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RESEARCH ARTICLE

Quasi-automorphisms on $\mathrm{B}(\mathcal{X})$

Yuping WANG, Guoxing JI

School of Mathematics and Information Science, Shaanxi Normal University,
Xi'an 710062, China

Abstract Let $\mathcal{X}$ be a Banach space with $\dim\mathcal{X}\geqslant 2$ and $\mathrm{B}(\mathcal{X})$ be the Banach algebra of all bounded linear operators on $\mathcal{X}$, $\forall A,B\in\mathrm{B}(\mathcal{X})$. Define a quasi-product by $A\circ B=A+B-AB$, and $(\mathrm{B}(\mathcal{X}),\circ)$ is a semigroup. In this paper, we mainly discuss the quasi-product automorphisms on $\mathrm{B}(\mathcal{X})$. It is proved that a bijective map φ on $\mathrm{B}(\mathcal{X})$ is a quasi-product automorphism if and only if φ is a ring automorphism.

Keywords Quasi-product, quasi-isomorphism, ring isomorphism
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0 Introduction

In recent years, much attention has been paid to the characterization of maps preserving certain properties on operator algebras, which aims to find characteristics that serve as isomorphism invariants for operator algebras. In particular, based on non-associative algebraic operations on operator algebras like the Lie product and Jordan product, many scholars have studied algebraic isomorphisms and preservers related to these operations, such as Lie isomorphisms, Lie derivations, Jordan isomorphisms, and Jordan derivations [7–10]. On one hand, it is noted that concepts like the Lie product and Jordan product are new non-associative operations derived from the addition and multiplication of the ring. On the other hand, one can define an associative binary operation on a ring based on addition and multiplication--the quasi-product operation. Let R be a ring. $\forall a,b\in\mathrm{R}$, and define $a\circ b=a+b-ab$ (or $a\circ b=a+b+ab$). The operation $\circ$ is called the quasi-product on R. It is seen that $(\mathrm{R},\circ)$ is a semigroup whose identity element is the zero element 0 of R. Let $[a,b]=ab-ba$ be the Lie product of a,b. Then there is $[a,b]=b\circ a-a\circ b$, namely the Lie product $[a,b]$ of a,b is the Lie

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Corresponding author: Yuping WANG, E-mail: wyp@snnu.edu.cn; Guoxing JI, E-mail: gxji@snnu.edu.cn

product of b, a in the sense of the quasi-product. Therefore, the quasi-product is closely related to the Lie product. Meanwhile, it is known that the quasi-product plays a very important role in studying the structure of ring or algebra, such as the Jacobson radical [1, 4, 5] and algebraic Lie quasi-nilpotency [2, 6]. Operator algebras are a class of algebras with topological properties. The quasi-product semigroups on operator algebras form an important class of operator semigroups [3]. Furthermore, as semigroups, the characterization of their algebraic isomorphisms is a fundamental problem. What is the relationship between such isomorphisms and ring isomorphisms? This paper aims to characterize the quasi-product semigroup isomorphisms on a class of operator algebras, namely, the Banach algebra consisting of all bounded linear operators on Banach space.

The following shows some concepts and notations. Let $\mathcal{X}$ be a complex Banach space, $\dim\mathcal{X}$ denotes the dimension of $\mathcal{X}$, and $\mathcal{X}'$ refers to the dual space of $\mathcal{X}$. $\mathrm{B}(\mathcal{X})$ and $\mathrm{F}(\mathcal{X})$ represent the Banach algebra of all bounded linear operators on $\mathcal{X}$ and the set of all finite-rank operators, respectively. M_n is the space of all $n \times n$ complex matrices. If $\dim\mathcal{X} = n$ is a Banach space with finite dimensions, then $\mathrm{B}(\mathcal{X})$ will be algebraically isomorphic to M_n. In this case, set $\mathrm{B}(\mathcal{X}) = M_n$. Let $A \in \mathrm{B}(\mathrm{H})$. If $A^2 = A$, then A is called an idempotent operator. If there exists a positive integer n such that $A^n = 0$, then A is called a nilpotent operator. For two idempotent operators P and Q, if $PQ = QP = P$, then there is $P \leqslant Q$. If $PQ = QP = 0$, then P and Q are said to be orthogonal, denoted by $P \perp Q$. $\forall x \in \mathcal{X}$, $f \in \mathcal{X}'$. $x \otimes f$ denotes the rank-one operator defined by $x \otimes f(z) = f(z)x$, $\forall z \in \mathcal{X}$. It is seen that $x \otimes f$ is an idempotent operator if and only if $f(x) = 1$, and a nilpotent operator if and only if $f(x) = 0$. $I_1(\mathcal{X})$ denotes the set of all rank-one idempotent operators in $\mathrm{B}(\mathcal{X})$. Furthermore, let $S \subseteq \mathcal{X}$ be a subset of $\mathcal{X}$, $[S]$ be the closed subspace generated by S, and M and N be closed subspaces of $\mathcal{X}$. If $M \cap N = 0$ and $M + N$ is a closed subspace, then $M \oplus N = M + N$ is called the Banach direct sum of M and N. When no confusion arises, I usually denotes the identity operator on the Banach space.

