Front. Math. China 2026, 21(1): 1–13
https://doi.org/10.3868/s140-DDD-026-0001-x

RESEARCH ARTICLE

Association schemes based on partial subspaces of type (2, 0, 1) in singular symplectic space

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Abstract Let $F_q^{(2\nu+l)}$ be the $(2\nu+l)$-dimensional singular symplectic space over the finite field F_q, K be a fixed maximal totally isotropic subspace in $F_q^{(2\nu+l)}$, and Ω be the set of all subspaces of type $(1,0,0)$ not contained in K. In this paper, we construct a class of association schemes by using all subspaces of type $(2,0,1)$ that contain a subspace from Ω, and compute all intersection numbers of the constructed schemes.

Keywords Association schemes, singular symplectic space, finite field
MR2010 05E30

0 Introduction

It is assumed that $\mathbb{F}_q$ is a finite field with q elements, where q is a prime power. Let $\mathbb{F}_q^{(2\nu+l)}$ be the $(2\nu+l)$-dimensional row vector space over the finite field $\mathbb{F}_q$ and

$$K_l = \begin{pmatrix} 0 & I^{(\nu)} & \\ -I^{(\nu)} & 0 & \\ & & 0^{(l)} \end{pmatrix}.$$

If all $(2\nu+l)\times(2\nu+l)$ matrices T over the finite field $\mathbb{F}_q$ satisfying $TK_lT^{\mathrm{T}} = K_l$ form a set, with matrix multiplication, there will be the singular symplectic group of degree $2\nu+l$ over the finite field $\mathbb{F}_q$, denoted by $\mathrm{Sp}_{2\nu+l}(\mathbb{F}_q)$.

Translated from *Advances in Mathematics* (*China*), 2016, 45(1): 48–56
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The vector space $\mathbb{F}_q^{(2\nu+l)}$ under the right multiplication action of $\mathrm{Sp}_{2\nu+l}(\mathbb{F}_q)$ is called the $(2\nu+l)$-dimensional singular symplectic space over the finite field $\mathbb{F}_q$.

Let P be an m-dimensional subspace in $\mathbb{F}_q^{(2\nu+l)}$ and E the subspace generated by $e_i(1 \leqslant i \leqslant 2\nu+l)$ in $\mathbb{F}_q^{(2\nu+l)}$, where $e_i(1 \leqslant i \leqslant 2\nu+l)$ is the row vector in $\mathbb{F}_q^{(2\nu+l)}$ whose ith component is 1 and the other components are 0. An m-dimensional subspace P in the $(2\nu+l)$-dimensional singular symplectic space refers to a subspace of type (m,s,k) if the rank of PK_lP^{T} is $2s$ and $\dim(P\cap E)=k$. In particular, a subspace of type $(\nu,0,0)$ is called a maximal totally isotropic subspace. A subspace of type (m,s,k) exists if and only if $0 \leqslant k \leqslant l$ and $2s \leqslant m-k \leqslant \nu+s$.

Association schemes can be viewed as edge colorings of complete graphs satisfying good regularity conditions, which have been widely applied in coding theory, design theory, graph theory and group theory and further studied in many chapters of books or books [1–6].

Definition 0.1 [1] Let X be a set of cardinality n and $R_i(i=0,1,\ldots,d)$ be subsets of $X\times X$ with the following properties:
(1) $R_0=\{(x,x) \mid x\in X\}$;
(2) $X\times X=R_0\cup R_1\cup\cdots\cup R_d$, and $R_i\cap R_j=\varnothing$, if $i\neq j$;
(3) For $i'\in\{0,1,\ldots,d\}$, there exists $R_i^{\mathrm{T}}=R_{i'}$, where $R_i^{\mathrm{T}}=\{(x,y) \mid (y,x)\in R_i\}$;
(4) For $i,j,k\in\{0,1,\ldots,d\}$ and any $(x,y)\in R_k$, the number $p_{ij}^k=|\{Z\in X \mid (x,z)\in R_i,(z,y)\in R_j\}|$ is a constant independent of the choice of (x,y) in R_k.

Such a configuration $\chi=(X,\{R_i\}_{0\leqslant i\leqslant d})$ is called an association scheme on X with d classes.

Association schemes play an important role in algebraic combinatorics. Wan et al. [10, 11] computed all parameters of the bipartite scheme. As a generalized bipartite scheme, Rieck [9] constructed association schemes with the subspaces of a given dimension in finite classical polar spaces. Wei and Wang (see [15, 16]) gave suborbits under the action of finite classical groups on the set of m-dimensional totally isotropic subspaces. Guo et al. [7, 8] constructed association schemes with the maximal totally isotropic subspaces in singular classical spaces. As generalized Grassmann schemes and bilinear forms schemes, Wang et al. [12, 13] constructed association schemes on attenuated spaces and singular linear spaces. Wang et al. [14] constructed a class of association schemes with minimal flats in classical polar spaces. Gao et al. [5] constructed association schemes with subspaces of type $(m,s,0)$ in singular symplectic space.

