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RESEARCH ARTICLE

Means and equations of means

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Abstract Over the past two decades, means and mean equations have been popular directions of research in the field of functional equations. This paper first introduces the definition and properties of mean as well as the development of the Gauss iteration, invariant equations, and the M-S problem. It then reviews the Bajraktarević mean, the Cauchy mean, and other related means, especially elaborating on the progress of researching corresponding equality and invariance problems. Both the Bajraktarević mean and the Cauchy mean contain two derivative functions, and the corresponding equation problems have been basically solved. However, the invariance of these two symmetric means, due to the presence of four unknown functions, has not been fully solved. Finally, this paper introduces the applications of means in other fields.

Keywords Mean, equality, invariance, Cauchy mean, Bajraktarević mean
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1 Introduction

"Mean" , also known as an average, is an important fundamental concept in mathematics with a long history of research. For example, the well-known arithmetic mean can be traced back to 7000 BC, when the Babylonians mentioned the arithmetic mean of two real numbers. Ancient Greek scientists Aristotle, Heron, Archimedes, and the Pythagorean school also mentioned and applied this mean. In 1821, Cauchy [12] gave the general definition of mean as follows.

Definition 1.1 Assume $I \subset \mathbb{R}$ is an open interval. If bivariate function $M : I^2 \to I$ satisfies

$$\min\{x, y\} \leqslant M(x, y) \leqslant \max\{x, y\},\ x, y \in I,$$

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M can be called the "mean" on the interval I. If the inequality above is strictly established when $x \neq y$, then M will be defined as the "strict mean".

For example, the arithmetic mean, geometric mean, harmonic mean, and inverse mean are respectively

$$A(x,y)=\frac{x+y}{2}, G(x,y)=\sqrt{xy}, H(x,y)=\frac{2xy}{x+y}, C(x,y)=\frac{x^2+y^2}{x+y}, x,y>0.$$

The $n(\geqslant 3)$-variate mean can be defined by the similar way mentioned above.

Some properties of mean will be introduced as follows. There is $M(J^2)=J$ for every subinterval of I, and the mean satisfies reflexivity, i.e., $M(x,x)=x$, $x\in I$. If for all $x,y\in I$, there is

$$M(x,y)=M(y,x),$$

then we say the mean $M(x, y)$ is symmetric. If

$$M(tx,ty)=tM(x,y),\ t,x,y>0,$$

then we say M is positive homogeneous. If

$$M(t+x,t+y)=t+M(x,y),\ x,y,t\in I,$$

we say M is transferable. If

$$M(M(x,y),M(u,v))=M(M(x,u),M(y,v)),\ x,y,u,v\in I,$$

then we say the mean satisfies bisymmetry.

The invariance of mean is closely related to its Gauss iteration. As is shown below, we will introduce the definition of the invariant equation and Gauss iteration, as well as the relationship between the two, followed by the research background and the results of Matkowski-Sutô problem (M-S problem).

Definition 1.2 [15] Let $K,M,N:I^2\to I$ be a mean. If invariant equation

$$K(M(x,y),N(x,y))=K(x,y),\ x,y\in I$$

holds, then the mean K is the invariant mean of mean mapping $(M,N):I^2\to I^2$, denoted by (M, N)-invariant.

Then the Gauss iteration of mean-type mapping is introduced.

Definition 1.3 [15] The sequence of Gauss iteration $((M_1,M_2)^n)_{n=0}^{\infty}$ of mean pair mapping $(M_1,M_2):I^2\to I^2$ is defined as follows:

$$(M_1,M_2)^0:=\mathrm{Id}|_{I^2}, (M_1,M_2)^{n+1}:=(M_1,M_2)\circ(M_1,M_2)^n. \tag{1.1}$$

If the Gauss iteration of (M_1, M_2) is convergent, the unique convergent mean $M_1\otimes M_2:I^2\to I$ will be the Gauss combination of M_1 and M_2.

