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RESEARCH ARTICLE

Generalized solution of nonlinear nonlocal singularly perturbed problems with two parameters

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Abstract In this paper, we discuss a class of higher order nonlinear nonlocal singularly perturbed boundary value problems with two parameters. Under suitable conditions, we use the fixed-point theorem to study the existence of a generalized solution. Moreover, with the help of the singular perturbation method, we gain the uniformly valid asymptotic representation of the solution.

Keywords Singular perturbation, asymptotic expansion, uniformly valid
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0 Introduction

Nonlinear singularly perturbed problems have received considerable attention in academia [1, 2]. Many approximation methods, including the boundary layer method, the averaging method, the method of matched asymptotic expansions, and the method of multiple scales, have been improved. Recently, many scholars, such as Samusenko [9], Skrynnikov [11], Tian and Zhu [12], Martínez and Wolanski [5], and Kellogg and Kopteva [4], have done plenty of work. Mo et al. [3, 6–8, 10] have studied a class of nonlinear problems using the singular perturbation method. This paper aims to study a class of singularly perturbed problems for elliptic equations with two parameters by the boundary layer method.

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A class of boundary value problems for nonlinear nonlocal elliptic equations with two parameters is

$$\begin{aligned}&\varepsilon^{2m}L_{2m}[u(x)]+\mu^{2l}L_{2k}[u(x)]+L_2[u(x)]=f(x,Tu(x)),\\&x=(x_1,x_2,\ldots,x_n)\in\Omega\subset \mathrm{R}^n;\end{aligned}\tag{1}$$

$$\frac{\partial^l u}{\partial n^l}(x)=g_l(x),\ x\in\partial\Omega,\ l=1,2,\ldots,m-1,\tag{2}$$

where ε,μ are small parameters with $0\leqslant\frac{\varepsilon^m}{\mu^l}\ll 1$, Ω is a bounded convex domain, $\partial\Omega$ is an infinitely smooth boundary of Ω, and

$$\begin{aligned}&L_{2l}\equiv\sum_{|\nu|,|\sigma|\leqslant l}(-1)^{|\nu|}D^\nu a_l^{\nu\sigma}(x)D^\sigma,\ 1\leqslant l\leqslant n,\\&D_j=\frac{\partial}{\partial x_j},\ D^\alpha=D_1^{\alpha_1}D_2^{\alpha_2}\cdots D_n^{\alpha_n},\ |\alpha|=\sum_{j=1}^n\alpha_j,\\&Tu=\int_\Omega K(x)u(x)\mathrm{d}x.\end{aligned}$$

Among them, $1<k<m, x^\alpha=(x_1^{\alpha_1},x_2^{\alpha_2},\ldots,x_n^{\alpha_n})$, and $a^{\nu\sigma},b^{\nu\sigma}$ are real-valued functions of $C^\infty(\Omega)$. K,f, and g_l are sufficiently smooth real-valued functions in their respective variable ranges, $L_u(1\leqslant l\leqslant m)$ are uniformly elliptic operators on $\overline{\Omega}$, and $\frac{\partial}{\partial n}$ is the outward normal derivative on $\partial\Omega$.

Considering the boundary value problems (1)–(2), we discuss the generalized boundary value problem in a functional space with two parameters:

$$\varepsilon^{2m}B_m[\psi,u]+\mu^{2k}B_k[\psi,u]+B_1[\psi,u]=(\psi,f(x,Tu)),\tag{3}$$

$$\left(\psi,\frac{\partial^l u}{\partial n^l}\right)=(\psi,g_l),x\in\partial\Omega,l=1,2,\ldots,m-1,\forall\psi\in C_0^\infty(\Omega),\tag{4}$$

where

$$B_j[\psi,u]\equiv\sum_{0<|\mu|,|\sigma|\leqslant j}(D^\mu\psi,a^{\mu\sigma}D^\sigma)u=(\psi,L_{2j}[u]),j=1,k,m.$$

$C_0^\infty(\Omega)$ denotes the compact subsets of $C^\infty(\Omega)$ in Ω, $B_j[v,u]$ is the bilinear operator associated with $L_{2j}[u]$,$L_{2j}[u]$ refers to the operator defined by bounded functions $a_j^{\nu\sigma}$ on Ω, and the bounded norms of v and u on the Sobolev space are

$$\|\psi\|_j=\left(\sum_{|\alpha|\leqslant j}\int_\Omega|D^\alpha\psi(x)|^2\mathrm{d}x\right)^{\frac{1}{2}},\ j=1,k,m,\ \forall\psi\in C_0^\infty(\Omega),$$

where (u,v) is the inner product on $H_0^j(\Omega)$.

