

# A survey on the Weinstein conjecture

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**Abstract** The Weinstein conjecture predicts that there is at least one periodic orbit for every compact contact hypersurface in a symplectic manifold, which turns out an important research direction in symplectic topology and contact topology. According to the methods of the research, this paper surveys the study on the Weinstein conjecture using the variational method and the pseudo-holomorphic curves method.

**Keywords** Weinstein conjecture, periodic orbit, contact manifold,  $J$ -holomorphic curves, Reeb flow

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## 1 The Weinstein conjecture

The research on finding periodic orbits of Hamiltonian systems has had a long history. It can be traced back to the studies in astronomy. In the solar system, planets all move periodically and stably along specific orbits around the sun. When physicists described this phenomenon, they found that the movements of planets in the solar system could be described by Hamiltonian systems. And the stability of the planets' movements can be interpreted as: the orbits of the planets lie on specific Hamiltonian energy surfaces. In mathematics, the research on the qualitative theory of Hamiltonian systems was first proposed by the French mathematician Poincaré. By the late 1970s, there had been many results regarding the existence of periodic solutions of Hamiltonian systems. Among them, the most important results were the existence of periodic orbits on convex energy surfaces obtained by Weinstein [42] and the existence of periodic orbits on star-shaped energy surfaces obtained by Rabinowitz [38]. These results, attracting the attention of many mathematicians, greatly promoted the research on periodic solutions of Hamiltonian systems and developed the nonlinear analysis.

However, the geometric assumptions made on the energy surfaces in these results cannot be preserved under symplectic diffeomorphisms, which made it impossible to generalize the research on periodic solutions of Hamiltonian systems to general symplectic manifolds. Through the study of the existing results, Weinstein discovered that the energy surfaces with periodic orbits all had a structure-the contact structure, which was invariant under symplectic diffeomorphisms. Thus, he put forward the following famous conjecture [43].

**The Weinstein conjecture** [43] (1978) *Any compact contact hypersurface  $S$  in the symplectic manifold  $(M, \omega)$  has at least one Hamiltonian periodic orbit.*

**Note** In the conjecture initially proposed by Weinstein, there was a topological condition of  $H^1(S; \mathbb{R}) = 0$  for the hypersurface  $S$ . However, there was no indication that this additional condition was necessary. As a result, the assumption of the triviality of the first cohomology group was removed.

First, some basic concepts are reviewed in symplectic geometry [33]. Let  $(M, \omega)$  be a symplectic manifold, and  $H : M \rightarrow \mathbb{R}$  be a Hamiltonian function on  $M$ , satisfying the condition

$$i_{X_H}\omega = -dH.$$

The vector field  $X_H$  corresponding to  $H$  is called a Hamiltonian vector field. Assume the map  $x: \mathbb{R} \rightarrow M$  satisfies the following Hamiltonian system

$$\frac{dx(t)}{dt} = X_H(x(t)). \quad (1.1)$$

If the bundle map  $J: TM \rightarrow TM$  satisfies  $J^2 = -\text{Id}$ , then  $J$  is called an almost complex structure on the manifold  $M$ . The almost complex structure  $J$  is said to be compatible with  $\omega$  if  $g(\cdot, \cdot) := \omega(\cdot, J\cdot)$  defines a Riemannian metric on  $M$ . The standard almost complex structure  $J_0$  in  $\mathbb{R}^{2n}$  can be expressed as  $J_0 = \begin{pmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{pmatrix}$ , where  $\text{Id}$  is the identity matrix in  $\mathbb{R}^n$ .  $J_0$  is compatible with the standard symplectic structure  $\omega_0 = \sum_{i=1}^n dx^i \wedge dy^i$  in  $\mathbb{R}^{2n}$ , and the Riemannian metric  $\omega_0(\cdot, J_0\cdot)$  is the usual inner product in  $\mathbb{R}^{2n}$ . Taking a compatible almost complex structure  $J$ , the Hamiltonian vector field  $X_H$  can be expressed as

$$X_H = J\nabla H. \quad (1.2)$$

Let the local coordinates near  $x(0) \in M$  be  $(p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_n)$ ,  $x(t) = (p_1(t), p_2(t), \dots, p_n(t), q_1(t), q_2(t), \dots, q_n(t))$ ,  $t \in (-\varepsilon, \varepsilon)$ , with  $\varepsilon$  being sufficiently small. From Equation (1.2), it can be obtained that, the Hamiltonian system (1.1) can be locally expressed as

$$\begin{cases} \dot{p}_i(t) = -\frac{\partial H}{\partial q_i}, \\ \dot{q}_i(t) = \frac{\partial H}{\partial p_i}, \forall i \in \{1, 2, \dots, n\}. \end{cases} \quad (1.3)$$

Assume  $S = \{x \in M \mid H(x) = 0\}$  and  $dH$  is non-zero everywhere  $S$ , then the Hamiltonian vector field  $X_H|_S$  is a tangent vector field on  $S$ . It is verified that the periodic orbits of  $X_H$  on  $S$  are determined by  $S$ , and only the parameterization of the orbits depends on  $H$ . The line bundle

$$\mathcal{L}_s = \{(x, \xi) \in T_x S \mid \omega_x(\xi, \eta) = 0, \forall \eta \in T_x S\}$$

is called the characteristic line bundle of the hypersurface  $S$ , and  $\mathcal{L}_s$  determines the direction of the Hamiltonian vector field on  $S$ . A closed characteristic (or Hamiltonian periodic orbit) of  $S$  is an embedded circle  $P$ , and  $TP = \mathcal{L}_s|_P$ .