In particular, the Gauss combination $A\otimes G$ of arithmetic mean and geometric mean is defined as [8, 18, 73]:

$$A \otimes G\,(x,y) = \left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}t}{\sqrt{x^2\cos^2 t + y^2\sin^2 t}}\right)^{-1},\ x, y \in \mathbb{R}_+.$$

This formula establishes the relationship between arithmetic-geometric mean and elliptic integral of the first kind. Many special algebraic functions and transcendental functions can be numerically calculated in an effective way with the help of arithmetic-geometric mean, which also serves as the basis for working out the irrational number π. Meanwhile, based on the fact that the Gauss iteration of arithmetic and geometric mean can quickly converge to the arithmetic-geometric mean $A \otimes G$, Gauss's discovery of the specific expression of arithmetic-geometric mean also plays an important role in number theory. In addition, Gauss found that the mean $A \otimes G$ satisfied the invariant equation:

$$A \otimes G\left(\frac{x+y}{2}, \sqrt{xy}\right) = A \otimes G\,(x,y)\,,\ x, y \in \mathbb{R}_+.$$

In terms of the relationship between the convergence and invariance of the general mean Gauss iteration, if the means M_1 and M_2 on interval I are continuous and at least one of them is strict, the Gauss iteration of the mean pair mapping $(M_1,\ M_2)$ will be convergent, and the convergent function will be an invariant mean [15]. In 2009, Matkowski [55] weakened the conditional assumption about strictness and obtained that if the means M and N on interval I were continuous and satisfied

$$\max\left(M(x,y), N(x,y)\right) - \min\left(M(x,y), N(x,y)\right) < \max(x,y) - \min(x,y),$$

then there would be the unique invariant mean K in the mean pair mapping (M, N), and the Gauss iteration of mapping $(M,N): I^2 \to I^2$ would converge to the mean pair mapping (K, K).

The research on the mean invariant equation can be traced back to 1914, when Sutô [70] discussed the equation

$$\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right) + \psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right) = x+y,\ x, y \in I, \tag{1.2}$$

and gave its solution under analytic conditions.

At the 5th International Conference on Functional Equations and Inequality held in Poland in 1995, Matkowski posed the question: under what circumstances did the sum of two quasi-arithmetic means equal twice the arithmetic mean?

In fact, according to the definition of quasi-arithmetic mean, the problem aimed to solve Equation (1.2), which later became known as the Matkowski-Sutô problem, namely M-S problem in short. Matkowski mentioned that he could solve the problem by making some regularity and smoothness assumptions about unknown functions, but these assumptions were not explicitly included in the equation of the problem.

Another noteworthy aspect of the M-S problem is its relevance to Hilbert's fifth problem. In 1900, Hilbert raised 23 open questions [27]. The famous Hilbert's fifth problem will be introduced in this paper: Is every local Euclidean group a necessary Lie group? Is there any functional equation where continuity implies differentiability like Lie group? They are fairly general issues still being studied today. Aczél [2] reviewed Hilbert's fifth problem, mentioning that some examples of Járai could be included in the solution of the problem, but these examples did not include iterations. The complete solution to the M-S problem can be seen as an example with iterations [15].

The M-S problem was completely solved by Daróczy and Páles [15] in 2002, but it went through a hard journey before that. In 1999, Matkowski [52] provided all solutions under quadratically differentiable conditions. Then, Daróczy and Maksa [13] found that if one of the two functions in the M-S problem was continuously differentiable, the problem could be solved. Another important conclusion was the extension theorem in reference [14]. However, the following researches on the problem encountered a bottleneck. Emerging new results on the M-S problem could not answer the initial problem. Finally, Daróczy and Páles [15], enlightened by the approach from reference [62], proved the regularity theorem in the M-S problem. At this time, the problem was solved. In fact, the M-S equation implies that unknown functions have monotonic properties. Therefore, with the help of famous Lebesgue theorem, it can be concluded that monotonic functions defined on an interval are almost differentiable nearly everywhere. Firstly, it is proved that the solution of the equation on a non-empty subinterval $J \subset I$ is differentiable and the derivative function is not zero, which can be extended to the entire interval I. The solution to M-S problem is shown below.

Theorem 1.1 [15] *The necessary and sufficient condition for strictly monotonic, continuous function $\varphi, \psi : I \to R$ to satisfy the functional equation*

$$\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)+\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)=x+y,\ x, y \in I \tag{1.3}$$

is that one of the following conclusions can be obtained:

(1) *there is non-zero constants a, c and constants b,d, such that*

$$\varphi(x)=ax+b,\ \psi(x)=cx+d,\ x \in I;$$

(2) *there exist real constants p, a, b, c, d satisfying $acp \neq 0$, such that*

$$\varphi(x)=ae^{px}+b,\ \psi(x)=ce^{-px}+d,\ x \in I.$$

Later, the M-S problem and other related issues were extensively studied. For example, references [10, 29, 33] discussed the equation for weighted quasi-arithmetic mean. References [11, 50, 51, 76, 77] respectively considered the homogeneity, transferability, equation problem, and invariant equation of Makó-Páles. Baják and Páles used Maple software to solve the invariant equations of Stolarsky mean or Gini mean [3–5]. Liu and Matkowski [42] defined a class of iterative means and studied their invariance. In 2018, Jarczyk [33] completed a review on mean invariance, mainly introducing the invariant equations of quasi-arithmetic mean and weighted quasi-arithmetic mean, as well as other related research results. References [46, 64] discussed the inequality problem of a class of generalized Bajraktarević means derived from two functions and one measure. References [52, 61] discussed the complementarity problem of means. References [21, 22, 36, 41] studied the iterative roots and embedding flow of means. Jarczyk [32] studied the convergence problem of parameterized mean and its Gauss iteration based on which reference [35] defined random mean and its Gauss algorithm. Barczy and Burai [7] continued to study the limit theorem of random means. For more information on means and their inequalities, please refer to references [8, 9, 26, 72].

This paper first introduces the definitions of various types of means, then focuses on several types derived from two abstract functions, namely Cauchy mean, Bajraktarević mean, and other related means. It also reviews the research progress on the equality and invariance of these means in the past decade at home and abroad, and finally presents the related applications of means.

2 Definition of various means

Nowadays, many widely used means have emerged with deepened research. Some means have clear inequality relationships [9, 72], such as arithmetic mean A, harmonic mean H, geometric mean G, contra harmonic mean C, logarithmic mean L, first Seiffert mean P, identity element mean I, Neumann-Sándor mean M, second Seiffert mean T, quadratic mean Q:

$$L\left(x,y\right)=\frac{x-y}{\ln x-\ln y}, P\left(x,y\right)=\frac{x-y}{2\mathrm{arcsin}\left(\dfrac{x-y}{x+y}\right)}, I\left(x,y\right)=\frac{1}{\mathrm{e}}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}},$$

$$M\left(x,y\right)=\frac{x-y}{2\mathrm{arcsinh}\left(\dfrac{x-y}{x+y}\right)}, T\left(x,y\right)=\frac{x-y}{2\mathrm{arctan}\left(\dfrac{x-y}{x+y}\right)}, Q\left(x,y\right)=\sqrt{\frac{x^2+y^2}{2}},$$

satisfy the following famous inequality chain:

$$H<G<L<P<I<A<M<T<Q<C.$$

For the two means m_1 and m_2, the notation $m_1 < m_2$ indicates that for all x, y and $x \neq y$, $m_1(x,y)<m_2(x,y)$ holds. The part $G \leqslant A$ can be illustrated by Fig. 1. In fact, from Fig. 1, it can be concluded that $(x+y)^2=4xy+(y-x)^2$, so $\frac{x+y}{2} \geqslant \sqrt{xy}$ can be established if and only if $x=y$.

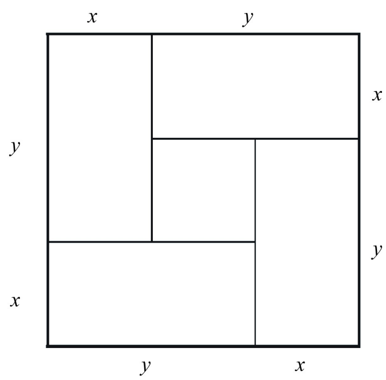

Fig. 1 $G \leqslant A$ graph

Next, more classes of means derived from abstract functions will be introduced. Firstly, we will present several types of classic means derived from one function.

Definition 2.1 [13] If $\varphi : I \to \mathbb{R}$ is a strictly monotonic, continuous function, the quasi-arithmetic mean $A_\varphi : I^2 \to I$ can be defined as follows:

$$A_\varphi(x, y) = \varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right), \ x, y \in I. \tag{2.1}$$

Quasi-arithmetic means cover arithmetic, geometric, and harmonic means. In fact, let $\varphi(x) = x$, $A_\varphi(x, y) = \frac{x+y}{2} = A(x, y)$; let $\varphi(x) = \ln x$, $A_\varphi(x, y) = \sqrt{xy} = G(x, y)$; let $\varphi(x) = \frac{1}{x}$, $A_\varphi(x, y) = \frac{2xy}{x+y} = H(x, y)$. Besides, it can be proved that the necessary and sufficient condition for the quasi-arithmetic mean to be homogeneous is that the quasi-arithmetic mean equals the power mean (or Hölder mean), which is defined as follows [15]:

$$H_p(x, y) := \begin{cases} \left(\dfrac{x^p + y^p}{2}\right)^{\frac{1}{p}}, & p \neq 0, \\ \sqrt{xy}, & p = 0, \end{cases}$$

where $x, y \in \mathbb{R}_+$, $p \in \mathbb{R}$.