In this paper, let K_0 be a fixed maximal totally isotropic subspace in the symplectic space $\mathbb{F}_q^{(2\nu)}$, $K=(K_0K_1)$ be a fixed maximal totally isotropic subspace in the singular symplectic space $\mathbb{F}_q^{(2\nu+l)}$, and Ω be the set of all subspaces of type $(1,0,0)$ in $\mathbb{F}_q^{(2\nu+l)}$ not contained in K. We construct a class of association schemes with subspaces of type $(2,0,1)$ in $\mathbb{F}_q^{(2\nu+l)}$ that contain a subspace from Ω. The following shows the results.

Theorem 0.1 *It is assumed that the characteristic of* $\mathbb{F}_q$ *is* 2 *and* $l \geqslant 2$. *Let* X *be the set of all subspaces of type* (2,0,1) *with matrix representation as follows*:

$$\begin{matrix} 2\nu & l \end{matrix}$$
$$\begin{pmatrix} u_1 & u_2 \\ 0 & u \end{pmatrix},$$

where $(u_1, u_2) \in \Omega$. *Assume* $t = 0$ *or* $t = 1$. *For any two elements in* X,

$$P = \begin{pmatrix} x_1 & x_2 \\ 0 & x \end{pmatrix}, \quad Q = \begin{pmatrix} y_1 & y_2 \\ 0 & y \end{pmatrix},$$

and the relations on X *are defined as follows*:

(1) $(P,Q) \in R_{(0,t)}$, *if* $x_1 = y_1$, $\dim(x \cap y) = 1 - t$, *and* $P + Q$ *is a subspace of type* $(2+t, 0, 1+t)$;

(2) $(P,Q) \in R_{(1,t)}$, *if* $x_1 = y_1, \dim(x \cap y) = 1 - t$, *and* $P + Q$ *is a subspace of type* $(3+t, 0, 2+t)$;

(3) $(P,Q) \in R_{(2,t)}$, *if* $x_1 + y_1$ *is a subspace of type* (2,1), $\dim((x_1 + y_1) \cap K_0) = 1, \dim(x \cap y) = 1 - t$, *and* $P + Q$ *is a subspace of type* $(3+t, 1, 1+t)$;

(4) $(P,Q) \in R_{(3,t)}$, *if* $x_1 + y_1$ *is a subspace of type* (2,1), $\dim((x_1 + y_1) \cap K_0) = 0, \dim(x \cap y) = 1 - t$, and $P + Q$ *is a subspace of type* $(3+t, 1, 1+t)$;

(5) $(P,Q) \in R_{(4,t)}$, *if* $x_1 + y_1$ *is a subspace of type* (2,0), $\dim((x_1 + y_1) \cap K_0) = 1, \dim(x \cap y) = 1 - t$, *and* $P + Q$ *is a subspace of type* $(3+t, 0, 1+t)$;

(6) $(P,Q) \in R_{(5,t)}$, *if* $x_1 + y_1$ *is a subspace of type* (2,0), $\dim((x_1 + y_1) \cap K_0) = 0, \dim(x \cap y) = 1 - t$, *and* $P + Q$ *is a subspace of type* $(3+t, 0, 1+t)$.

Then a symmetric association scheme can be obtained. The parameters d, v, and $n_{(r,t)} (r = 0, 1, 2, 3, 4, 5, t = 0, 1)$ are determined by Lemma 1.1, and the intersection numbers $p^{(r,t)}_{(i,j)(\lambda,\mu)}$ are gained from Eqs. (6)–(29) in Section 1.

1 Proof of Theorem 0.1

In this section, we prove Theorem 0.1 and compute all parameters of the obtained association schemes.

Let q be a prime power and m_1, m_2 be two integers. For simplicity, we use the Gaussian coefficient:

$$\begin{bmatrix} m_2 \\ m_1 \end{bmatrix}_q = \frac{\prod_{i=m_2-m_1+1}^{m_2} (q^i - 1)}{\prod_{i=1}^{m_1} (q^i - 1)}.$$

It is defined that when $m_1 = m_2$, $\begin{bmatrix} m_2 \\ m_1 \end{bmatrix}_q = 1$; when $m_1 < 0$ or $m_2 < m_1$, $\begin{bmatrix} m_2 \\ m_1 \end{bmatrix}_q = 0$.