1 Existence of the generalized solution

We first discuss the solution of problems (3)–(4). Assume:

[H$_1$] There exist constants $C_{ji}(j=1,k,m;i=1,2)$ independent of v and u, such that

$$\left|B_j\left[v,u\right]\right| \leqslant C_{j1}\|v\|_j \cdot \|u\|_j,\ \left|B_j\left[v,v\right]\right| \leqslant C_{j2}\|v\|_j^2,\ \forall v,u \in H_0^j,\ j=1,k,m;$$

[H$_2$] For $1 \leqslant |\nu|,|\sigma| \leqslant j, j=1,k,m$, the coefficients $a_j^{\nu\sigma}$ are bounded on Ω, and there exist c_j satisfying

$$\left|a_j^{\nu\sigma}\left(x\right) - a_j^{\nu\sigma}\left(y\right)\right| \leqslant c_j\left(|x-y|\right),\ |\nu|=|\sigma|=j,\ j=1,l,m,\ \forall x,y \in \Omega,$$

and when $|x-y| \to 0,\ c_j(|x-y|) \to 0$.

[H$_3$] There exist positive constants δ_1,δ_2, such that

$$\delta_1 \leqslant \frac{\partial f}{\partial u}\left(x,Tu\right) \leqslant \delta_2,\ \forall x \in \overline{\Omega}, \forall u \in H_0^j,\ 0<\varepsilon \ll 1.$$

[H$_4$] There exists a generalized solution $w_{00} \in H_0^m(\Omega)$: $(\psi,u)=(\psi,g_0)\,, x \in \partial\Omega$, $\forall \psi \in C_0^\infty(\Omega)$ of the boundary value problem $B_1[\psi,u]=(\psi,f(x,Tu))$.

The following shows the way to prove the theorem.

Theorem 1 *Under the assumptions* [H$_1$]–[H$_4$], *there is a solution* $u(x) \in H_0^m(\Omega)$ *for the generalized boundary value problems* (3)–(4).

Proof Considering an arbitrary function $u_0(x) \in H_0^m(\Omega)$, we discuss the following generalized boundary value problem:

$$\begin{aligned}&\varepsilon^{2m}B_m[\psi,u]+\mu^{2k}B_k[\psi,u]+B_1[\psi,u]=\left(\psi,f\left(x,Tu_0\right)\right),\\&\left(\psi,\frac{\partial^l u}{\partial n^l}\right)=\left(\psi,g_l\right),\quad x\in\partial\Omega,\quad l=0,\ 1,\ \ldots,\ m-1,\quad \forall\psi\in C_0^\infty\left(\Omega\right).\end{aligned}$$

According to the Lax-Milgram theorem [2] and assumptions [H$_1$]–[H$_4$], for the operator of bounded functions on the Hilbert space $H_0^m(\Omega)$, there is

$$F[v]=\varepsilon^{2m}B_m[v,u]+\mu^{2k}B_k[v,u]+B_1[v,u],$$

where $F[v]=(v,f\left(x,Tu_0(x)\right))$, and there exists a generalized solution $u_1(x) \in H_0^m(\Omega)$ satisfying

$$\begin{aligned}&\varepsilon^{2m}B_m\left[\psi,u_1\right]+\mu^{2k}B_k\left[\psi,u_1\right]+B_1\left[\psi,u_1\right]=\left(\psi,f\left(x,Tu_0\right)\right),\\&\left(\psi,\frac{\partial^l u_1}{\partial n^l}\right)=\left(\psi,g_l\right),\quad x\in\partial\Omega,\quad l=1,\ 2,\ \ldots,\ m-1,\quad \forall\psi\in C_0^\infty(\Omega).\end{aligned}$$

Considering iterative method and the following equation

$$\begin{aligned}&\varepsilon^{2m}B_m\left[\psi,u_i\right]+\mu^{2k}B_k\left[\psi,u_i\right]+B_1\left[\psi,u_i\right]=\left(\psi,f\left(x,Tu_{i-1}\right)\right),\\&\left(\psi,\frac{\partial^l u_i}{\partial n^l}\right)=0,\quad x\in\partial\Omega,\quad l=1,\ 2,\ \ldots,\ m-1,\quad\forall\psi\in C_0^\infty(\Omega),\end{aligned}$$

we can obtain the solution $u_i(x)\in H_0^m(\Omega)$ as well as a sequence of functions $\{u_j(x)\in H_0^m(\Omega),j=0,1,\ldots\}$. Therefore, there exists a generalized solution $u(x)\in H_0^m(\Omega)$ of the boundary value problems (3)–(4), such that

$$\lim_{j\to\infty}\left(\psi,u_j\right)=\left(\psi,u\right),\quad\forall\psi\in C_0^\infty\left(\Omega\right).$$

Theorem 1 is proved.