1 Main results

Let $\dim\mathcal{X} \geqslant 2$, $\mathrm{B}(\mathcal{X})$ be the Banach algebra of all bounded linear operators on $\mathcal{X}$, and $\circ$ be the quasi-product on $\mathrm{B}(\mathcal{X})$, namely $A \circ B = A + B - AB$, $\forall A, B \in \mathrm{B}(\mathcal{X})$, $(\mathrm{B}(\mathcal{X}), \circ)$ is a semigroup. Let φ be a map on $\mathrm{B}(\mathcal{X})$. If $\varphi(A \circ B) = \varphi(A) \circ \varphi(B)$, $\forall A, B \in \mathrm{B}(\mathcal{X})$, then φ is called a quasi-homomorphism. If φ is bijective and a quasi-homomorphism, then φ is called a quasi-isomorphism.

Proposition 1.1 *Let φ be a quasi-isomorphism on $\mathrm{B}(\mathcal{X})$. Then $\varphi(I) = I$ and*

(1) φ preserves idempotent operators and rank-one idempotent operators in both directions;

(2) φ preserves the order and orthogonality of idempotent operators in both

directions;
(3) φ *preserves rank-one operators in both directions.*

Proof $\forall A \in \mathrm{B}(\mathcal{X})$, since $I = A \circ I = I \circ A$, there is

$$\begin{aligned}\varphi(I) &= \varphi(A \circ I) = \varphi(A) \circ \varphi(I) = \varphi(A) + \varphi(I) - \varphi(A)\varphi(I) \\ &= \varphi(I \circ A) = \varphi(I) \circ \varphi(A) = \varphi(A) + \varphi(I) - \varphi(I)\varphi(A),\end{aligned}$$

then $\varphi(A) = \varphi(A)\varphi(I) = \varphi(I)\varphi(A)$. Since φ is surjective, it follows that $\varphi(I) = I$.
(1) Let $P \in \mathrm{B}(\mathcal{X})$ be an idempotent operator. Then $P \circ P = P$, so $\varphi(P) = \varphi(P \circ P) = \varphi(P) + \varphi(P) - \varphi(P)^2$, namely $\varphi(P)$ is idempotent. Hence φ preserves idempotent operators in both directions.
(2) It is first proved that φ preserves the order and orthogonality of idempotent operators in both directions. Let P, $Q \in \mathrm{B}(\mathcal{X})$. If $P \leqslant Q$, then $P \circ Q = Q \circ P = P + Q - PQ = Q$. Therefore,

$$\begin{aligned}\varphi(P) \circ \varphi(Q) &= \varphi(Q) \circ \varphi(P) \\ &= \varphi(P) + \varphi(Q) - \varphi(P)\varphi(Q) \\ &= \varphi(Q) + \varphi(P) - \varphi(Q)\varphi(P) \\ &= \varphi(Q),\end{aligned}$$

namely $\varphi(P) = \varphi(Q)\varphi(P) = \varphi(P)\varphi(Q)$. As a result, $\varphi(P) \leqslant \varphi(Q)$, and φ preserves the order of idempotent operators. Since φ^{-1} has the same property, φ preserves the order of idempotent operators in both directions. If $P \perp Q$, it is seen that $\varphi(P)\varphi(Q) = \varphi(Q)\varphi(P)$. Let $E = \varphi(P)\varphi(Q) = \varphi(Q)\varphi(P)$. If $E \neq 0$, let $F = \varphi^{-1}(E)$. Then $E \leqslant \varphi(P)$ and $E \leqslant \varphi(Q)$. As φ preserves order, $F \leqslant P, F \leqslant Q$, then $F = 0$, which is contradictory. Therefore, φ preserves the orthogonality of operators. By the property of φ^{-1}, φ preserves the orthogonality of operators in both directions.