For quasi-arithmetic mean, there exists the following famous Aczél theorem.

Theorem 2.1 [1] *Assume an interval $I \subset \mathbb{R}, M : I^2 \to I$. The necessary and sufficient condition for the bivariate mean M to be the arithmetic mean over interval I is that the bivariate mean M is continuous and monotonic for each variable and satisfies reflexivity, symmetry, and bisymmetry.*

After weighting, we can obtain the weighted quasi-arithmetic mean $A_{\varphi,\lambda} : I^2 \to I$, which can be defined as follows [29]:

$$A_{\varphi,\lambda}(x, y) := \varphi^{-1}(\lambda\varphi(x) + (1 - \lambda)\varphi(y)), \ x, y \in I, \tag{2.2}$$

where $\varphi : I \to \mathbb{R}$ is a strictly monotonic, continuous function, called derivative function; $\lambda \in (0, 1)$ is a real number, called a weight. Specifically, when $\lambda = \frac{1}{2}$, $A_{\varphi,\lambda} = A_\varphi$, they are quasi-arithmetic mean.

Definition 2.2 [54] If $\varphi : I \to \mathbb{R}$ is a strictly monotonic, continuous function on I, then Lagrange mean $L_\varphi : I^2 \to I$ can be defined as follows:

$$L_\varphi(x,y) := \begin{cases} \varphi^{-1}\left(\frac{1}{y-x}\int_x^y \varphi(t)\,\mathrm{d}t\right), & x \neq y, \\ x, & x = y, \end{cases}$$

where $x, y \in I$.

Combining weighted arithmetic mean and Lagrange mean, in 2008 Makó and Páles [50] proposed a new mean, later known as the Makó-Páles mean. Assume $\varphi : I \to \mathbb{R}$ is a strictly monotonic, continuous function, and μ is the Borel probability measure of the interval [0,1], Makó-Páles mean $M_{\varphi,\mu} : I^2 \to I$ is defined as follows:

$$M_{\varphi,\mu}(x,y) := \varphi^{-1}\left(\int_0^1 \varphi(tx + (1-t)y)\,\mathrm{d}\mu(t)\right),\ x, y \in I.$$

In particular, if $\mu = \lambda\delta_1 + (1-\lambda)\delta_0$, then $\mathcal{M}_{\varphi,\mu} = A_{\varphi,\lambda}$ is the weighted quasi-arithmetic mean. If μ is the Lebesgue measure on [0,1], then the mean $M_{\varphi,\mu} = L_\varphi$ is a Lagrange mean.

Next, we will introduce several generalized forms of the above means, which are derived from two functions.

Matkowski mean [41] is a generalized form of quasi-arithmetic mean, and its definition is

$$A_{f,g}(x,y) = (f+g)^{-1}(f(x) + f(y)),\ x, y \in I,$$

where $f, g : I \to \mathbb{R}$ is continuous with the same monotonicity, and $f + g$ is strictly monotonic. When $f = \lambda\varphi,\ g = (1-\lambda)\varphi$, the mean becomes a weighted quasi-arithmetic mean, that is $A_{f,g} = A_{\varphi,\lambda}$.

Bajraktarević mean is another generalization of quasi-arithmetic mean, defined as follows.

Definition 2.3 [6] If $\varphi, \psi : I \to \mathbb{R}$ is a continuous function and satisfies $\psi(x) \neq 0,\ x \in I,\ \frac{\varphi}{\psi}$ is a one-to-one mapping, then the Bajraktarević mean generated by φ and ψ is defined as

$$B_{\varphi,\psi}(x,y) := \left(\frac{\varphi}{\psi}\right)^{-1}\left(\frac{\varphi(x) + \varphi(y)}{\psi(x) + \psi(y)}\right),\ x, y \in I. \tag{2.3}$$

Let $\alpha = \frac{\varphi}{\psi}$. Bajraktarević mean can be rewritten as

$$B_\psi^\alpha(x,y) = \alpha^{-1}\left(\frac{\psi(x)}{\psi(x) + \psi(y)}\alpha(x) + \frac{\psi(y)}{\psi(x) + \psi(y)}\alpha(y)\right).$$

So, Bajraktarević mean is also called functional weighted quasi-arithmetic mean.