2 Outer solution

The reduced problems of (3)–(4) is considered as following:

$$B_1[\psi,u]=(\psi,f(x,Tu)),\tag{5}$$

$$\left(\psi,u\right)=\left(\psi,g_0\left(x\right)\right),\quad x\in\partial\Omega,\quad\forall\psi\in C_0^\infty\left(\Omega\right).\tag{6}$$

According to assumption [H$_4$], there is a solution $w_{00}(x)\in H_0^m(\Omega)$ for problems (5)–(6). Let the outer solution $w(x,\varepsilon,\mu)$ of the generalized boundary value problems (3)–(4) be

$$w\left(x,\varepsilon,\mu\right)=\sum_{i,j=0}^{\infty}w_{ij}\left(x\right)\varepsilon^i\mu^j.\tag{7}$$

Substitute (7) into equations (3)–(4), expand ε,μ, combine the coefficients of powers $\varepsilon^i\mu^j$, and let the coefficients of like power $\varepsilon^i\mu^j(i,j=1,2,\ldots,i+j\neq 0)$ be zero. Considering the solution w_{00} of problems (5)–(6), there are

$$\begin{aligned}&B_1\left[\psi,w_{ij}\right]-\left(\psi,f_u\left(x,Tw_{00}\right)Tw_{ij}\right)\\&=-B_m\left[\psi,w_{(i-2m)(j-2m)}\right]-B_k\left[\psi,w_{(i-2k)(j-2k)}\right]+\left(\psi,h_{ij}\right),\end{aligned}\tag{8}$$

$$(\psi,w_{ij})=0,x\in\partial\Omega,\forall\psi\in C_0^\infty(\Omega).\tag{9}$$

Here and hereafter, terms with negative subscripts are identically zero, and h_{ij} refers to functions successively known in terms of $w_{rs}(r\leqslant i,s\leqslant j,r+s\neq i+j)$. The solution $w_{ij}(x)$ can be obtained successively, and the outer solution (7) is obtained, which may not satisfy the boundary condition (4) for $l=1,2,\ldots,m-1$. As a result, the correction terms of boundary layer need to be built near $\partial\Omega$.

3 Correction terms of boundary layer

According to [6], a local coordinate system (ρ, ϕ) is built at each point of a neighborhood of $\partial\Omega$. In the neighborhood $0 \leqslant \rho \leqslant \rho_0$ of $\partial\Omega$, we discuss the generalized nonlinear nonlocal boundary value problem with two parameters:

$$\varepsilon^{2m}\overline{B}_m [\psi, u] + \mu^{2k}\overline{B}_k [\psi, u] + \overline{B}_1 [\psi, u] = (\psi, f(\rho, \phi, Tu)), \tag{10}$$

$$\left(\psi, \frac{\partial^l u}{\partial \rho^l}\right) = (\psi, \overline{g}_l), x \in \partial\Omega, l = 0, 1, \ldots, m-1, \forall \psi \in C_0^\infty (0 \leqslant \rho \leqslant \rho_0), \tag{11}$$

where

$$\begin{aligned}
\overline{B}_l [\psi, u] &\equiv \sum_{1 \leqslant |\mu|, |\sigma| \leqslant l} (D^\mu \psi, \overline{a}^{\mu\sigma} D^\sigma) u = \big(\psi, \overline{L}[u]\big), \\
\overline{L}_{2m} &\equiv \sum_{1 \leqslant |\nu|, |\sigma| \leqslant m} (-1)^{|\nu|} D^\nu (\overline{a}_m^{\nu\sigma}(x)) D^\sigma + \sum_{1 \leqslant |\nu| \leqslant m} D^\nu \overline{b}_m^\nu, \\
\overline{L}_{2k} &\equiv \sum_{1 \leqslant |\nu|, |\sigma| \leqslant k} (-1)^{|\nu|} D^\nu (\overline{a}_k^{\nu\sigma}(x)) D^\sigma + \sum_{1 \leqslant |\nu| \leqslant k} D^\nu \overline{b}_k^\nu,
\end{aligned}$$