Finally, It requires to prove that φ preserves rank-one idempotent operators in both directions. Let P be a rank-one idempotent operator on $B(\mathrm{H})$ and $Q = \varphi(P)$. If $\operatorname{rank} Q > 1$, then there exists a rank-one idempotent operator $Q_1 \in B(\mathrm{H})$ such that $Q_1 < Q$. Then $\varphi^{-1}(Q_1) \leqslant \varphi^{-1}(Q) = P$ is an idempotent operator, and $\varphi^{-1}(Q_1) = P$, namely $Q_1 = Q$, which is contradictory. Hence Q is a rank-one operator. By the property of φ^{-1}, φ preserves rank-one idempotent operators in both directions.
(3) Let $A = x \otimes f$ be a rank-one operator and $\|x\| = 1$. If A is nilpotent, namely $f(x) = 0$, take $g \in \mathcal{X}'$ such that $g(x) = 1$. Then $B = x \otimes g$ and $A + B$ are rank-one idempotent operators. It is seen that $A + B = (A + B) \circ A$, and there is $\varphi(A + B) = \varphi(A + B) \circ \varphi(A) = \varphi(A + B) + \varphi(A) - \varphi(A + B)\varphi(A)$, namely $\varphi(A) = \varphi(A + B)\varphi(A)$ is a rank-one operator. If $f(x) \neq 0$, then $A = aP$, where a is a constant and P is a rank-one idempotent operator. Then from $P = A \circ P$, it can be seen that $\varphi(A) = \varphi(A)\varphi(P)$ is a rank-one operator. Therefore, φ

preserves rank-one operators. Similarly, φ^{-1} also preserves rank-one operators.

Lemma 1.1 *Let* $\dim\mathcal{X}\geqslant 3$ *and* φ *be a quasi-isomorphism on* $\mathrm{B}(\mathcal{X})$. *If* $\varphi(P)=P, \forall P\in I_1(\mathcal{X})$, *then* $\varphi(A)=A$, $\forall A\in\mathrm{B}(\mathcal{X})$.

Proof Since φ is a semigroup isomorphism of $(\mathrm{B(H)},\circ)$ and 0 is the semigroup identity, there is $\varphi(0)=0$. By Proposition 1.1, it is known that $\varphi(P)=P$ for any idempotent operator P. The following shows the proof step by step.

(1) For any rank-one operator A, there is $\varphi(A)=A$.

First, it is supposed that A is a rank-one nilpotent operator, namely $A=x\otimes g$, where $g\in\mathcal{X}'$ such that $g(x)=0$. Take any $y\in\mathcal{X}$ such that $g(y)=1$, and choose a subspace M such that $\ker g=[x]\oplus M$. Then there is $\mathcal{X}=[x]\oplus[y]\oplus M$. Take $f\in\mathcal{X}'$ such that $f(x)=1$ and $\ker f=[y]\oplus M$. Let $P_1=x\otimes f$, $P_2=y\otimes g$, and $P=P_1+P_2$. Then P_1, P_2, and P are idempotent operators, and $P\circ A=A\circ P=P$. Therefore, $P\circ\varphi(A)=\varphi(A)\circ P=P$. In particular, $\varphi(A)=P\varphi(A)=\varphi(A)P$. On the other hand, $P_1\circ A=P_1$, $P_2\circ A=P_2+A$ is a rank-one idempotent, so $P_1\circ\varphi(A)=P_1$, $P_2\circ\varphi(A)=P_2+A$, namely $\varphi(A)=P_1\varphi(A)$, $\varphi(A)-P_2\varphi(A)=A$. As a result, $\varphi(A)=A$.