The hypersurface  $S$  in a symplectic manifold  $(M, \omega)$  is of contact type, which indicates that there exists a 1-form  $\lambda$  on  $S$  satisfying  $d\lambda = j^* \omega$  and  $\lambda(\xi) \neq 0$  always holds for any  $0 \neq \xi \in \mathcal{L}_s$ , where  $j: S \rightarrow M$  is an inclusion map. The 1-form  $\lambda$  is called the contact form. The contact form  $\lambda$  on the contact hypersurface  $S \subset M$  determines one contact structure  $\xi\lambda := \ker(\lambda) \subset TS$  on  $S$ , and there exists a unique vector  $X_\lambda$  satisfying  $i_{X_\lambda} d\lambda \equiv 0$  and  $i_{X_\lambda} \lambda = 1$ . This vector field  $X_\lambda$  is called the Reeb vector field of  $\lambda$ . It is noted that the periodic orbits of the Hamiltonian vector field  $X_H$  on  $S$  coincide with the closed orbit trajectory of the Reeb vector field. Thus, the following is general Weinstein conjecture.

**Weinstein conjecture** (general case) *For every compact contact manifold  $(S, \lambda)$ , the Reeb vector field  $X_\lambda$  of  $\lambda$  has a closed orbit.*

In the three-dimensional case, the Weinstein conjecture was proven to be true by Taubes [40]. In many other special cases, the Weinstein conjecture has also been proven to be correct. The following introduces these results in detail. In addition, the Weinstein conjecture remains an open problem until today. Researches on the Weinstein conjecture have greatly developed Hamiltonian systems, Reeb dynamics, and symplectic topology.

It should be pointed out that if the contact condition of the hypersurface  $S$  is removed, then the Weinstein conjecture will not hold. In fact, without the contact condition, there are many counter examples to the Weinstein conjecture. One of the most representative ones was  $\mathbb{R}^{2n}$  constructed by Herman [19], where in  $n \geq 4$  there exists a smooth hypersurface  $S$  without Hamiltonian periodic orbits. Ginzburg and Gürel [17] constructed a  $C^2$  Hamiltonian function  $F$  intrinsic to  $\mathbb{R}^4$  without periodic orbits on at least one regular energy surface. Kerman [28] constructed a smooth Hamiltonian function  $H$  in  $\mathbb{R}^{2n}$ ,  $n \geq 3$ , and  $H$  has regular energy surfaces without periodic solutions.

## 2 Variational method in symplectic topology

The variational method plays a very important role in both symplectic topology and Hamiltonian systems, which is also reflected in the study of the Weinstein conjecture. Let the symplectic manifold  $(M, \omega)$  be  $(\mathbb{R}^{2n}, \omega_0)$  or  $(T^*N, \omega_{st})$ ,

where  $\omega_{st}$  is the canonical symplectic form on  $T^*N$ . Define a Hamiltonian function  $H : M \rightarrow \mathbb{R}$  such that the contact hypersurface  $S$  is a regular energy surface of  $H$  (i.e.,  $X_H$  is non-zero everywhere on  $S$ ). The Weinstein conjecture asserts that the Hamiltonian system (1.1) has at least one periodic solution on  $S$ . For manifolds with a linear structure themselves, the power of variational method is particularly remarkable. The general idea is to define a functional  $\mathcal{A}$  to transform the existence problem of periodic solutions of the Hamiltonian system (1.1) into the existence problem of critical points of the functional  $\mathcal{A}$ .

The first breakthrough regarding the Weinstein conjecture was obtained by Viterbo [41] in 1987.

**Theorem 2.1** [41] *If  $S \subset (\mathbb{R}^{2n}, \omega_0)$  is a compact contact hypersurface, then  $S$  possesses at least one closed characteristic.*

We take this result as an example to illustrate the application of the variational method in the study of the Weinstein conjecture. In [41], Viterbo associated a closed characteristic on  $S$  with the periodic solution of

$$\begin{cases} \dot{x} = J_0 N(x), \\ x(t) \in S, \end{cases} \quad (2.1)$$

where  $N(x)$  represents the outer normal vector of  $S$  at  $x$ . For star-shaped energy surfaces, Rabinowitz [38] constructed one Hamiltonian function  $H$  such that  $H|_S = 1$ , and the periodic solutions in (2.1) were transformed into the nontrivial solutions of a Hamiltonian system with fixed period

$$\begin{cases} \dot{x} = J\nabla H(x), \\ x(0) = x(T). \end{cases} \quad (2.2)$$

