely prime ideals and associated radicals, In: Ring Theory (Jain, S.K. and Rizvi, S.T. eds.), Singapore: World Scientific, 1993, 102–129
4. Chen, Y.Q., Fan, Y. and Hao, Z.F., Ideals in Morita rings and Morita semigroups, Acta Math. Sin., Engl. Ser., 2005, 21(4): 893–898
5. Ferrero, M. and Puczyłowski, E.R., On rings which are sums of two subrings, Arch. Math. (Basel), 1989, 53(1): 4–10
6. Ferrero, M. and Puczyłowski, E.R., The singular ideal and radicals, J. Aust. Math. Soc., 1998, 64(2): 195–209
7. Gardner, B.J., Radical classes of regular rings with Artinian primitive images, Pacific J. Math., 1982, 99(2): 337–349
8. Guo, X.Z., On the periodic radical of a ring, Canad. Math. Bull., 1995, 38(2): 215–217
9. Handelman, D. and Lawrence, J., Strongly prime rings, Trans. Amer. Math. Soc., 1975, 211: 209–223
10. Hwang, S.U., Jeon, Y.C. and Lee, Y., Structure and topological conditions of NI rings, J. Algebra, 2006, 302(1): 186–199
11. Jaegermann, M., Morita contexts and radicals, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., 1972, 20(8): 619–623
12. Khatri, C.G. and Mitra, S.K., Hermitian and nonnegative definite solutions of linear matrix equations, SIAM J. Appl. Math., 1976, 31(4): 579–585
13. Nicholson, W.K. and Watters, J.F., Normal radicals and normal classes of rings, J. Algebra, 1979, 59(1): 5–15
14. Poole, D.G. and Stewart, P.N., Classical quotient rings of generalized matrix rings, Int. J. Math. Math. Sci., 1995, 18(2): 311–316
15. Rege, M.B., On von Neumann regular rings and SF-rings, Math. Japonica, 1986, 31(6): 927–936
16. Ren, Y.L. and Wang, Y., Nilpotency for Morita context rings, Math. Pract. Theory, 2007, 27(9): 148–152 (in Chinese)
17. Szász, F.A., Radicals of Rings, New York: John Wiley & Sons, 1981
18. Wang, Y. and Ren, Y.L., Morita context rings with a pair of zero homomorphisms (I), J. Jilin Univ. Sci., 2006, 44(3): 318–324 (in Chinese)