Next, it is assumed that $A=\lambda P$, $\lambda\in C$, where P is a rank-one idempotent operator and $\lambda\neq 0$. Since $\lambda P\circ P=P\circ\lambda P=P$, there is $\varphi(\lambda P)=\varphi(\lambda P)P=P\varphi(\lambda P)$. Therefore, there exists a constant $\mathrm{h}_P(\lambda)\in C$ such that $\varphi(\lambda P)=\boldsymbol{h}_P(\lambda)P$. Let $P=x\otimes f$ and $\lambda\neq 1$. Take $N=x\otimes g$ as a rank-one idempotent operator, where x, f, g are as above. Then $\lambda P\circ N=\lambda\left(x\otimes\left(f+\frac{1-\lambda}{\lambda}g\right)\right)=\lambda Q$, where $Q=x\otimes\left(f+\frac{1-\lambda}{\lambda}g\right)$ is a rank-one idempotent operator. Therefore, $f_P(\lambda)P\circ N=f_Q(\lambda)Q$, and $f_P(\lambda)=f_Q(\lambda)=\lambda$. Hence, $\varphi(\lambda P)=\lambda P$.

(2) For any finite-rank operator $A\in\mathrm{B}(\mathcal{X}),\varphi(A)=A$.

$\forall A\in\mathrm{F}(\mathcal{X})$, it is known that there exist closed subspaces $M,N,\dim M<\infty$, such that $\mathcal{X}=M\oplus N$, and under this decomposition, $A=B\oplus 0$. It is noted that $\dim M<\infty$, so there exists a basis e_1, e_2, $\ldots$, e_n of M such that B can be written as an $n\times n$ upper triangular matrix $B=(b_{ij})$ with $b_{ij}=0,\forall j<i$. Let $B_i=(t_{kl})$, where $t_{il}=b_{il}, t_{kl}=0,\forall k\neq i$, $i=1,2,\ldots,n$. Then $B=B_n\circ B_{n-1}\circ\cdots\circ B_1$. Let $A_i=B_i\oplus 0$, $i=1,2,\ldots,n$. Then each A_i is a rank-one operator and $A=A_n\circ A_{n-1}\circ\cdots\circ A_1$. Therefore,

$$\begin{aligned}\varphi(A)&=\varphi\left(A_n\circ A_{n-1}\circ\cdots\circ A_1\right)\\&=\varphi\left(A_n\right)\circ\varphi\left(A_{n-1}\right)\circ\cdots\circ\varphi\left(A_1\right)\\&=A_n\circ A_{n-1}\circ\cdots\circ A_1\\&=A.\end{aligned}$$

(3) If $\dim\mathcal{X}=\infty$, then for any operator $A\in\mathrm{B}(\mathcal{X}),\varphi(A)=A$.

Let $A\in\mathrm{B}(\mathcal{X})$. Then for any finite-rank idempotent $P\in\mathrm{B}(\mathcal{X})$, there is $A\circ(I-P)=AP+(I-P)=AP\circ(I-P)$. Therefore,

$$\begin{aligned}
&\varphi(A\circ(I-P))=\varphi(AP\circ(I-P)),\\
&\varphi(A)\circ(I-P)=\varphi(AP)\circ(I-P),\\
&\varphi(A)P=\varphi(AP)P=AP.
\end{aligned}$$

Hence $\varphi(A)=A$.

The following shows notations and the main theorem of this paper. If $\dim\mathcal{X}=n<\infty$, there is $\mathrm{B}(\mathcal{X})=M_n$. Let τ be a ring automorphism on C. For any $A=(a_{ij})\in M_n$, let $A_\tau=(\tau(a_{ij}))$.

Theorem 1.1 *Let* $\mathcal{X}$ *be a complex Banach space with* $\dim\mathcal{X}\geqslant 2$ *and* φ *be a quasi-isomorphism on* $\mathrm{B}(\mathcal{X})$.
(1) If $\dim\mathcal{X}=\infty$, then there exists a bounded invertible linear or conjugate linear operator T on $\mathcal{X}$, such that

$$\varphi(A)=TAT^{-1},\ \forall A\in B(\mathcal{X}).$$

(2) If $2\leqslant\dim\mathcal{X}<\infty$, then there exists an invertible matrix $T\in M_n$ and a ring automorphism τ on C, such that

$$\varphi(A)=TA_\tau T^{-1},\ \forall A\in M_n.$$

Proof (1) By Proposition 1.1, φ preserves rank-one idempotent operators and their orthogonality in both directions. By [11, Theorem 2.4], there exists a bounded invertible linear or conjugate linear operator T on $\mathcal{X}$, such that $\varphi(P)=TPT^{-1}$, $\forall P\in I_1(\mathcal{X})$, or a bounded invertible linear or conjugate linear operator $T:\mathcal{X}'\to\mathcal{X}$, such that $\varphi(P)=TP'T^{-1}$, $\forall P\in I_1(\mathcal{X})$, where P' denotes the adjoint operator of P. In the second case $\mathcal{X}$ is reflexive. In fact, the second case cannot occur. Otherwise, there will exist $P,\ Q\in I_1(\mathcal{X})$ such that $P\circ Q\neq Q\circ P$. Then