Proposition 1.1 [13] *For $1 \leqslant m \leqslant n$ and $0 \leqslant i \leqslant \min\{m, n-m\}$, let P' and Q' be two fixed m-dimensional subspaces in $\mathbb{F}_q^{(n)}$ such that* $\dim(P' \cap Q') = m - i$. *Then the number of m-dimensional subspaces S' in $\mathbb{F}_q^{(n)}$ satisfying* $\dim(P' \cap S') = m - s$ *and* $\dim(S' \cap Q') = m - u$ *is*

$$p_{su}^{i}(m,n) = \sum_{\rho+\alpha=u,\beta+\gamma=m-u,\rho+\gamma\leqslant s} q^{\omega} \prod_{k=i+\rho-s+1}^{i-\gamma} \left(q^k - 1\right)$$
$$\times \begin{bmatrix} \alpha \\ s-\rho-\gamma \end{bmatrix}_q \begin{bmatrix} i \\ \alpha \end{bmatrix}_q \begin{bmatrix} i \\ \gamma \end{bmatrix}_q \begin{bmatrix} m-i \\ \beta \end{bmatrix}_q \begin{bmatrix} n-m-i \\ \rho \end{bmatrix}_q,$$

where $\omega = \frac{1}{2}(s-\gamma-\rho)(s-\gamma-\rho-1) + (m-\beta)(m-\beta-i) + \rho(2i-\alpha-\gamma)$.

Let $\mathbb{F}_q^{(2\nu)}$ be the 2ν-dimensional vector space over the finite field $\mathbb{F}_q$, W be a fixed maximal totally isotropic subspace in $\mathbb{F}_q^{(2\nu)}$, and Θ be the set of all 1-dimensional subspaces not contained in W.

Proposition 1.2 [14] *It is assumed that the characteristic of $\mathbb{F}_q$ is 2. A partition of $\Theta \times \Theta$ is defined as follows:*

(1) $R_0 = \{(P,P) \mid P \in \Theta\}$,

(2) $R_1 = \{(P,Q) \mid P,Q \in \Theta, P+Q$ *is non-isotropic and* $\dim((P+Q)\cap W) = 1\}$,

(3) $R_2 = \{(P,Q) \mid P,Q \in \Theta, P+Q$ *is non-isotropic and* $\dim((P+Q)\cap W) = 0\}$,

(4) $R_3 = \{(P,Q) \mid P,Q \in \Theta, P+Q$ *is totally isotropic and* $\dim((P+Q)\cap W) = 1\}$,

(5) $R_4 = \{(P,Q) \mid P,Q \in \Theta, P+Q$ *is totally isotropic and* $\dim((P+Q)\cap W) = 0\}$.

Then $\chi = \left(\Theta, \{R_i\}_{i=0}^{4}\right)$ is a symmetric association scheme with parameters:

$$\begin{cases} k_1 = (q-1)q^{\nu-1}, \\ k_2 = q^{\nu}(q^{\nu-1}-1), \\ k_3 = q^{\nu-1}-1, \end{cases} \quad \begin{cases} p_{11}^1 = (q-2)\,q^{\nu-1}, \\ p_{13}^1 = q^{\nu-1}-1, \\ p_{24}^1 = (q^{\nu-1}-1)\,q^{\nu-1}, \end{cases} \quad \begin{cases} p_{23}^2 = (q-1)\,q^{\nu-2}-1, \\ p_{24}^2 = (q^{\nu}-2q+1)\,q^{\nu-2}, \\ p_{33}^3 = q^{\nu-1}-2, \\ p_{12}^1 = p_{23}^1 = p_{33}^1 = p_{33}^2 = 0. \end{cases}$$

According to Proposition 1.2, the construction in Theorem 0.1 yields a symmetric association scheme.

In this section, assume

$$K = \begin{pmatrix} \nu & \nu & l \\ I^{(\nu)} & 0 & 0 \end{pmatrix}, \ Q = \begin{pmatrix} e_{\nu+1} \\ e_{2\nu+1} \end{pmatrix}.$$

Lemma 1.1 *The parameters of the association schemes determined by Theorem* 0.1 *are*

$$d = 11,$$
$$v = \frac{q^{\nu+l-1}\left(q^{\nu}-1\right)\left(q^{l}-1\right)}{\left(q-1\right)^2},$$

$$n_{(0,t)} = q^{t}\begin{bmatrix} l-1 \\ t \end{bmatrix}_q,$$

$$n_{(1,t)} = q^{2t}\left(q^{l-t-1}-1\right)\begin{bmatrix} l-1 \\ t \end{bmatrix}_q,$$

$$n_{(2,t)} = (q-1)\,q^{\nu+l+t-2}\begin{bmatrix} l-1 \\ t \end{bmatrix}_q,$$

$$n_{(3,t)} = q^{\nu+l+t-1}\left(q^{\nu-1}-1\right)\begin{bmatrix} l-1 \\ t \end{bmatrix}_q,$$

$$n_{(4,t)} = q^{l+t-1}\left(q^{\nu-1}-1\right)\begin{bmatrix} l-1 \\ t \end{bmatrix}_q,$$

$$n_{(5,t)} = q^{\nu+l+t-1}\begin{bmatrix} \nu-1 \\ 1 \end{bmatrix}_q\begin{bmatrix} l-1 \\ t \end{bmatrix}_q.$$

