

The integral representations of the products for the harmonic polynomials and the Poisson kernel and their growth properties in the half space

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Abstract In this paper, we study the integrals and growth properties of the products multiplying types of polynomials and the Poisson kernel in the half space. By gradually weakening the convergence conditions and redefining the measure, two special cases, namely the growth properties of the integrals multiplying two types of harmonic polynomials and the Poisson kernel, are obtained to solve the Dirichlet boundary value problem of the polyharmonic equation. Moreover, we generalize the results concerning the modified Poisson integral in the half space.

Keywords Harmonic polynomial, growth property, modified Poisson kernel

MR2010 31B15, 31B25

1 Introduction and main results

In recent years, more and more attention has been paid on the Poisson integral and its growth properties on the half space $H = \{x = (x', x_n) \in R^{n-1} \times R: x_n > 0\}$ of R^n [2, 5, 7, 8]. Scholars at home and abroad use Gegenbauer polynomials [1] to modify the Poisson kernel and study the modified Poisson integral and its

growth properties [1, 3, 4].

For the Dirichlet boundary value problem of polyharmonic equation

$$\Delta^m U(x) = 0, \quad U \in W^{m,p}(H), \quad (1)$$

with boundary conditions

$$\left(\frac{\partial}{\partial x_n} \right)^k U(x', 0) = f_k(x'), \quad 0 \leq k \leq m-1,$$

when $1 < p \leq q < \infty$, $\frac{n}{p} - \frac{n-1}{q} < 1$, and

$$\int_{R^n} (1 + |y|)^{|\lambda|-l} |f(y)| dy < \infty.$$

It is proved in reference [6] that the K -potential

$$U_k f(x) = \int_{R^n} \frac{(x-y)^\lambda}{|x-y|^n} f(y) dy$$

is an integral representation of the polyharmonic Dirichlet problem (1), where λ is a multi-index of length m , $l > 0$.

Reference [9] studied the simplest case of the K -potential, where the modified Poisson kernel is

$$P_1(x) = \frac{2x_n}{n\omega_n} \frac{x_1}{|x|^n}, \quad x \in H,$$

and under certain convergence conditions, the generalized Poisson integral is as follows:

$$U[P_1] f(x) = \frac{2x_n}{n\omega_n} \int_{R^{n-1}} \frac{(x_1 - y_1) f(y')}{|x - y'|^n} dy'.$$

$U[P_1] f(x)$ satisfies the growth property $o(x_1 |x|)$, $|x| \rightarrow \infty$ when removing an exceptional set. Reference [10] modified the Poisson kernel and, under corresponding convergence conditions, the integral of the product multiplying the harmonic polynomial $(x_2 - y_2)^2 - (x_1 - y_1)^2$ and the Poisson kernel is as follows:

$$U[P_2] f(x) = \frac{2x_n}{n\omega_n} \int_{R^{n-1}} \frac{(x_2 - y_2)^2 - (x_1 - y_1)^2}{|x - (y', 0)|^n} f(y') dy'.$$

$U[P_2] f(x)$ satisfies the growth property

$$\lim_{|x| \rightarrow \infty, x \in H \setminus G} |x|^3 U[P_2] f(x) = 0,$$

when removing an exceptional set G .

To complete the growth properties of the K -potential, this paper studies special multi-indices. First, modify the Poisson kernel as

$$P_3(x, y') = \frac{2x_n}{n\omega_n} \frac{x_1x_2 - y_1y_2}{|x - (y', 0)|^n}, \quad x_1, y_1, x_2, y_2 \in \mathbb{R}.$$

When the continuous function f on \mathbb{R}^{n-1} satisfies

$$\int_{\mathbb{R}^{n-1}} \frac{(1 + y_1y_2) f(y')}{1 + |y'|^n} dy' < \infty, \quad (2)$$

the integral of the product multiplying the harmonic polynomial $x_1x_2 - y_1y_2$ and the Poisson kernel is

$$U[P_3]f(x) = \frac{2x_n}{n\omega_n} \int_{\mathbb{R}^{n-1}} \frac{(x_1x_2 - y_1y_2) f(y')}{|x - (y', 0)|^n} dy'. \quad (3)$$

Note that (2) is equivalent to the condition $U[P_3]f(x) \not\equiv \infty$. If $x_1x_2 - y_1y_2 = 1$, then $P_3(x, y')$ is the classical Poisson kernel in the upper half space.