$$\begin{aligned}
\varphi(P)\circ\varphi(Q)&=\varphi(P)+\varphi(Q)-\varphi(P)\varphi(Q)\\
&=TP'T^{-1}+TQ'T^{-1}-TP'Q'T^{-1}\\
&=T(Q\circ P)'T^{-1}\\
&\neq T(P\circ Q)'T^{-1}\\
&=\varphi(P\circ Q),
\end{aligned}$$

contradicting that φ is a quasi-isomorphism. Hence only the first case occurs. Define $\psi(A)=T^{-1}\varphi(A)T$, $\forall A\in\mathrm{B}(\mathcal{X})$. Then ψ is a quasi-isomorphism on $\mathrm{B}(\mathcal{X})$ satisfying $\psi(P)=P$, $\forall P\in I_1(\mathcal{X})$. By Lemma 1.2, $\psi(A)=A$, $\forall A\in\mathrm{B}(\mathcal{X})$. Therefore, $\varphi(A)=T\psi(A)T^{-1}=TAT^{-1}$, $\forall A\in\mathrm{B}(\mathcal{X})$.
(2) If $n\geqslant 3$, by [11, Theorem 2.3], there exists an invertible matrix $T\in M_n$ and a ring automorphism τ on C such that $\varphi(P)=TP_\tau T^{-1},\forall P\in I_1(M_n)$, or $\varphi(P)=TP_\tau^T T^{-1},\forall P\in I_1(M_n)$, where P^{T} denotes the transpose of P. Similarly, it can be shown that the second case does not occur. According to (1), it is known that (2) holds.

When $n=2$, let $E_{ij}(i,j=1,2)$ be the standard basis of M_2, namely $E_{ij}=(e_{kl})$, where $e_{ij}=1$, $e_{kl}=0$, $\forall(k,l)\neq(i,j)$. By Proposition 1.1, $\varphi(E_{11})$ and $\varphi(E_{22})$ are two orthogonal rank-one idempotent matrices. Hence, there exists an invertible matrix S_1 such that $\varphi(E_{11})=S_1E_{11}S_1^{-1}$, $\varphi(E_{22})=S_1E_{22}S_1^{-1}$. Define $\phi(A)=S_1^{-1}\varphi(A)S_1$, $\forall A\in M_2$. Then ϕ is a quasi-isomorphism on M_2 and $\phi(E_{ii})=E_{ii}, i=1,2$.

It is first proved that there exists an invertible matrix S_2 and a ring automorphism τ on C such that $\forall\lambda\in$ C, $\phi(\lambda E_{ij})=\tau(\lambda)S_2E_{ij}S_2^{-1}$, $i,\ j=1,\ 2$. Since $\lambda E_{ii}\circ E_{jj}=E_{jj}\circ\lambda E_{ii}(i,j=1,2)$ and ϕ preserves rank-one operators, it is known that there exist bijections τ_{ii} on C such that $\forall\lambda\in$ C, $\phi(\lambda E_{ii})=\tau_{ii}(\lambda)E_{ii}$, $i=1,2$. On the other hand, since $E_{11}\circ\lambda E_{12}=E_{11}$, it is shown that $\phi(\lambda E_{12})=E_{11}\phi(\lambda E_{12})$. So there exist $a_{11},\ a_{12}\in$ C such that

$$\phi(\lambda E_{12})=\phi\left(\begin{bmatrix}0&\lambda\\0&0\end{bmatrix}\right)=\begin{bmatrix}a_{11}&a_{12}\\0&0\end{bmatrix}.$$

$E_{22}+\lambda E_{12}=E_{22}\circ\lambda E_{12}$, then

$$\phi(E_{22}+\lambda E_{12})=\phi\left(\begin{bmatrix}0&\lambda\\0&1\end{bmatrix}\right)=E_{22}\circ\phi(\lambda E_{12})=\begin{bmatrix}a_{11}&a_{12}\\0&1\end{bmatrix}.$$

By Proposition 1.1, ϕ preserves rank-one operators, so $a_{11}=0$, $\phi(\lambda E_{12})=a_{12}E_{12}$. As a result, there exists a bijection τ_{12} on C such that $\forall\lambda\in$ C, $\phi(\lambda E_{12})=\tau_{12}(\lambda)E_{12}$. Similarly, there exists a bijection τ_{21} on C such that $\forall\lambda\in$ C, $\phi(\lambda E_{21})=\tau_{21}(\lambda)E_{21}$.