Proof According to the definition of $R_{(i,j)}$, the number of the class in the association scheme is

$$d = 11. \tag{1}$$

Let $V = \begin{pmatrix} u_1 & u_2 \\ 0 & u \end{pmatrix}$ be an arbitrary element in X. The number of ways to choose the subspace u is $\frac{q^l-1}{q-1}$. Since the action of $\mathrm{Sp}_{2\nu+l}(\mathbb{F}_q)$ on the set of subspaces in the same type is transitive, the number of V does not depend on the specific choice of u. Without losing generality and assuinge $u = \begin{pmatrix} 1 & 0 \end{pmatrix}$, V has a matrix representation

$$V = \begin{pmatrix} u_1 & 0 & u_{22} \\ 0 & 1 & 0 \end{pmatrix},$$

where u_{22} is an arbitrary $1 \times (l-1)$ matrix. According to [12, Lemma 2.2], the number of u_1 is $q^{\nu}\frac{q^{\nu}-1}{q-1}$, and the number of V is $\frac{q^{\nu+l-1}(q^{\nu}-1)\left(q^{l}-1\right)}{(q-1)^2}$. Therefore,

$$v = \frac{q^{\nu+l-1}\left(q^{\nu}-1\right)\left(q^{l}-1\right)}{(q-1)^2}. \tag{2}$$

Next, we compute $n_{(i,t)}(0 \leqslant i \leqslant 5,\ 0 \leqslant t \leqslant 1)$. We only compute $n_{(2,t)}$ in detailand the others can be calculated similarly.

$n_{(2,0)}$ is the number of subspaces P such that $(Q, P) \in R_{(2,0)}$. Assume

$$P = \begin{pmatrix} x_1 & 0 & x_2 \\ 0 & 1 & 0 \end{pmatrix}.$$

According to Proposition 1.2, the number of x_1 is $(q-1)q^{\nu-1}$. Then the number of P is $q^{l-1}(q-1)q^{\nu-1} = (q-1)q^{\nu+l-2}$. So

$$n_{(2,0)}=(q-1)q^{\nu+l-2}. \tag{3}$$

$n_{(2,1)}$ is the number of subspaces U such that $(Q,U)\in R_{(2,1)}$. Assume

$$U=\begin{pmatrix} y_1 & y_2\\ 0 & y\end{pmatrix}.$$

According to Proposition 1.2, the number of y_1 is $(q-1)q^{\nu-1}$. The number of ways to choose the subspace y is $\begin{bmatrix} l\\ 1\end{bmatrix}_q-1$. Then the number of P is $(q-1)q^{\nu-1}\left(\frac{q^l-1}{q-1}-1\right)q^{l-1}=\left(q^{l-1}-1\right)q^{\nu+l-1}$. So

$$n_{(2,1)}=\left(q^{l-1}-1\right)q^{\nu+l-1}. \tag{4}$$

According to Eqs. (3) and (4), there is

$$n_{(2,t)}=(q-1)\,q^{\nu+l+t-2}\begin{bmatrix} l-1\\ t\end{bmatrix}_q. \tag{5}$$

Next, we compute the intersection numbers. According to Proposition 1.1, there is

$$p^{(0,t)}_{(0,j)(0,\nu)}=p^{t}_{j\nu}\left(1,l\right). \tag{6}$$

Assume

$$W=\begin{pmatrix} e_1+e_{\nu+1}\\ e_{2\nu+1}\end{pmatrix},$$

then $(W,Q)\in R_{(2,0)}$.

$p^{(2,0)}_{(3,0)(5,0)}$ is the number of subspaces A such that $(W,A)\in R_{(3,0)}$ and $(Q,A)\in R_{(5,0)}$. Assume

$$A=\begin{pmatrix} 2\nu & 1 & l-1\\ a_1 & 0 & a_2\\ 0 & 1 & 0\end{pmatrix}.$$

According to Proposition 1.2, the number of a_1 is $\left(q^{\nu-1}-1\right)q^{\nu-1}$. Then the number of A is $q^{l-1}\left(q^{\nu-1}-1\right)q^{\nu-1}=\left(q^{\nu-1}-1\right)q^{\nu+l-2}$. So

$$p^{(2,0)}_{(3,0)(5,0)}=\left(q^{\nu-1}-1\right)q^{\nu+l-2}.$$

$p^{(2,0)}_{(3,1)(5,1)}$ is the number of subspace B such that $(W,B)\in R_{(3,1)}$ and $(Q,B)\in R_{(5,1)}$. Assume

$$B=\begin{pmatrix} 2\nu & l\\ b_1 & b_2\\ 0 & b\end{pmatrix}.$$

According to Proposition 1.2, the number of b_1 is $(q^{\nu-1}-1)\,q^{\nu-1}$. The number of ways to choose the subspace b is $\begin{bmatrix} l \\ 1 \end{bmatrix}_q - 1$. Therefore, the number of B is

$$q^{l-1}\left(q^{\nu-1}-1\right)q^{\nu-1}\left(\begin{bmatrix} l \\ 1 \end{bmatrix}_q - 1\right) = \frac{\left(q^{\nu-1}-1\right)\left(q^{l-1}-1\right)q^{\nu+l-1}}{q-1}.$$