For a general contact hypersurface, Rabinowitz's construction method is not feasible. Viterbo improved Rabinowitz's method and constructed a Hamiltonian function  $H$  on  $\mathbb{R}^{2n}$ , thus transforming the periodic solutions of (2.1) into nontrivial solutions of the Hamiltonian system with fixed period (2.2). Let  $K$  be a positive number, and assume  $H_K(z) = H(z) + \frac{K}{2}|z|^2$ . In the Fenchel sense, the dual functional of  $H_K(z)$  is  $H_K^*(y) = \sup_{z \in \mathbb{R}^{2n}} [(z, y) - H_K(z)]$ . Viterbo defined the functional

$$F_K = \int_0^T ((J\dot{x} - Kx, x) + H_K^*(-J\dot{x} + Kx)) ds$$

to correspond the solutions of system (2.2) to the critical points of functional  $F_K$ . By performing a finite-dimensional reduction on  $F_K$ , we get a functional  $f_K$  acting on the finite-dimensional space, which satisfies the (P.S.) compactness and is also coercive. More importantly, the critical points of  $F_K$  and  $f_K$  are in one-to-one correspondence, and they have the same critical values. Thus, the existence of nontrivial critical points is obtained. The proof in Viterbo [41] was simplified by Hofer and Zehnder [23].

A class of parameterized hypersurfaces modeled on  $S$  means that there exists a diffeomorphism  $\psi : S \times I \rightarrow U \subset M$  to the bounded neighborhood  $U$  of  $S$ , such that for any  $x \in S$ ,  $\psi(x, 0) = x$ , where  $I$  is an open interval containing 0. Denote  $S_\varepsilon = \psi(S \times \{\varepsilon\})$ . A compact hypersurface  $S \subset (M, \omega)$  is called stable if  $S$  admits a class of parameterized hypersurfaces  $(S_\varepsilon)$  modeled on  $S$ , such that the characteristic line bundle  $\mathcal{L}_\varepsilon$  of  $S_\varepsilon$  is independent of  $\varepsilon$ . Contact hypersurfaces are stable. In fact, a hypersurface  $S$  in a symplectic manifold  $(M, \omega)$  is contact if and only if there exists a vector field  $Y$  on a neighborhood  $U$  of  $S$ , such that  $L_Y \omega = \omega$  and  $Y$  is transverse to  $S$ . Such a vector field  $Y$  is called a Liouville vector field. The isomorphism between the line bundle  $\mathcal{L}_S$  and  $\mathcal{L}_{S_t}$  can be defined via the flow of the Liouville vector field. However, a stable hypersurface is not necessarily contact. Hofer and Zehnder [25] provided a counterexample.

In 1988, Hofer and Viterbo [21] proved a result of the Weinstein conjecture in a cotangent bundle.

**Theorem 2.2** [21] *Let  $N$  be a compact manifold with  $\dim N \geq 2$ , and  $M$  be endowed with a canonical symplectic structure by  $M = T^*N$ . Suppose  $S$  is a smooth, compact, and connected hypersurface in  $M$ , such that the bounded part of  $M \setminus S$  contains the zero section of  $T^*N$ . Then if  $S$  is of contact type, it has a closed characteristic.*

Ekeland and Hofer [9, 10] cooperatively introduced the concept of symplectic capacity in  $\mathbb{R}^{2n}$  with the help of variational principle. Later, Hofer and Zehnder [24] generalized the concept of symplectic capacity to general symplectic manifolds.

Symplectic capacity refers to a map with the following properties  $c: \{(M, \omega) \mid \dim(M) = 2n\} \rightarrow [0, \infty]$ .

- Monotonicity: if there exists a symplectic embedding  $i: (M, \omega) \hookrightarrow (N, \tau)$ , then  $c(M, \omega) \leq c(N, \tau)$ .

- Conformity: For all  $\alpha \in \mathbb{R}$ ,  $\alpha \neq 0$ ,  $c(M, \alpha\omega) = |\alpha| c(M, \omega)$ .

- Non-triviality:  $c(B(1), \omega_0) = \pi = c(Z(1), \omega_0)$ , where  $B(r) = \{(x, y) \in \mathbb{R}^{2n} \mid |x|^2 + |y|^2 < r^2\}$ ,  $Z(r) = \{(x, y) \in \mathbb{R}^{2n} \mid x_1^2 + y_1^2 < r^2\}$ .

Hofer and Zehnder [24] explicitly constructed a symplectic capacity  $c_0$  as follows:

Let  $\mathcal{H}(M, \omega)$  denote the set consisting of smooth functions that satisfy the following properties:

- There exist open set  $U_H$ , compact set  $K_H$  and constant  $m(H)$ , such that  $U_H \subset K_H \subset (M \setminus \partial M)$ ,  $H(U_H) = 0$ ,  $H(M \setminus K_H) = m(H)$ .

- $0 \leq H(x) \leq m(H)$ ,  $\forall x \in M$ ,

where  $m(H) := \max H - \min H$  is the amplitude of the Hamiltonian function  $H$ . The function  $H$  is called admissible, indicating that the periodic solutions of the Hamiltonian vector field  $X_H$  are either constant or have a period  $T > 1$ . Let  $\mathcal{Ha}(M, \omega) \subset \mathcal{H}(M, \omega)$  denote all admissible functions. Define

$$c_0(M, \omega) := \sup_{\mathcal{H}_a(M, \omega)} m(H).$$

If there exists a constant  $C \geq 0$ , such that  $c_0(M, \omega) \leq C$ , then  $c_0(M, \omega)$  represents the infimum of all numbers possessing the following properties:  $\forall H \in \mathcal{H}(M, \omega)$ , if  $m(H) > c_0(M, \omega)$ , there is the  $T$ -periodic solution  $0 < T \leq 1$  to  $X_H$ .