$\forall x,y\in$ C, and according to $xE_{12}\circ yE_{12}=(x+y)E_{12}$, there is $\phi(xE_{12})\circ\phi(yE_{12})=\phi((x+y)E_{12})$, namely $\tau_{12}(x)E_{12}\circ\tau_{12}(y)E_{12}=\tau_{12}(x+y)E_{12}$, so $\tau_{12}(x)+\tau_{12}(y)=\tau_{12}(x+y)$. From $yE_{22}\circ(x-xy)E_{12}=xE_{12}\circ yE_{22}$ and $(x-xy)E_{12}\circ yE_{11}=yE_{11}\circ xE_{12}$, it is known $\tau_{12}(xy)=\tau_{22}(y)\tau_{12}(x)$ and $\tau_{12}(xy)=\tau_{11}(y)\tau_{12}(x)$, so $\tau_{11}(x)=\tau_{22}(x)$. Denote $\tau=\tau_{11}=\tau_{22}$. Since $\tau_{12}(xy)=\tau_{11}(y)\tau_{12}(x)$, set $x=1$ gives $\tau_{12}(y)=\tau(y)\tau_{12}(1)$. Similarly, $\tau_{21}(y)=\tau(y)\tau_{21}(1)$, and there is $\tau_{21}(1)\tau_{12}(1)=1$. Let

$$A=xE_{12}\circ xE_{21}=\begin{bmatrix}-x^2&x\\x&0\end{bmatrix}\text{ and }\phi(A)=\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}.$$

Since

$$A\circ E_{11}=\begin{bmatrix}1&x\\0&0\end{bmatrix},\ A\circ E_{22}=\begin{bmatrix}-x^2&0\\x&1\end{bmatrix},$$

there is

$$\phi\left(\begin{bmatrix}1&x\\0&0\end{bmatrix}\right)=\phi(A)\circ E_{11}=\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}\circ E_{11}=\begin{bmatrix}1&b_{12}\\0&b_{22}\end{bmatrix},$$

$$\phi\left(\begin{bmatrix}-x^2&0\\x&1\end{bmatrix}\right)=\phi(A)\circ E_{22}=\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}\circ E_{22}=\begin{bmatrix}b_{11}&0\\b_{21}&1\end{bmatrix}.$$

On the other hand, since

$$\phi\left(\begin{bmatrix}1 & x\\ 0 & 0\end{bmatrix}\right)=\phi\left(\begin{bmatrix}0 & x\\ 0 & 0\end{bmatrix}\circ\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right)=\begin{bmatrix}\tau(1) & \tau_{12}(x)\\ 0 & 0\end{bmatrix},$$
$$\phi\left(\begin{bmatrix}-x^2 & 0\\ x & 1\end{bmatrix}\right)=\phi\left(\begin{bmatrix}-x^2 & 0\\ 0 & 0\end{bmatrix}\circ\begin{bmatrix}0 & 0\\ x & 0\end{bmatrix}\circ\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)=\begin{bmatrix}\tau\left(-x^2\right) & 0\\ \tau_{21}(x) & \tau(1)\end{bmatrix},$$

there is $b_{11}=\tau\left(-x^2\right),\ b_{21}=\tau_{21}(x),\ b_{12}=\tau_{12}(x),\ b_{22}=0$, namely

$$\phi\left(\begin{bmatrix}-x^2 & x\\ x & 0\end{bmatrix}\right)=\begin{bmatrix}\tau\left(-x^2\right) & \tau_{12}(x)\\ \tau_{21}(x) & 0\end{bmatrix}.$$

As

$$\phi(A)=\phi\left(xE_{12}\right)\circ\phi\left(xE_{21}\right)=\begin{bmatrix}-\tau_{21}(x)\tau_{12}(x) & \tau_{12}(x)\\ \tau_{21}(x) & 0\end{bmatrix},$$

there is $\tau\left(-x^2\right)=-\tau_{21}(x)\tau_{12}(x)=-\tau(x)^2\tau_{21}(1)\tau_{12}(1)$, namely $\tau_{21}(1)\tau_{12}(1)=1$.