Hence

$$p^{(2,0)}_{(3,1)(5,1)} = \frac{\left(q^{\nu-1}-1\right)\left(q^{l-1}-1\right)q^{\nu+l-1}}{q-1}.$$

It is known that $p^{(2,0)}_{(3,0)(5,1)} = p^{(2,0)}_{(3,1)(5,0)} = 0$. So

$$p^{(2,0)}_{(3,j)(5,\mu)} = \begin{cases} \left(q^{\nu-1}-1\right)q^{\nu+l-2}, & \text{if } j=\mu=0, \\ 0, & \text{if } j\neq\mu, \\ \dfrac{\left(q^{\nu-1}-1\right)\left(q^{l-1}-1\right)q^{\nu+l-1}}{q-1}, & \text{if } j=\mu=1. \end{cases}$$

Assume

$$H = \begin{pmatrix} e_1 + e_{\nu+1} \\ e_{2\nu+2} \end{pmatrix},$$

then $(Q,H)\in R_{(2,1)}$.

$p^{(2,1)}_{(3,0)(5,1)}$ is the number of subspace C such that $(Q,C)\in R_{(3,0)}$ and $(H,C)\in R_{(5,1)}$. Assume

$$C = \begin{pmatrix} 2\nu & 1 & l-1 \\ c_1 & 0 & c_2 \\ 0 & 1 & 0 \end{pmatrix}.$$

According to Proposition 1.2, the number of c_1 is $(q^{\nu-1}-1)\,q^{\nu-1}$. Therefore, the number of C is $(q^{\nu-1}-1)\,q^{\nu-1}q^{l-1} = (q^{\nu-1}-1)\ q^{\nu+l-2}$. So

$$p^{(2,1)}_{(3,0)(5,1)} = \left(q^{\nu-1}-1\right)q^{\nu+l-2}.$$

Similar to the calculation of $p^{(2,1)}_{(3,0)(5,1)}$, we obtain

$$p^{(2,1)}_{(3,1)(5,0)} = \left(q^{\nu-1}-1\right)q^{\nu+l-2}.$$

$p^{(2,1)}_{(3,1)(5,1)}$ is the number of subspace D such that $(Q,D)\in R_{(3,1)}$ and $(H,D)\in R_{(5,1)}$. Assume

$$D = \begin{pmatrix} 2\nu & l \\ d_1 & d_2 \\ 0 & d \end{pmatrix}.$$

According to Proposition 1.2, the number of d_1 is $(q^{\nu-1}-1)\,q^{\nu-1}$. The number of ways to choose the subspace d is $\begin{bmatrix} l \\ 1 \end{bmatrix}_q - 2$. Therefore, the number of D is

$$q^{l-1}\left(q^{\nu-1}-1\right)q^{\nu-1}\left(\begin{bmatrix} l \\ 1 \end{bmatrix}_q - 2\right) = \frac{\left(q^{\nu-1}-1\right)\left(q^{l}-2q+1\right)q^{\nu+l-2}}{q-1}.$$

And

$$p^{(2,1)}_{(3,1)(5,1)} = \frac{\left(q^{\nu-1}-1\right)\left(q^{l}-2q+1\right)q^{\nu+l-2}}{q-1}.$$

It is seen that

$$p^{(2,1)}_{(3,0)(5,0)} = 0.$$

Therefore,

$$p^{(2,1)}_{(3,j)(5,\mu)} = \begin{cases} 0, & \text{if } j=\mu=0, \\ \left(q^{\nu-1}-1\right)q^{\nu+l-2}, & \text{if } j\neq\mu, \\ \dfrac{\left(q^{\nu-1}-1\right)\left(q^{l}-2q+1\right)q^{\nu+l-2}}{q-1}, & \text{if } j=\mu=1. \end{cases}$$

There is

$$p^{(2,1)}_{(3,0)(5,0)} = p^{(2,0)}_{(3,0)(5,1)} = p^{(2,1)}_{(3,1)(5,0)} = 0,$$

and

$$p^{(2,t)}_{(3,j)(5,\mu)} = \begin{cases} \dfrac{\left(q^{\nu-1}-1\right)\left(q^{l-1}-1\right)q^{\nu+l-1}}{q-1}, & \text{if } t=0, j=\mu=1, \\ \left(q^{\nu-1}-1\right)q^{\nu+l-2}, & \text{if } t=1, j\neq\mu, \text{or } t=j=\mu=0, \\ \dfrac{\left(q^{\nu-1}-1\right)\left(q^{l}-2q+1\right)q^{\nu+l-2}}{q-1}, & \text{if } t=j=\mu=1. \end{cases} \tag{7}$$