With the above symplectic capacity  $c_0(M, \omega)$ , Hofer and Zehnder [25] proved the following theorems.

**Theorem 2.3** [25] (Hofer-Zehnder) *Let  $S$  be a regular compact energy surface of  $H$ . Assume there exists a neighborhood  $U$  of  $S$  with finite capacity:  $c_0(U, \omega) < \infty$ , and  $S \times (-\varepsilon, \varepsilon)$  can be embedded into  $U$ . Then there exists a dense subset  $\Sigma \subset (-\varepsilon, \varepsilon)$ , such that  $\forall \varepsilon_0 \in \Sigma$ , and  $S_{\varepsilon_0}$  has periodic solutions of  $X_H$ .*

It can be seen that this theorem does not necessarily yield periodic orbits on  $S$ . However, Hofer and Zehnder [25] pointed out that if the periods  $T_{\varepsilon_0}$  of the periodic orbits on  $S_{\varepsilon_0}$  were bounded, then periodic orbits on  $S$  could be obtained. According to the definition of a stable hypersurface, the following existence results can be obtained from the theorem of Hofer and Zehnder.

**Theorem 2.4** [25] *Suppose  $S \subset (M, \omega)$  is a compact and stable hypersurface with a neighborhood  $U$  of finite capacity, then  $S$  has a closed orbit. In particular, if  $S$  is of contact type, then  $S$  has a closed orbit.*

Based on this theorem and the definition of symplectic capacity, we can obtain Viterbo's result regarding the Weinstein conjecture in  $\mathbb{R}^{2n}$ . Symplectic capacity has numerous applications in the Weinstein conjecture. For example, Lu demonstrated in [32] that the Weinstein conjecture held in the vicinity of a closed symplectic sub-manifold of any symplectic manifold by applying symplectic capacity.

### 3 Pseudo-holomorphic curve and symplectic topology

In 1985, Gromov [18] introduced pseudo-holomorphic curve into symplectic manifolds, which significantly promoted the development of symplectic topology research. The theory of pseudo-holomorphic curve plays a crucial role in symplectic topology and mathematical physics, contributing to the discovery of mathematical theories such as Floer homology, Gromov-Witten invariants, and mirror symmetry.

Let  $(\Sigma, j)$  be a Riemann surface and  $J$  be an almost complex structure on the symplectic manifold  $(M, \omega)$ . The map  $u : \Sigma \rightarrow M$  is called a pseudo-holomorphic curve (or  $J$ -holomorphic curve) if  $du$  is complex linear, that is,  $J \circ du = du \circ j$ .  $u$  is a  $J$ -holomorphic curve if and only if

$$\bar{\partial}_J u := \frac{1}{2}(du + J \circ du \circ j) = 0. \quad (3.1)$$

Under the local coordinates  $z = s + it$  of  $\Sigma$ , Equation (3.1) can be written as

$$\partial_s u + J \partial_t u = 0.$$

This is the Cauchy-Riemann equation, which is an elliptic partial differential equation. By considering the properties of the almost complex structure on the symplectic manifold, specifically the properties of the moduli space formed by the  $J$ -holomorphic curves on the symplectic manifold, Gromov introduced elliptic and geometric methods into the study of symplectic topology.

After Gromov introduced the method of pseudo-holomorphic curves, the researches on the Weinstein conjecture mainly proceed from the following two perspectives: one is to regard the contact hypersurface as the energy surface of a certain Hamiltonian function on the symplectic manifold, and study the Weinstein conjecture by using the properties of the symplectic manifold and the Hamiltonian system; the other is to directly consider the contact hypersurface as a contact manifold, and study the Weinstein conjecture by using the contact structure and the Reeb dynamics. The common point of these two perspectives is to transform the existence problem of solutions to the ordinary differential Equation (1.1) into the study of the properties of the moduli space composed of (perturbed)  $J$ -holomorphic curves (disks, strips) that satisfy certain conditions. In this way, the existence problem of solutions to the ordinary differential equation is transformed into a problem of first-order partial differential equations. Superficially, this seems to complicate the problem. In fact, this shift in perspective enables the application of analytical tools such as elliptic theory and Fredholm theory to the problem at hand, which has been proven effective [4, 37]. For the obtained partial differential equations, instead of directly solving them, we consider the moduli space  $\mathcal{M}$  formed by their solutions. Usually, certain conditions need to be added to make the moduli space a compact manifold. This requires to overcome two technical challenges: transversality and compactness. The transversality problem is generally overcome by perturbing the pseudo-holomorphic curves. According to Gromov's compactness theorem, it is often necessary to add some singular elements to the moduli space to achieve compactness. Based on the topological properties of the symplectic manifold  $(M, \omega)$  and the properties of the moduli space, a sequence of special pseudo-holomorphic cylinders (disks) is found. The asymptotic limit of this sequence of pseudo-holomorphic cylinders (disks) is the Hamiltonian periodic orbit we are looking for. If the requirements for the moduli space are relaxed, that is, the requirement that the moduli space  $\mathcal{M}$  is a manifold is relaxed to an orbifold, the difficulty of transversality can also be overcome with the help of the virtual fundamental class, the virtual neighborhood technique, and the polyfold theory. These methods have also been applied to the study of the Weinstein conjecture and achieved good results. The following

further introduces the relevant progress in the study of the Weinstein conjecture.