and

$$\begin{aligned}
&\overline{D}_n = \frac{\partial}{\partial \rho}, \overline{D}_j = \frac{\partial}{\partial \phi_j}, j = 1, 2, \ldots, n-1, \\
&\overline{D}^\alpha = \overline{D}_1^{\alpha_1} \overline{D}_2^{\alpha_2} \cdots \overline{D}_n^{\alpha_n}, \alpha = \sum_{j=1}^n \alpha_j, \\
&\overline{g}_l(\rho, \phi) = (-1)^l g_l(x), \quad \overline{a}_m^{nn} = \sum_{i,j=1}^n a_m^{ij} \frac{\partial \rho}{\partial x_i} \frac{\partial \rho}{\partial x_j} > 0, \quad \overline{a}_k^{nn} = \sum_{i,j=1}^n a_k^{ij} \frac{\partial \rho}{\partial x_i} \frac{\partial \rho}{\partial x_j} > 0.
\end{aligned}$$

Expressions for $\overline{a}_m^{ij}, \overline{a}_k^{ij}, \overline{b}_m^j, \overline{b}_k^j$ are omitted, $C_0^\infty (0 \leqslant \rho \leqslant \rho_0)$ is the compact subset of $C^\infty (0 \leqslant \rho \leqslant \rho_0)$, and $\overline{B}_j[v, u]$ denotes the bilinear operator defined on the Sobolev space $H^j (0 \leqslant \rho \leqslant \rho_0)$ for bounded functions v and u.

Now construct the first boundary layer correction term V_1. Introduce the stretched variable [2]

$$\xi = \frac{\rho}{\mu} \tag{12}$$

and

$$u \sim \sum_{i,j=0}^\infty w_{ij}(x) \varepsilon^i \mu^j + V_1, \tag{13}$$

where

$$V_1 \sim \sum_{i,j=0}^\infty v_{ij}(\xi, \phi) \sigma^i \mu^j, \ 0 < \sigma = \frac{\varepsilon}{\mu} \ll 1. \tag{14}$$

Substitute (12)–(14) into equations (10)–(11), expand σ, μ, and combine the coefficients of power $\sigma^i\mu^j(i,j=0,1,2,\ldots)$. Set the coefficients of like power $\sigma^i\mu^j$ to zero. For $\sigma^i\mu^j$ and $\forall\psi\in C_0^\infty\,(0\leqslant\mu\xi\leqslant\rho_0)$, there are

$$\left(\widetilde{D}_n\psi,\overline{a}_k^{nn}\widetilde{D}_nv_{100}\right)+\left(\widetilde{D}_1\psi,\overline{a}^{11}\widetilde{D}_1v_{100}\right)=\left(\psi,f_u\left(0,\phi,Tv_{100}\right)Tv_{100}\right),\qquad(15)$$

$$\left(\psi,\frac{\partial^l v_{100}}{\partial\xi^l}\right)=\left(\psi,\overline{g}_l\right),\quad\xi=0,\qquad(16)$$

$$\left(\widetilde{D}_n\psi,\overline{a}_k^{nn}\widetilde{D}_nv_{1ij}\right)+\left(\widetilde{D}_1\psi,\overline{a}^{11}\widetilde{D}_1v_{1ij}\right)=\left(\psi,\overline{h}_{ij}\right)+\widetilde{G}_{ij},\qquad(17)$$

$$\left(\psi,\frac{\partial^l v_{1ij}}{\partial\xi^l}\right)=0,\xi=0,\qquad(18)$$

where $l=0,1,\ldots,k-1,i,j=0,1,\ldots,i+j\neq0$, $\widetilde{G}_{ij}$ and $\overline{\mathrm{h}}_{ij}$ are successively known terms, and

$$\widetilde{D}_n=\frac{\partial}{\partial\xi},\ \widetilde{D}_j=\frac{\partial}{\partial\phi_j},\ j=1,2,\ldots,n-1,$$
$$\widetilde{D}^\alpha=\widetilde{D}_1^\alpha\widetilde{D}_2^\alpha\cdots\widetilde{D}_n^\alpha,\ \alpha=\sum_{j=1}^m\alpha_j.$$

According to (15)–(18), there is $v_{1ij}(i,j=0,1,\ldots)$. From (14), we obtain the correction term V_1 of the first boundary layer in the neighborhood of $\partial\Omega$, and there is

$$v_{1ij}=O\left(\exp\left(-k_{ij}\frac{\rho}{\mu}\right)\right),\ i,j=0,1,\ldots,\ 0\leqslant\rho\leqslant\rho_0,\ 0<\mu\ll1,\qquad(19)$$

where $k_{ij}(i,j=0,1,\ldots)$ are positive constants.