According to Proposition 1.2 and the calculations mentioned above, we obtain the following intersection numbers:

$$p^{(i,1)}_{(r,0)(\lambda,0)} = p^{(i,0)}_{(r,0)(\lambda,1)} = p^{(i,0)}_{(r,1)(\lambda,0)} = 0 \ \ (0\leqslant i,r,\lambda\leqslant 5), \tag{8}$$

$$p^{(0,t)}_{(0,j)(1,\mu)} = \begin{cases} q-1, & \text{if } t=j=1, \mu=0, \\ q^{l}-2q+1, & \text{if } t=j=\mu=1, \\ 0, & \text{if } t=\mu=1, j=0, \text{or } t=0, \end{cases} \tag{9}$$

$$p_{(i,j)(i,\mu)}^{(0,t)} = \begin{cases} n_{(i,k)}, & \text{if } t=0, j=\mu=k, 0\leqslant k\leqslant 1, 1\leqslant i\leqslant 5, \\ q^{l-1}-q, & \text{if } i=1, t=1, j\neq\mu, \\ \dfrac{\left(q^{l}-2q+1\right)\left(q^{l-1}-2q+1\right)}{q-1}, & \text{if } i=1, t=j=\mu=1, \\ (q-1)q^{\nu+l-2}, & \text{if } i=2, t=1, j\neq\mu, \\ \left(q^{l}-2q+1\right)q^{\nu+l-2}, & \text{if } i=2, t=j=\mu=1, \\ \left(q^{\nu-1}-1\right)q^{\nu+l-1}, & \text{if } i=3, t=1, j\neq\mu, \\ \dfrac{\left(q^{\nu-1}-1\right)\left(q^{l}-2q+1\right)q^{\nu+l-1}}{q-1}, & \text{if } i=3, t=j=\mu=1, \\ \left(q^{\nu-1}-1\right)q^{l-1}, & \text{if } i=4, t=1, j\neq\mu, \\ \dfrac{\left(q^{\nu-1}-1\right)\left(q^{l}-2q+1\right)q^{l-1}}{q-1}, & \text{if } i=4, t=j=\mu=1, \\ \dfrac{\left(q^{\nu-1}-1\right)q^{\nu-l-1}}{q-1}, & \text{if } i=5, t=1, j\neq\mu, \\ \dfrac{\left(q^{\nu-1}-1\right)\left(q^{l}-2q+1\right)q^{\nu+l-1}}{(q-1)^{2}}, & \text{if } i=5, t=j=\mu=1, \end{cases} \tag{10}$$

$$p_{(1,j)(1,\mu)}^{(1,t)} = \begin{cases} q^{l-1}-2, & \text{if } t=j=\mu=0, \\ \dfrac{q^{2}\left(q^{l-2}-1\right)\left(q^{l-1}-q-1\right)}{q-1}, & \text{if } t=0, j=\mu=1, \\ q^{l-1}-q-1, & \text{if } t=1, j\neq\mu, \\ q\left(q^{l-1}-2q+1\right)+\left(\dfrac{q^{l}-q^{2}}{q-1}-1\right)\left(q^{l-1}-2q\right), & \text{if } t=j=\mu=1, \end{cases} \tag{11}$$

$$p_{(2,j)(2,\mu)}^{(1,t)} = \begin{cases} \left(q^{l-1}-1\right)q^{\nu+l-1}, & \text{if } t=0, j=\mu=1, \\ (q-1)\,q^{\nu+l-2}, & \text{if } t=1, j\neq\mu, \text{or } t=j=\mu=0, \\ \left(q^{l}-2q+1\right)q^{\nu+l-2}, & \text{if } t=j=\mu=1, \end{cases} \tag{12}$$

$$p_{(3,j)(3,\mu)}^{(1,t)} = \begin{cases} \dfrac{\left(q^{l-1}-1\right)\left(q^{\nu-1}-1\right)q^{\nu+l}}{q-1}, & \text{if } t=0, j=\mu=1, \\ \left(q^{\nu-1}-1\right)q^{\nu+l-1}, & \text{if } t=1, j\neq\mu, \text{or } t=j=\mu=0, \\ \dfrac{\left(q^{l}-2q+1\right)\left(q^{\nu-1}-1\right)q^{\nu+l-1}}{q-1}, & \text{if } t=j=\mu=1, \end{cases} \tag{13}$$

$$p^{(1,t)}_{(4,j)(4,\mu)}=\begin{cases}\dfrac{\left(q^{l-1}-1\right)\left(q^{\nu-1}-1\right)q^{l}}{q-1}, & \text{if } t=0, j=\mu=1,\\ \left(q^{\nu-1}-1\right)q^{l-1}, & \text{if } t=1, j\neq\mu, \text{or } t=j=\mu=0,\\ \dfrac{\left(q^{l}-2q+1\right)\left(q^{\nu-1}-1\right)q^{l-1}}{q-1}, & \text{if } t=j=\mu=1,\end{cases} \tag{14}$$