### 3.1 Symplectic manifold and Hamiltonian system

Floer et al. [14] proved that the Weinstein conjecture on  $(P \times \mathbb{C}^l, \omega \oplus \sigma)$  ( $l \geq 1$ ) held, where  $\sigma$  was a standard symplectic structure on  $\mathbb{C}^l$ .

**Theorem 3.1** [14] *Let  $(P, \bar{\omega})$  be a compact symplectic manifold such that  $[\bar{\omega}]$  is 0 in  $\pi_2(P)$ . Then, the compact contact hypersurface  $\Sigma$  in  $(P \times \mathbb{C}^l, \omega \oplus \sigma)$  ( $l \geq 1$ ) has at least one closed orbit.*

Floer et al. defined  $(V, \omega) = (P \times \mathbb{C}, \bar{\omega} \oplus \sigma)$  in [14]. Let

$$C_0^1(S^1, V) = \{z \in C^1(S^1, V) \mid \exists \bar{z} \in C^1(D^2, V), \text{ such that } z = \bar{z}|_{\partial D}\}.$$

For the given Hamiltonian function  $H : V \rightarrow \mathbb{R}$ , define the functional  $\mathcal{A}_H : C_0^1(S^1, V) \rightarrow \mathbb{R}$ ,

$$\mathcal{A}_H(z) = \int_D \bar{z}^* \omega - \int_{S^1} H(z(\theta)) d\theta.$$

Since  $[\omega]$  is 0 in  $\pi_2(V)$ , it can be seen that the definition of  $\mathcal{A}_H$  is independent on the choice of  $\bar{z}$ . There are no nontrivial  $J$ -holomorphic curves in  $V$ . Let  $\Sigma$  be a compact contact hypersurface in  $V$  and  $u_0$  be a special fixed point in  $\Sigma$ , such that the constant path at  $u_0$  is a critical point of  $\mathcal{A}_H$ . Consider the unstable manifold of  $\mathcal{A}_H$

$$W_H^U(u_0) = \{u \in W^{1,p}((-\infty, 0] \times S^1, V) \mid \bar{\partial}u + \nabla H(u) = 0, \lim_{s \rightarrow -\infty} u(s, \cdot) = u_0\},$$

where we assume  $p > 2$ . For the vector  $c$  on  $V$ , let

$$I_c = \{u \in W^{1,p}(D^2, V) \mid \partial u = c\},$$

where  $\partial = \frac{\partial}{\partial x} - J \frac{\partial}{\partial y}$ .  $\dot{I}_c = \{u|_{S^1} \mid u \in I_c\}$ ,  $\dot{W}_H^U = \{u|_{S^1} \mid u \in W_H^U\}$ . With Fredholm theory, Floer et al. [14] proved if the Hamiltonian system

$$\begin{cases} \dot{z} = X_H(z), \\ z(0) = z(2\pi) \end{cases} \quad (3.2)$$

had no nontrivial solution, then the intersection number  $|\dot{I}_c \cap \dot{W}_H^U|$  was non-zero. However, if  $|c|$  is large enough,  $\dot{I}_c \cap \dot{W}_H^U = \emptyset$ . This indicates that (3.2) must have a nontrivial solution, and such a nontrivial solution corresponds to the periodic solution of (1.1).

When there are nontrivial  $J$ -holomorphic curves in  $V$ , it is necessary to consider the properties of the moduli space of  $J$ -holomorphic curves. Hofer and Viterbo applied the method of  $J$ -holomorphic curves in [22] and proved that the Weinstein conjecture held in  $\mathbb{C}p^n$  and  $S^2 \times P$ , where  $P$  was a compact manifold satisfying certain conditions.

**Theorem 3.2** [22] *The compact, connected, and stable hypersurface  $S$  in  $(\mathbb{C}P^n, \omega)$  has a Hamiltonian periodic orbit.*

**Theorem 3.3** [22] *If there is  $m(P, \omega, J) \geq \int_{S^2} \sigma$  for certain  $J \in \mathcal{F}(P, \omega)$ , then the compact, connected, and stable hypersurface  $S$  in  $(P \times S^2, \omega \oplus \sigma)$  that separates  $P \times \{0\}$  and  $P \times \{\infty\}$  has a Hamiltonian periodic solution.*