In Equation (2.3), suppose $\frac{\varphi}{\psi} = \mathrm{id}|_I$, and Bajraktarević mean $B_{\varphi,\psi}$ turns into Beckenbach-Gini mean [9]:

$$B_{\psi}(x,y) := \frac{x\psi(x) + y\psi(y)}{\psi(x) + \psi(y)}. \tag{2.4}$$

In Equation (2.3), let $\varphi(x) = x^s, \psi(x) = x^r$. Then Bajraktarević mean $B_{\varphi,\psi}$ turns into Gini mean

$$G_{s,r}(x,y) := \begin{cases} \left(\dfrac{x^s + y^s}{x^r + y^r}\right)^{\frac{1}{s-r}}, & s \neq r, \\ \exp\left(\dfrac{x^s \ln x + y^s \ln y}{x^s + y^s}\right), & s = r, \end{cases}$$

where $x, y \in R_+$.

Cauchy mean is a kind of generalized Lagrange mean, and its definition is shown below.

Definition 2.4 [53] If $\varphi, \psi : I \to \mathbb{R}$ is a differentiable function satisfying $\psi'(x) \neq 0, x \in I$, and $\frac{\varphi'}{\psi'}$ is bijective, then Cauchy mean $C_{\varphi,\psi} : I^2 \to I$ can be defined as follows:

$$C_{\varphi,\psi}(x,y) = \begin{cases} \left(\dfrac{\varphi'}{\psi'}\right)^{-1} \left(\dfrac{\varphi(x) - \varphi(y)}{\psi(x) - \psi(y)}\right), & x \neq y, \\ x, & x = y. \end{cases} \tag{2.5}$$

Let $f = \varphi'$, $g = \psi'$, Cauchy mean can be equivalently written as

$$C_{f,g}(x,y) := \left(\frac{f}{g}\right)^{-1} \left(\frac{\int_0^1 f(tx + (1-t)y)\,\mathrm{d}t}{\int_0^1 g(tx + (1-t)y)\,\mathrm{d}t}\right), \ x, y \in I.$$

In particular, when $\psi = id$ or $g = 1$, this mean turns into Lagrange mean. In addition, if in Equation (2.5), we take

$$\varphi(x) := \begin{cases} x^p, & p \neq 0, \\ \ln x, & p = 0; \end{cases} \qquad \psi(x) := \begin{cases} x^q, & q \neq 0, \\ \ln x, & q = 0, \end{cases}$$

then Cauchy mean is Stolarsky mean.

$$S_{p,q}(x,y) := \begin{cases} \left(\dfrac{q(x^p - y^p)}{p(x^q - y^q)}\right)^{\frac{1}{p-q}}, & pq(p-q) \neq 0, \\ \left(\dfrac{x^p - y^p}{p(\ln x - \ln y)}\right)^{\frac{1}{p}}, & p \neq 0, q = 0, \\ \left(\dfrac{q(\ln x - \ln y)}{x^q - y^q}\right)^{-\frac{1}{q}}, & q \neq 0, p = 0, \\ \exp\left(\dfrac{x^p \ln x - y^p \ln y}{x^p - y^p} - \dfrac{1}{p}\right), & q \neq 0, p = q, \\ \sqrt{xy}, & p = q = 0, \end{cases}$$

where $S_{0,1} = \frac{x-y}{\ln x - \ln y}$ is a logarithmic mean, $S_{2p,p} = (\frac{x^p+y^p}{2})^{\frac{1}{p}}$ is a power mean, and $S_{1,1} = \frac{1}{\mathrm{e}}(\frac{x^x}{y^y})^{\frac{1}{x-y}}$ is an identity element mean.

In 2008, Losonczi and Páles [46] introduced a new generalized quasi-arithmetic mean with integral measures containing two derived functions and a probability measure. It is a broader class of means, including the quasi-arithmetic mean, Lagrange mean, Bajraktarević mean, and Cauchy mean that we mentioned above. The specific definition is as follows.

Definition 2.5 [46] Let $f, g : I \to \mathbb{R}$ be a continuous function on a given interval I, g is a monotonic positive function, $\frac{f}{g}$ is a monotonic continuous function, and μ is the Borel probability measure on [0,1]. The quasi-arithmetic mean in the broad sense (also called generalized Bajraktarević mean) $M_{f,g;\mu} : I^2 \to I$ is defined as follows:

$$M_{f,g;\mu}(x,y) := \left(\frac{f}{g}\right)^{-1}\left(\frac{\int_0^1 f(tx+(1-t)y)\,\mathrm{d}\mu(t)}{\int_0^1 g(tx+(1-t)y)\,\mathrm{d}\mu(t)}\right),\ x,y\in I. \tag{2.6}$$

Specially, when $\mu = \frac{\delta_0+\delta_1}{2}$, this mean turns into a Bajraktarević mean; when μ is a Lebesgue measure, this mean turns into a Cauchy mean.