$$p^{(1,t)}_{(5,j)(5,\mu)}=\begin{cases}\dfrac{\left(q^{l-1}-1\right)\left(q^{\nu-1}-1\right)q^{\nu+l}}{(q-1)^2}, & \text{if } t=0, j=\mu=1,\\ \dfrac{\left(q^{\nu-1}-1\right)q^{\nu+l-1}}{q-1}, & \text{if } t=1, j\neq\mu, \text{or } t=j=\mu=0,\\ \dfrac{\left(q^{l}-2q+1\right)\left(q^{\nu-1}-1\right)q^{\nu+l-1}}{(q-1)^2}, & \text{if } t=j=\mu=1,\end{cases} \tag{15}$$

$$p^{(2,t)}_{(2,j)(2,\mu)}=\begin{cases}\dfrac{(q-2)\left(q^{l-1}-1\right)q^{\nu+l-1}}{q-1}, & \text{if } t=0, j=\mu=1,\\ (q-2)\,q^{\nu+l-2}, & \text{if } t=1, j\neq\mu, \text{or } t=j=\mu=0,\\ \dfrac{\left(q^{l-2}-2q+1\right)(q-2)\,q^{\nu+l-2}}{q-1}, & \text{if } t=j=\mu=1,\end{cases} \tag{16}$$

$$p^{(2,t)}_{(2,j)(4,\mu)}=\begin{cases}\dfrac{\left(q^{\nu-1}-1\right)\left(q^{l-1}-1\right)q^{l}}{q-1}, & \text{if } t=0, j=\mu=1,\\ \left(q^{\nu-1}-1\right)q^{l-1}, & \text{if } t=1, j\neq\mu, \text{or } t=j=\mu=0,\\ \dfrac{\left(q^{l}-2q+1\right)\left(q^{\nu-1}-1\right)q^{l-1}}{q-1}, & \text{if } t=j=\mu=1,\end{cases} \tag{17}$$

$$p^{(2,t)}_{(3,j)(5,\mu)}=\begin{cases}\dfrac{\left(q^{\nu-1}-1\right)\left(q^{l-1}-1\right)q^{\nu+l-1}}{q-1}, & \text{if } t=0, j=\mu=1,\\ \left(q^{\nu-1}-1\right)q^{\nu+l-2}, & \text{if } t=1, j\neq\mu, \text{or } t=j=\mu=0,\\ \dfrac{\left(q^{l}-2q+1\right)\left(q^{\nu-1}-1\right)q^{\nu+l-2}}{q-1}, & \text{if } t=j=\mu=1,\end{cases} \tag{18}$$

$$\begin{aligned}&p^{(3,t)}_{(3,j)(4,\mu)}\\ &=\begin{cases}\dfrac{(q-1)\left(q^{l-1}-1\right)q^{\nu+l-2}}{q-1}-\dfrac{\left(q^{l-1}-1\right)q^{l}}{q-1}, & \text{if } t=0, j=\mu=1,\\ (q-1)q^{\nu+l-3}-q^{l-1}, & \text{if } t=1, j\neq\mu, \text{or } t=j=\mu=0,\\ \dfrac{(q-1)\left(q^{l}-2q+1\right)q^{\nu+l-3}}{q-1}-\dfrac{\left(q^{l}-2q+1\right)q^{l-1}}{q-1}, & \text{if } t=j=\mu=1,\end{cases}\end{aligned} \tag{19}$$

$$p^{(3,t)}_{(3,j)(5,\mu)} = \begin{cases} \dfrac{(q^{\nu}-2q+1)\left(q^{l-1}-1\right)q^{\nu+l-2}}{q-1}, & \text{if } t=0, j=\mu=1, \\ (q^{\nu}-2q+1)\,q^{\nu+l-3}, & \text{if } t=1, j\neq\mu, \text{or } t=j=\mu=0, \\ \dfrac{(q^{\nu}-2q+1)\left(q^{l}-2q+1\right)q^{\nu+l-3}}{q-1}, & \text{if } t=j=\mu=1, \end{cases} \tag{20}$$

$$p^{(4,t)}_{(4,j)(4,\mu)} = \begin{cases} \dfrac{(q^{\nu-1}-2)\left(q^{l-1}-1\right)q^{l}}{q-1}, & \text{if } t=0, j=\mu=1, \\ (q^{\nu-1}-2)\,q^{l-1}, & \text{if } t=1, j\neq\mu, \text{or } t=j=\mu=0, \\ \dfrac{(q^{\nu-1}-2)\left(q^{l}-2q+1\right)q^{l-1}}{q-1}, & \text{if } t=j=\mu=1, \end{cases} \tag{21}$$