Regarding the compact symplectic manifold  $(V, \omega)$ , Hofer and Viterbo [22] defined

$$m(V, \omega, J) = \inf \{ \langle \omega, [u] \rangle \mid u \text{ is a non-constant } J\text{-holomorphic sphere} \}$$

and

$$m(V, \omega) = \inf \{ \langle \omega, \alpha \rangle \mid \alpha \in [S^2, V], \langle \omega, \alpha \rangle > 0 \},$$

where  $[S^2, V]$  represents the set of free homotopy classes from  $S^2$  to  $V$ . Assume  $\alpha \in [S^2, V]$  satisfies  $\langle \omega, \alpha \rangle = m(V, \omega, J)$ . Suppose the set  $\mathcal{H}(a, J, \Sigma_0, \Sigma_\infty)$  is composed of all  $J$ -holomorphic spheres satisfying the following conditions:

$$[u] = \alpha, u(*) \in \Sigma_*, * \in \{0, \infty\}, \text{ and } \int_{|z| \leq 1} u^* \omega = \frac{1}{2} \langle \omega, \alpha \rangle, \quad (3.3)$$

where  $\Sigma_0$  and  $\Sigma_\infty$  are mutually disjoint compact sub-manifolds in  $V$ .

Hofer and Viterbo [22] proved that for any  $J \in \mathcal{F}(V, \omega)$ , there was a small neighborhood  $\mathcal{U}$  of  $J$ , and within  $\mathcal{U}$ , there was a regular almost complex structure  $\tilde{J}$  arbitrarily close to  $J$  such that the set  $\mathcal{H}(\alpha, \tilde{J}, \Sigma_0, \Sigma_\infty)$  was a compact smooth manifold. If  $\tilde{J}_1, \tilde{J}_2$  is close enough to  $J$ , then the compact manifold  $\mathcal{H}(\alpha, \tilde{J}_1, \Sigma_0, \Sigma_\infty)$  and  $\mathcal{H}(\alpha, \tilde{J}_2, \Sigma_0, \Sigma_\infty)$  are cobordant. Define the cobordism class  $d(\alpha, J, \Sigma_0, \Sigma_\infty) := [\mathcal{H}(\alpha, \tilde{J}, \Sigma_0, \Sigma_\infty)]$ , then the definition of  $d(\alpha, J, \Sigma_0, \Sigma_\infty)$  is independent of the choice of  $\tilde{J}$ . There is a very important connection between the  $d$ -index and  $J$ -holomorphic spheres. Define the parametric section  $f_\lambda(u) = \bar{\partial}_J u + \lambda h(u)$ , and consider the set  $\mathcal{C} = \{(\lambda, u) \in [0, +\infty) \times \mathcal{B} \mid f_\lambda(u) = 0\}$  consisting of perturbed  $J$ -holomorphic spheres, where  $\mathcal{B}$  is the Hilbert manifold consisting of  $u \in H^{2,2}(S^2, V)$  that satisfies (3.3). Hofer and Viterbo [22] proved that if  $\mathcal{C}$  was compact, then the  $d$ -index  $d(\alpha, J, \Sigma_0, \Sigma_\infty) = [\emptyset]$ . For the compact manifold  $\mathbb{C}P^n, S^2 \times P$ , the super-minimal holomorphic spheres on it exist. By choosing an appropriate regular almost complex structure, the  $d$ -index  $d(\alpha, J, \Sigma_0, \Sigma_\infty) \neq [\emptyset]$  can be calculated. This leads to the set  $\mathcal{C}$  to be non-compact, and the energies of all the perturbed  $J$ -holomorphic spheres in  $\mathcal{C}$  are bounded. Such non-compactness will result in the existence of a sequence of infinitely long tubes. There is a sequence of periodic orbits on this sequence of pseudo-holomorphic cylinders that asymptotically converges to a non-constant periodic solution of the system (1.1). This periodic solution is the closed orbit sought in the Weinstein conjecture.

In [22], Hofer and Viterbo required the manifolds to be compact. By making a

priori compactness estimates for geometrically bounded symplectic manifolds, Lu [30] extended the results of [22] to non-compact geometrically bounded symplectic manifolds. In particular, he proved that the Weinstein conjecture held in  $S^2 \times T^*N$ , where  $N$  was a closed manifold or a non-compact manifold of finite topological type.

Hofer and Viterbo [22] as well as Lu [30] considered the Weinstein conjecture in product manifolds of the type  $S^2 \times P$ . In such product manifolds, for the almost complex structures  $J_1$  on  $P$ ,  $J = i \times J_1$  is regular, where  $i$  is the standard complex structure on  $S^2$ . When using the method of Hofer and Viterbo [22] to deal with the Weinstein conjecture on common product manifold  $P_1 \times P_2$ , to overcome the transversality, it is necessary to construct a product regular almost complex structure  $J = J_1 \times J_2$ , where  $J_1$  and  $J_2$  are regular almost complex structures on  $P_1$ ,  $P_2$ , respectively. However, constructing an almost complex structure is very difficult. Ding and Hu [7], by using the regularity criterion in [34], constructed such a regular product almost complex structure and thus proved the Weinstein conjecture in  $P_1 \times P_2$  under the condition that  $P_1$  was a 4-dimensional geometrically bounded symplectic manifold and there was an embedded  $J$ -holomorphic sphere representing the minimal homotopy class. In particular, it was proven that the Weinstein conjecture held in  $\mathbb{C}p^2 \times T^*N$ , where  $N$  was either a closed manifold or a non-compact manifold of finite topological type. Using the same idea, when  $P_1$  is a Kähler manifold with a minimal homotopy class represented by a  $J$ -holomorphic sphere, Ding [6] constructed a class of regular product almost complex structures. Specifically, it was proven that the Weinstein conjecture held in  $\mathbb{C}p^n \times T^*N$ , where  $N$  was either a closed manifold or a non-compact manifold of finite topological type.