Theorem 1 *Let f be a measurable function on \mathbb{R}^{n-1} satisfying (2) and $U[P_3]f(x)$, defined by (3), be the integral of the product multiplying the harmonic polynomial $x_1x_2 - y_1y_2$ and the Poisson kernel. The function $U[P_3]f(x)$ satisfies*

$$U[P_3]f(x) = o(x_1x_2|x|), \quad |x| \rightarrow \infty, \quad x \in H \setminus G, \quad (4)$$

where $G = \bigcup_{j=1}^{\infty} B(x_j, \rho_j)$, $B(x_j, \rho_j)$ denotes the ball with center x_j and radius ρ_j .

Note 1 When $x_1 = x_2 = 1$, Theorem 1 reduces to the classical growth result for the Poisson integral in [8].

Considering the complexity of the multi-index λ , the Poisson kernel is further modified as

$$P_4(x, y') = \frac{2x_n}{n\omega_n} \frac{x_1^{\lambda_1} x_2^{\lambda_2} - y_1^{\lambda_1} y_2^{\lambda_2}}{|x - y'|^n}, \quad x_1, y_1, x_2, y_2 \in \mathbb{R},$$

where λ_1, λ_2 are positive real numbers. When the measurable function f on \mathbb{R}^{n-1} satisfies

$$\int_{\mathbb{R}^{n-1}} \frac{(1 + y_1^{\lambda_1} y_2^{\lambda_2}) f(y')}{|1 + (y', 0)|^n} dy' < \infty, \quad (5)$$

the integral of the product multiplying the polynomial $(x_1^{\lambda_1} x_2^{\lambda_2} - y_1^{\lambda_1} y_2^{\lambda_2})$ and the Poisson kernel is

$$U[P_4]f(x) = \frac{2x_n}{n\omega_n} \int_{\mathbb{R}^{n-1}} \frac{(x_1^{\lambda_1} x_2^{\lambda_2} - y_1^{\lambda_1} y_2^{\lambda_2}) f(y')}{|x - y'|^n} dy'. \quad (6)$$

Note that condition (5) is equivalent to $U[P_4]f(x) \not\equiv \infty$.

When $\lambda_1 = \lambda_2 = 1$, P_4 is the modified Poisson kernel P_3 in the upper half space H ; when $x_1^{\lambda_1} x_2^{\lambda_2} - y_1^{\lambda_1} y_2^{\lambda_2} = 1$, P_4 is the classical Poisson kernel in the upper half space. Theorem 2 is obtained by generalizing Theorem 1.

Theorem 2 *Let f be a measurable function on \mathbb{R}^{n-1} satisfying (5) and $U[P_4]f(x)$, defined by (6), be the integral of the product multiplying the polynomial $x_1^{\lambda_1}x_2^{\lambda_2} - y_1^{\lambda_1}y_2^{\lambda_2}$ and the Poisson kernel $U[P_4]f(x)$ satisfies*

$$U[P_4]f(x) = o(x_1^{\lambda_1}x_2^{\lambda_2}|x|), \quad |x| \rightarrow \infty, \quad x \in H \setminus G, \quad (7)$$

where G is defined as in Theorem 1.

Note 2 Theorems 1 and 2 establish growth properties while removing exceptional sets. The sizes of these exceptional sets can be controlled, and the results cannot be further improved. As a result, the growth properties of the K -potential [6] and other corresponding results are partly generalized.

2 Proof of theorems

Let μ be a positive Borel measure on \mathbb{R}^{n-1} , $\alpha \geq 0$. The measurable maximal function $M(d\mu)(x)$ on order α is defined in [9, 10].