The following proves that τ is a ring automorphism. Since $\tau_{12}(1)\neq 0$, when $x=y=1$, according to $\tau_{12}(xy)=\tau_{11}(y)\tau_{12}(x)$, there is $\tau(1)=1$. As $\tau(xy)\tau_{12}(1)=\tau_{12}(xy)=\tau(x)\tau(y)\tau_{12}(1)$, there is $\tau(x)\tau(y)=\tau(xy),\ \forall x,y\in\mathrm{C}$. Moreover,

$$\begin{aligned}(\tau(x)+\tau(y))\tau_{12}(1)&=\tau(x)\tau_{12}(1)+\tau(y)\tau_{12}(1)\\&=\tau_{12}(x)+\tau_{12}(y)\\&=\tau_{12}(x+y)\\&=\tau(x+y)\tau_{12}(1),\forall x,y\in\mathrm{C}\end{aligned}$$

and there is $\tau(x+y)=\tau(x)+\tau(y),\ \forall x,\ y\in\mathrm{C}$. Therefore, τ is a ring automorphism. Let

$$S_2=\begin{bmatrix}\tau_{12}(1)^{\frac{1}{2}} & 0\\ 0 & \tau_{21}(1)^{\frac{1}{2}}\end{bmatrix}.$$

There is $\phi\left(\lambda E_{ij}\right)=\tau(\lambda)S_2E_{ij}S_2^{-1}, i,j=1,2$.

Next, define $\psi(A)=S_2^{-1}\phi(A)S_2,\forall A\in M_2$. Then ψ is a quasi-isomorphism and $\psi\left(\lambda E_{ij}\right)=\tau(\lambda)E_{ij}, i,j=1,2$. Let

$$A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\text{ and }\psi(A)=\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}.$$

Since

$$A\circ E_{11}=\begin{bmatrix}1 & a_{12}\\ 0 & a_{22}\end{bmatrix},\ A\circ E_{22}=\begin{bmatrix}a_{11} & 0\\ a_{21} & 1\end{bmatrix},$$

there is

$$\psi\left(\begin{bmatrix}1 & a_{12}\\ 0 & a_{22}\end{bmatrix}\right)=\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}\circ E_{11}=\begin{bmatrix}1 & b_{12}\\ 0 & b_{22}\end{bmatrix},$$
$$\psi\left(\begin{bmatrix}a_{11} & 0\\ a_{21} & 1\end{bmatrix}\right)=\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}\circ E_{22}=\begin{bmatrix}b_{11} & 0\\ b_{21} & 1\end{bmatrix}.$$

On the other hand, since

$$\psi\left(\begin{bmatrix}1 & a_{12}\\ 0 & a_{22}\end{bmatrix}\right)=\psi\left(\begin{bmatrix}0 & 0\\ 0 & a_{22}\end{bmatrix}\circ\begin{bmatrix}0 & a_{12}\\ 0 & 0\end{bmatrix}\circ\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right)=\begin{bmatrix}\tau(1) & \tau\left(a_{12}\right)\\ 0 & \tau\left(a_{22}\right)\end{bmatrix},$$
$$\psi\left(\begin{bmatrix}a_{11} & 0\\ a_{21} & 1\end{bmatrix}\right)=\psi\left(\begin{bmatrix}a_{11} & 0\\ 0 & 0\end{bmatrix}\circ\begin{bmatrix}0 & 0\\ a_{21} & 0\end{bmatrix}\circ\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)=\begin{bmatrix}\tau\left(a_{11}\right) & 0\\ \tau\left(a_{21}\right) & \tau(1)\end{bmatrix},$$

there is $b_{11}=\tau\left(a_{11}\right),\ b_{21}=\tau\left(a_{21}\right),\ b_{12}=\tau\left(a_{12}\right),\ b_{22}=\tau\left(a_{22}\right)$, namely $\psi(A)=A_\tau,\forall A\in M_2$. Let $T=S_1S_2$. Then $\varphi(A)=TA_\tau T^{-1},\forall A\in M_2$.

Note If φ is a quasi-isomorphism on C, namely $\varphi(x\circ y)=\varphi(x)\circ\varphi(y)$, $\forall x,y\in$ C, is φ necessarily a ring automorphism on C ?

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