The ideas in references [14] and [22] essentially involve examining the relationship between the existence of genus zero  $J$ -holomorphic curves and the existence of periodic solutions of the Hamiltonian system (1.1). Due to the reliance on the  $d$ -index theory in the proof, the problem of transversality must be addressed. This requires some technical restrictions on the symplectic manifold. The development of Floer homology and Gromov-Witten theory has provided tools for overcoming the transversality. Liu and Tian [29] utilized the virtual moduli cycle to establish the relationship between the existence of pseudo-holomorphic curves of arbitrary genus and the existence of periodic solutions of the Hamiltonian system (1.1).

**Theorem 3.4** [29] *If there exist  $A \in H_2(V, \mathbb{Z})$  and  $\alpha_+, \alpha_- \in H_*(V, \mathbb{Q})$ , such that*

- (1)  $\text{Supp}(\alpha_+) \hookrightarrow \overset{\circ}{V}_-$  and  $\text{Supp}(\alpha_-) \hookrightarrow \overset{\circ}{V}_+$ ;
- (2)  $GW$ -invariant  $\Psi_A(C; \alpha_-, \alpha_+, \beta_1, \beta_2, \dots, \beta_n) \neq 0$ ,

*then hypersurface  $S$  contains at least one closed orbit.*

**Theorem 3.5** [29] *The Weinstein conjecture holds on  $(M \times \mathbb{C}^n, \omega \oplus \omega_0)$  and  $(\prod_{i=1}^k \mathbb{C}P^{n_i}, \oplus_{i=1}^k \omega_i)$ . Here,  $(M, \omega)$  is a closed symplectic manifold,  $(\mathbb{C}^n, \omega_0)$  is the  $n$ -dimensional complex space endowed with the standard symplectic structure, and*

$\omega_i$  is the standard symplectic structure in  $\mathbb{C}P^{ni}$ .

Inspired by reference [29], Lu [31] utilized the properties of Gromov-Witten invariants and the techniques proposed by Viterbo to prove that the Weinstein conjecture held in uniruled symplectic manifolds and product manifolds of uniruled symplectic manifolds and compact symplectic manifolds. By using the product formula for Gromov-Witten invariants of Hamiltonian fibrations in [26]. Hyvrier [27] extended Lu's results in [31] to the context of Hamiltonian fibrations, and proved that the Weinstein conjecture held in certain Hamiltonian fibrations  $P$ .

### 3.2 Contact manifolds and Reeb dynamics

First, we introduce some basic concepts regarding contact structures. The contact form  $\lambda$  in a three-dimensional closed manifold  $M$  is said to be over-twisted if there exists an embedded disk  $F \simeq D$ ,  $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ , such that

$$T(\partial F) \subset \xi_\lambda|_{\partial F}, \quad T_x F \not\subset \xi_{\lambda,x}, \quad \forall x \in \partial F.$$

Here,  $\xi_\lambda = \ker(\lambda)$ . The contact form that is not overtwisted is called tight. Based on different contact structures on three-dimensional closed manifolds, Hofer [20] applied the method of  $J$ -holomorphic curves to obtain important results of the Weinstein conjecture on three-dimensional manifolds. For overtwisted contact forms, Hofer [20] proved the following results:

**Theorem 3.6** [20] *If  $\lambda$  is an overtwisted contact form in a three-dimensional closed manifold  $M$ , then the corresponding Reeb vector field has at least one contractible periodic orbit.*

Hofer's [20] idea was to place the contact manifold  $(M, \lambda)$  into its symplectization  $(\mathbb{R} \times M, d(e^t \lambda))$ . By examining suitable  $J$ -holomorphic curves from a "punctured" Riemann surface into the symplectization of  $M$ , the images of perturbed  $J$ -holomorphic curves would asymptotically converge to periodic orbits of the Reeb vector field near these "punctures" under certain assumptions. Let the pair  $(a, u)$  to  $\mathbb{R} \times M$  satisfy the following conditions:

$$\begin{cases} \pi \frac{\partial u}{\partial s} + J(u) \pi \frac{\partial u}{\partial t} = 0 \text{ is in } \mathring{D}, \\ (u^* \lambda) \circ i = da \text{ is in } \mathring{D}, \\ a|_{\partial D} = 0, \\ u(\partial D) \subset F^*. \end{cases} \quad (3.4)$$

Here,  $a : D \rightarrow \mathbb{R}$ ,  $u : D \rightarrow M$  are  $J$ -holomorphic disks,  $\pi : TM \rightarrow \xi$ ,  $F$  is a Riemann surface with singularities, and  $F^*$  is the part of  $F$  with singularities removed. Let  $\mathcal{B}$  be the set consisting of all pairs that satisfy Equation (3.4). For the contact form of the over-twisted structure, the Riemann surface  $F$  has an elliptic singularity  $\{e\}$ . By using the bubbling-off analysis, a sequence of  $J$ -holomorphic disks that enable  $u|_{\partial D}$  to wind once around the singularity  $\{e\}$  can be found. And the asymptotic limit of the boundaries of this sequence of  $J$ -holomorphic

disks is the nontrivial periodic orbit we are looking for.