Lemma 1 *If f is a measurable function on \mathbb{R}^{n-1} satisfying (1), and*

$$U_1[P_3]f(x) \triangleq \frac{2x_n}{n\omega_n} \int_{\mathbb{R}^{n-1}} \frac{x_1x_2f(y')}{|x - (y', 0)|^n} dy',$$

then

$$U_1[P_3]f(x) = o(x_1x_2|x|), \quad |x| \rightarrow \infty, \quad x \in H \setminus G. \quad (8)$$

Note 3 The result can be directly obtained with the method in [8].

Lemma 2 *If f is a measurable function on \mathbb{R}^{n-1} satisfying (1), and*

$$U_2[P_3]f(x) \triangleq \frac{2x_n}{n\omega_n} \int_{\mathbb{R}^{n-1}} \frac{y_1y_2f(y')}{|x - (y', 0)|^n} dy',$$

then

$$U_2[P_3]f(x) = o(|x|), \quad |x| \rightarrow \infty, \quad x \in H \setminus G. \quad (9)$$

Proof Let

$$K(x, y') = \frac{2x_n}{n\omega_n} \frac{1 + |(y', 0)|^n}{|x - (y', 0)|^n};$$

$$dm(y') = \frac{y_1y_2|f(y')|dy'}{1 + |(y', 0)|^n}.$$

Then the function $U_2[P_3]f(x)$ can be rewritten as

$$U_2 [P_3] f(x) = \int_{R^{n-1}} K(x, y') dm(y').$$

Let E be an arbitrary Lebesgue measurable set in R^n . Define $m^{(\varepsilon)}(E)$ and $E(\varepsilon)$ as in [10], namely

$$\begin{aligned} m^{(\varepsilon)}(E) &= m(E \cap \{x' \in R^{n-1} : |x'| > R_\varepsilon\}), \\ E(\varepsilon) &= \left\{ x' \in R^n : |x'| > R_\varepsilon, \text{ there exists } t > 0, \right. \\ &\quad \left. \text{such that } m^{(\varepsilon)}(B(x, t) \cap R^{n-1}) > \varepsilon \frac{t^{n-\alpha}}{|x|^{n-\alpha}} \right\}, \end{aligned}$$

where R_ε is a sufficiently large positive number related to ε . Let

$$U_2^{(\varepsilon)} [P_3] f(x) = \int_{R^{n-1}} K(x, y') dm^{(\varepsilon)}(y').$$

Define

$$\begin{aligned} U_{21} [P_3] f(x) &\triangleq \int_{\{y' \in R^{n-1} : \delta < |x-y'| \leq \frac{|x|}{2}\}} K(x, y') dm^{(\varepsilon)}(y'), \\ U_{22} [P_3] f(x) &\triangleq \int_{\{y' \in R^{n-1} : \frac{|x|}{2} < |x-y'| \leq 2|x|\}} K(x, y') dm^{(\varepsilon)}(y'), \\ U_{23} [P_3] f(x) &\triangleq \int_{\{y' \in R^{n-1} : |x-y'| \geq 2|x|\}} K(x, y') dm^{(\varepsilon)}(y'). \end{aligned}$$

Then

$$U_2^{(\varepsilon)} [P_3] f(x) = U_{21} [P_3] f(x) + U_{22} [P_3] f(x) + U_{23} [P_3] f(x).$$

When $|x| \geq 2R_\varepsilon$ and $x \notin E(\varepsilon)$, where R_ε is sufficiently large, there is

$$|U_{21} [P_3] f(x)| \leq \frac{2^{n+1}x_n(1+|x|^n)}{n\omega_n} \int_{\{y' \in R^{n-1} : \delta < |x-y'| \leq \frac{|x|}{2}\}} \frac{dm^{(\varepsilon)}(y')}{|x-y'|^n} \leq M\varepsilon|x|.$$

Similarly,

$$|U_{22} [P_3] f(x)| \leq \frac{2^{n+1}x_n(1+|x|^n)}{n\omega_n} \int_{\{y' \in R^{n-1} : \frac{|x|}{2} < |x-y'| \leq 2|x|\}} \frac{dm^{(\varepsilon)}(y')}{|x-y'|^n} \leq M\varepsilon|x|.$$

Likewise,

$$|U_{23} [P_3] f(x)| \leq \int_{\{y' \in R^{n-1} : |x-y'| \geq 2|x|\}} \frac{2x_n(1+|y'|^n)}{n\omega_n|x-y'|^n} dm^{(\varepsilon)}(y') \geq M\varepsilon|x|.$$

Here M is a positive constant related to ε , which may represent different constants in different places. According to the proof method of Theorem 1 in [8], it is known that (9) holds when removing the exceptional set G . The proof of Lemma 2 is completed.