$$p^{(2,t)}_{(5,j)(5,\mu)} = \begin{cases} \dfrac{(q^{\nu-1}-1)\left(q^{l-1}-1\right)q^{\nu+l-1}}{(q-1)^2}, & \text{if } t=0, j=\mu=1, \\ \dfrac{(q^{\nu-1}-1)\,q^{\nu+l-2}}{q-1}, & \text{if } t=1, j\neq\mu, \text{or } t=j=\mu=0, \\ \dfrac{\left(q^{l}-2q+1\right)(q^{\nu-1}-1)\,q^{\nu+l-2}}{(q-1)^2}, & \text{if } t=j=\mu=1, \end{cases} \tag{22}$$

$$p^{(3,t)}_{(3,j)(3,\mu)} = \begin{cases} \dfrac{(q-1)(q^{\nu}-2q+1)\left(q^{l-1}-1\right)q^{\nu+l-2}}{q-1}, & \text{if } t=0, j=\mu=1, \\ (q-1)(q^{\nu}-2q+1)\,q^{\nu+l-3}, & \text{if } t=1, j\neq\mu, \text{or } t=j=\mu=0, \\ \dfrac{\left(q^{l}-2q+1\right)(q-1)(q^{\nu}-2q+1)\,q^{\nu+l-3}}{q-1}, & \text{if } t=j=\mu=1, \end{cases} \tag{23}$$

$$p^{(3,t)}_{(4,j)(5,\mu)} = \begin{cases} \dfrac{\left(q^{l-1}-1\right)q^{\nu+l-2}}{q-1}, & \text{if } t=0, j=\mu=1, \\ q^{\nu+l-3}, & \text{if } t=1, j\neq\mu, \text{or } t=j=\mu=0, \\ \dfrac{\left(q^{l}-2q+1\right)q^{\nu+l-3}}{q-1}, & \text{if } t=j=\mu=1, \end{cases} \tag{24}$$

$$p^{(3,t)}_{(5,j)(5,\mu)} = \begin{cases} \dfrac{\left(q^{l-1}-1\right)(q^{\nu}-2q+1)\,q^{\nu+l-2}}{(q-1)^2}, & \text{if } t=0, j=\mu=1, \\ \dfrac{(q^{\nu}-2q+1)\,q^{\nu+l-3}}{q-1}, & \text{if } t=1, j\neq\mu, \text{or } t=j=\mu=0, \\ \dfrac{\left(q^{l}-2q+1\right)(q^{\nu}-2q+1)\,q^{\nu+l-3}}{(q-1)^2}, & \text{if } t=j=\mu=1, \end{cases} \tag{25}$$

$$p^{(4,t)}_{(5,j)(5,\mu)} = \begin{cases} \dfrac{\left(q^{l-1}-1\right)\left(q^{\nu-2}-1\right)q^{\nu+l}}{(q-1)^2}, & \text{if } t=0, j=\mu=1, \\ \dfrac{\left(q^{\nu-2}-1\right)q^{\nu+l-1}}{q-1}, & \text{if } t=1, j\neq\mu, \text{or } t=j=\mu=0, \\ \dfrac{\left(q^{l}-2q+1\right)\left(q^{\nu-2}-1\right)q^{\nu+l-1}}{(q-1)^2}, & \text{if } t=j=\mu=1, \end{cases} \tag{26}$$

$$p^{(5,t)}_{(5,j)(5,\mu)} = \begin{cases} \dfrac{\left(q^{l-1}-1\right)\left(q^{\nu}-2\mathrm{q}+1\right)q^{\nu+l-2}}{(q-1)^2}, & \text{if } t=0, j=\mu=1, \\ \dfrac{\left(q^{\nu}-2\mathrm{q}+1\right)q^{\nu+l-3}}{q-1}, & \text{if } t=1, j\neq\mu, \text{or } t=j=\mu=0, \\ \dfrac{\left(q^{l}-2q+1\right)\left(q^{\nu}-2\mathrm{q}+1\right)q^{\nu+l-3}}{(q-1)^2}, & \text{if } t=j=\mu=1, \end{cases} \tag{27}$$

$$p^{(2,t)}_{(2,j)(3,\mu)} = p^{(2,t)}_{(3,j)(4,\mu)} = p^{(2,t)}_{(4,j)(4,\mu)} = p^{(3,t)}_{(4,j)(4,\mu)} = 0, \tag{28}$$

$$p^{(2,t)}_{(2,j)(5,\mu)} = p^{(2,t)}_{(4,j)(5,\mu)} = p^{(4,t)}_{(4,j)(5,\mu)} = 0. \tag{29}$$

Since $n_{(i,t)}p^{(i,t)}_{(r,j)(\lambda,\mu)} = n_{(r,j)}p^{(r,j)}_{(i,t)(\lambda,\mu)}$, all the remaining intersection numbers of this association scheme can be determined by Eqs. (6)–(29), and Theorem 0.1 is proved.

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