It can be seen from Theorem 3.6 that the Weinstein conjecture holds in  $S^3$ . In fact, through Eliashberg's [11, 12] classification of three-dimensional over-twisted contact forms, the tight contact forms in  $S^3$  can be transformed into Rabinowitz's results in star-shaped regions. For general tight contact forms, Hofer provided the following theorem.

**Theorem 3.7** [20] *If  $M$  is a closed three-dimensional manifold and  $\pi_2(M) \neq 0$ , then the Weinstein conjecture holds on  $M$ .*

When a contact manifold has some special structures, it is possible to use these structures to obtain information about periodic orbits. The open-book decomposition is a very important structure. An open-book decomposition  $(pr, B)$  of a manifold  $M$  means that there exist:

- A submanifold  $B \subset M$  of codimension 2, and  $B$  has a tubular neighborhood diffeomorphic to  $B \times \mathbb{C}$ .
- A fibration  $pr : (M \setminus B) \rightarrow S^1$  such that the map  $pr$  coincides with the argument coordinates of the  $\mathbb{C}$ -factor on the neighborhood  $B \times \mathbb{C}$ .

The submanifold  $B$  refers to the binding of the open-book, and the fibers of  $pr$  are called the pages of the open-book. The closure of a page  $P$  in  $M$  is a compact manifold with boundary  $B$ .

Abbas et al. [1] proved that the strong Weinstein conjecture held for planar contact structures in three-dimensional manifolds. The strong Weinstein conjecture with respect to the contact structure  $\lambda$  means that for every Reeb vector field  $X$  on the contact manifold  $S$ , there exist finitely many periodic orbits  $x_i, i = 1, 2, \dots, k$ , such that the sum of the first homology classes  $[x_1], [x_2], \dots, [x_k]$  induced by the closed orbits  $x_i$  is zero. If the strong Weinstein conjecture holds, then the Weinstein conjecture must hold.

In 2007, the Weinstein conjecture in the three-dimensional case was completely solved by Taubes [40] using the theory of Seiberg-Witten equations.

**Theorem 3.8** [40] *The Weinstein conjecture holds on any three-dimensional closed manifold.*

Under certain conditions, Hofer's [20] results were extended to high-dimensional cases. Albers and Hofer [3], by using the concept of Plastikstufe overtwisted (Ps-overtwisted) contact structure proposed by Niederkrüger [35], proved that the Weinstein conjecture held on high-dimensional contact manifolds with Ps-overtwisted contact structures. Recently, Borman et al. [5] provided the existence and classification results of overtwisted contact structures on high-dimensional contact manifolds. The method of Albers and Hofer [3] was also applicable to high-dimensional over-twisted contact structures.

Niederkrüger and Rechtman [36] improved Hofer's method in proving Theorem 3.7 for the three-dimensional case: Assume that  $(M^{2n+1}, \xi)$  contains an  $(n+1)$ -dimensional submanifold  $N$ , where  $N$  represents a nontrivial homology class in

$H_{n+1}(M; \mathbb{Z}_2)$ , and the contact structure  $\xi$  induces a Legendrian open-book on  $N$ . Then any contact form  $\lambda$  of  $\xi$  has a contractible Reeb periodic solution. In particular, it is proved that there must exist a contractible closed orbit on the connected sum of the real projective space and any contact manifold [2, 13].

Geiges and Zehmisch [15] refined Hofer's [20] approach. By constructing a symplectic cap on the convex boundary of the symplectic cobordism  $W$  of the contact manifold  $M$ , they made it possible to consider a moduli space composed of holomorphic spheres rather than a moduli space composed of holomorphic disks. It is proved that there exists a 0-homologous Reeb periodic orbit encircling the contact manifold  $M$  which serves as the concave boundary of  $W$ . In particular, they demonstrated that the strong Weinstein conjecture held in sub-critical Stein manifolds and  $T^*(Q \times S^1)$  (where  $Q$  was a closed manifold).

Using the same idea, Dörner et al. [8] proved that if the contact structure  $\xi$  was supported under the open-book  $(pr, B)$ , then any arbitrarily defined contact form  $\lambda$  of  $\xi$  had a contractible Reeb periodic solution, i.e., the Weinstein conjecture held in this case. By using contact surgery, Geiges and Zehmisch [16] proved that if a nontrivial contact connected sum had a nontrivial fundamental group or a torsion free homology group, then the Weinstein conjecture held on it. Using the polyfold technique, Suhr and Zehmisch [39] proved that the strong Weinstein conjecture held on a contact manifold which was the concave boundary of a symplectic cobordism and had a local foliation structure with pseudo-holomorphic spheres as leaves.

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