From Lemma 1 and Lemma 2, Theorem 1 holds.

The proof of Theorem 2 requires Lemma 3 and Lemma 4.

Lemma 3 *Let f be a measurable function on \mathbb{R}^{n-1} satisfying (5). Then*

$$U_1 [P_4] f(x) \triangleq \frac{2x_n}{n\omega_n} \int_{\mathbb{R}^{n-1}} \frac{x_1^{\lambda_1} x_2^{\lambda_2} f(y') dy'}{|x - (y', 0)|^n}$$

satisfies

$$U_1 [P_4] f(x) = o(x_1^{\lambda_1} x_2^{\lambda_2} |x|), \quad |x| \rightarrow \infty, \quad x \in H \setminus G. \quad (10)$$

Proof Define

$$L(x, y') = \frac{2x_n}{n\omega_n} \frac{1 + |y'|^n}{|x - y'|^n};$$

$$dm_1(y') = \frac{y_1 y_2 |f(y')| dy'}{1 + |y'|^n}.$$

Let E and $E(\varepsilon)$ be respectively defined as in Lemma 2, and

$$m_1^{(\varepsilon)}(E) \triangleq m_1(E \cap \{x' \in \mathbb{R}^{n-1} : |x'| \geq R_\varepsilon\}).$$

Then the function $U_1 [P_4] f(x)$ can be rewritten as

$$U_1 [P_4] f(x) = \int_{\mathbb{R}^{n-1}} L(x, y') dm_1(y').$$

Let

$$U_1^{(\varepsilon)} [P_4] f(x) = \int_{\mathbb{R}^{n-1}} L(x, y') dm_1^{(\varepsilon)}(y').$$

Define

$$U_{11} [P_4] f(x) \triangleq \int_{\{y' \in \mathbb{R}^{n-1} : \delta < |x - y'| \leq \frac{|x|}{2}\}} L(x, y') dm_1^{(\varepsilon)}(y'),$$

$$U_{12} [P_4] f(x) \triangleq \int_{\{y' \in \mathbb{R}^{n-1} : \frac{|x|}{2} < |x - y'| \leq 2|x|\}} L(x, y') dm_1^{(\varepsilon)}(y'),$$

$$U_{13} [P_4] f(x) \triangleq \int_{\{y' \in \mathbb{R}^{n-1} : |x - y'| \geq 2|x|\}} L(x, y') dm_1^{(\varepsilon)}(y').$$

Then

$$U_1^{(\varepsilon)} [P_4] f(x) = U_{11} [P_4] f(x) + U_{12} [P_4] f(x) + U_{13} [P_4] f(x).$$

(10) is proved by the proof method similar to Lemma 2 in this paper.

Lemma 4 *Under the conditions of Lemma 3, the integral of the product multiplying the polynomial $y_1^{\lambda_1} y_2^{\lambda_2}$ and the Poisson kernel*

$$U_2 [P_4] f(x) \triangleq \frac{2x_n}{n\omega_n} \int_{\mathbb{R}^{n-1}} \frac{y_1^{\lambda_1} y_2^{\lambda_2} f(y') dy'}{|x - (y', 0)|^n}$$

satisfies the growth property

$$U_2 [P_4] f(x) = o(|x|), \quad |x| \rightarrow \infty, \quad x \in H \setminus G. \quad (11)$$

Note 4 Although the results of Lemma 2 and Lemma 4 are the same, their respective convergence conditions are different, so the removed exceptional sets are different.

According to Lemma 3 and Lemma 4, Theorem 2 holds.

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