Then construct correction term V_2 of the second boundary layer. Introduce the stretched variable [2]

$$\eta=\frac{\rho}{\zeta},\zeta=\left(\frac{\varepsilon^m}{\mu^l}\right)^{\frac{1}{m+k}}\qquad(20)$$

and

$$u\sim\sum_{i,j=0}^\infty w_{ij}\varepsilon^i\mu^j+V_2,\qquad(21)$$

where

$$V_2=\sum_{i,j=0}^\infty v_{2ij}\left(\eta,\phi\right)\zeta^i\mu^j,\ 0<\zeta\ll1.\qquad(22)$$

Substitute (20)–(22) into equations (10)–(11), expand ζ,μ, and combine coefficients of power $\zeta^i\mu^j(i,j=0,1,2,\ldots)$. Set the coefficients of like power $\zeta^i\mu^j$ to zero. For $\zeta^i\mu^j$ and $\forall\psi\in C_0^\infty\,(0\leqslant\zeta\eta\leqslant\rho_0)$, there are

$$\left(\hat{D}_n\psi, \overline{a}_k^{nn}\hat{D}_n v_{200}\right) + \left(\hat{D}_1\psi, \overline{a}^{11}\hat{D}_1 v_{200}\right) = \left(\psi, f_u\left(0, \phi, Tv_{200}\right)Tv_{200}\right), \tag{23}$$

$$\left(\psi, \frac{\partial^l v_{200}}{\partial \xi^l}\right) = 0, \quad \eta = 0, \tag{24}$$

$$\left(\hat{D}_n\psi, \overline{a}_k^{nn}\hat{D}_n v_{2ij}\right) + \left(\hat{D}_1\psi, \overline{a}^{11}\hat{D}_1 v_{2ij}\right) = \left(\psi, \hat{h}_{ij}\right) + \hat{G}_{ij}, \tag{25}$$

$$\left(\psi, \frac{\partial^l v_{2ij}}{\partial \xi^l}\right) = 0, \quad \eta = 0, \tag{26}$$

where $l = 0, 1, \ldots, k-1, i, j = 0, 1, \ldots, i+j \neq 0$, $\hat{G}_{ij}$ and $\hat{\mathrm{h}}_{ij}$ are successively known terms, and

$$\hat{D}_n = \frac{\partial}{\partial \eta},\ \hat{D}_j = \frac{\partial}{\partial \phi_j},\ j = 1, 2, \ldots, n-1,$$
$$\hat{D}\alpha = \hat{D}_1^{\alpha_1}\hat{D}_2^{\alpha_2}\cdots\hat{D}_n^{\alpha_n},\ \alpha = \sum_{j=1}^{m}\alpha_j.$$

According to (23)–(26), there is $v_{2ij}(i, j = 0, 1, \ldots)$. Then from (22), we obtain the correction term V_2 of the second boundary layer in the neighborhood of $\partial\Omega$, and there is

$$v_{2ij} = O\left(\exp\left(-\overline{k}_{ij}\frac{\rho}{\zeta}\right)\right), i, j = 0, 1, \ldots, 0 \leqslant \rho \leqslant \rho_0, 0 < \zeta \ll 1, \tag{27}$$

where $\overline{k}_{ij}(i, j = 0, 1, \ldots)$ are positive constants.

Note From (22) and (27), as well as $\frac{\varepsilon}{\mu} \to 0$, the thickness of the thin layer for the correction term V_2 on the second boundary layer is smaller than that of the correction term V_1 on the first boundary layer.

As a result, we obtain the asymptotic solution of the boundary value problems (3)–(4) with two parameters:

$$u \sim \sum\nolimits_{i,j=0}^{\infty} w_{ij}\left(x\right)\varepsilon^i\mu^j + \varsigma\left(\rho\right)\sum\nolimits_{i,j=0}^{\infty}\left(v_{1ij}(\xi, \phi)\sigma^i\mu^j + v_{2ij}(\eta, \phi)\zeta^i\mu^j\right), \tag{28}$$

where $\varsigma(\rho) \in C^\infty[0, \infty]$ and it satisfies

$$\varsigma(\rho) = \begin{cases} 1, & 0 \leqslant \rho \leqslant \frac{1}{3}\rho_{0,} \\ 0, & \rho \geqslant \frac{2}{3}\rho_{0.} \end{cases}$$

4 Conclusion

Define the remainder $\overline{z}$, such that

$$u = \overline{w} + \overline{V}_1 + \overline{V}_2 + \overline{z}, \tag{29}$$

where

$$\overline{w} = \sum\nolimits_{i,j=0}^{M} w_{ij}\varepsilon^i\mu^j, \quad \overline{V}_1 = \sum\nolimits_{i,j=0}^{M} v_{1ij}\left(\xi,\varphi\right)\xi^i\mu^j, \quad \overline{V}_2 = \sum\nolimits_{i,j=0}^{M} v_{2ij}\left(\eta,\phi\right)\eta^i\mu^j,$$

and $\overline{w}, \overline{V}_1, \overline{V}_2$ respectively refer to the outer solution w of the generalized boundary value problems (3)–(4), the M th order asymptotic representations of the correction term V_1 on the first boundary layer, and the M th order asymptotic representations of the correction term V_2 on the second boundary layer.

A priori estimate for $\overline{z}$ is as follows:

$$\begin{aligned}&\varepsilon^{2m}B_m[\psi,\overline{z}]+\mu^{2k}B_k[\psi,\overline{z}]+B_1[\psi,\overline{z}]\\ &\quad= \varepsilon^{2m}B_m\left[\psi, u-\left(\overline{w}+\overline{V}_1+\overline{V}_2\right)\right] + \mu^{2k}B_k\left[\psi, u-\left(\overline{w}+\overline{V}_1+\overline{V}_2\right)\right]\\ &\qquad+ B_1\left[\psi, u-\left(\overline{w}+\overline{V}_1+\overline{V}_2\right)\right] + f\left(x, T\left(\overline{w}+\overline{V}_1+\overline{V}_2+\overline{z}\right)\right), \forall\psi\in C_0^\infty(\Omega).\end{aligned}$$

According to $\overline{z}\in H_0^m(\Omega)$, there are

$$\begin{aligned}&\varepsilon^{2m}B_m[\overline{z},\overline{z}]+\mu^{2k}B_k[\overline{z},\overline{z}]+B_1[\overline{z},\overline{z}]\\ &\quad= \left(\overline{z}, -\varepsilon^{2m}L_{2m}\left[\overline{w}+\overline{V}_1+\overline{V}_2\right] - \mu^{2k}L_{2k}\left[\overline{w}+\overline{V}_1+\overline{V}_2\right] - L_2\left[\overline{w}+\overline{V}_1+\overline{V}_2\right]\right)\\ &\qquad+ f\left(x, T\left(\overline{w}+\overline{V}_1+\overline{V}_2+\overline{z}\right)\right) - f\left(x, T\left(\overline{w}+\overline{V}_1+\overline{V}_2\right)\right).\end{aligned}$$

Thus, there exists a positive constant C independent of ε and μ, such that

$$\varepsilon^{2m}\|\overline{z}\|_m^2 + \mu^{2k}\|\overline{z}\|_k^2+\|\overline{z}\|_1^2 \leqslant C_1\left\{\varepsilon^{2m}B_m[\overline{z},\overline{z}]+\mu^{2k}B_k[\overline{z},\overline{z}]+B_1[\overline{z},\overline{z}]\right\}.$$

And

$$\varepsilon^{2m}\|\overline{z}\|_m^2 + \mu^{2k}\|\overline{z}\|_k^2+\|\overline{z}\|_1^2 = O\left(\lambda^{M+1}\right), 0 < \lambda = \max(\varepsilon,\mu,\zeta) \ll 1.$$

Hence, there is the following theorem:

Theorem 2 *Under assumptions* $[\mathrm{H}_1]$–$[\mathrm{H}_4]$, *for sufficiently small* ε *and* μ, *the generalized solution of the boundary value problems* (3)–(4) *satisfies the relations*

$$\begin{aligned}&\left\|u-\left(\overline{w}+\overline{V}_1+\overline{V}_2\right)\right\|_k = O\left(\lambda^{M-2k+1}\right),\\ &\left\|u-\left(\overline{w}+\overline{V}_1+\overline{V}_2\right)\right\|_m = O\left(\lambda^{M-2m+1}\right), 0 < \lambda \ll 1.\end{aligned}$$

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