3 Equality problems of means

3.1 Prerequisites

To facilitate the following description, some definitions, notations, and lemmas will be focused on first.

Let $I \subset \mathbb{R}$ be a non-empty open interval, and $\mathcal{CM}(I)$ and $\mathcal{CP}(I)$ respectively represent the sets of all continuous, strictly monotonic functions and continuous, positive functions on interval I.

If there are constants $a, b\in\mathbb{R}, a \neq 0$, such that the function f, g satisfies $g = af + b$, then f and g are called equivalent, abbreviated as

$$f \sim g.$$

If there are $a, b, c, d\in\mathbb{R}$, with $ad - bc \neq 0$, such that

$$h = af + bg,\ k = cf + dg, \tag{3.1}$$

then we say function pairs (f,g) and (h,k) are equivalent, notated as

$$(f, g) \sim (h, k).$$

Assume that f, g is an nth-order continuously differentiable function and $f'g - fg'$ is non-zero everywhere on interval I, then for any $0 \leqslant i, j \leqslant n$, define the Wronski determinant as

$$W_{f,g}^{i,j} := \begin{vmatrix} f^{(i)} & f^{(j)} \\ g^{(i)} & g^{(j)} \end{vmatrix},\ i,j \in \mathbb{N}.$$

Let

$$\Phi_{f,g} := \frac{W_{f,g}^{(0,2)}}{W_{f,g}^{(0,1)}},\ \Psi_{f,g} := -\frac{W_{f,g}^{(1,2)}}{W_{f,g}^{(0,1)}}. \tag{3.2}$$

Since f, g is the solution to equation

$$\begin{vmatrix} Y & Y' & Y'' \\ f & f' & f'' \\ g & g' & g'' \end{vmatrix} = 0, \tag{3.3}$$

and the determinant at the left side of Equation (3.3) can be expanded in the first row to obtain $Y'' = \Phi_{f,g}Y' + \Psi_{f,g}Y$, so f, g forms the basic set of solutions to the second-order homogeneous linear differential equation

$$Y'' = \Phi_{f,g}Y' + \Psi_{f,g}Y.$$

Given the real number α, define function $S_\alpha, C_\alpha : \mathbb{R} \to \mathbb{R}$

$$S_\alpha(t) := \sum_{k=0}^{\infty} \frac{\alpha^k t^{2k+1}}{(2k+1)!} = \begin{cases} \dfrac{\sin\left(\sqrt{-\alpha}t\right)}{\sqrt{-\alpha}}, & \alpha < 0, \\ t, & \alpha = 0, \\ \dfrac{\sinh\left(\sqrt{\alpha}t\right)}{\sqrt{\alpha}}, & \alpha > 0; \end{cases} \tag{3.4}$$

$$C_\alpha(t) := \sum_{k=0}^{\infty} \frac{\alpha^k t^{2k}}{(2k)!} = \begin{cases} \dfrac{\cos\left(\sqrt{-\alpha}t\right)}{\sqrt{-\alpha}}, & \alpha < 0, \\ 1, & \alpha = 0, \\ \dfrac{\cosh\left(\sqrt{\alpha}t\right)}{\sqrt{\alpha}}, & \alpha > 0. \end{cases} \tag{3.5}$$

Note that the above functions form a basic set of solutions to the second-order homogeneous linear differential equation

$$Y'' = \alpha Y.$$

According to the famous trigonometric and hyperbolic identities, it can be verified that functions S_α and C_α satisfy the identity

$$C_\alpha^2 - \operatorname{sign}(\alpha) S_\alpha^2 = 1. \tag{3.6}$$

Suppose that the first derivative of function $f : I \to \mathbb{R}$ is non-zero and the third derivative is differentiable, the definition of Schwarz derivative is as follows:

$$S_f = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{3.7}$$

By applying the knowledge of determinants and homogeneous linear equations, it can be concluded that the Schwarz derivative of a function is a constant, which is equivalent to the ratio of the solutions of two harmonic oscillator equations.

Lemam 3.1 [25] *Let $\gamma \in \mathbb{R}$, $f : I \to \mathbb{R}$ be a third-order differentiable function, where f' cannot take 0 on I. Then the necessary and sufficient condition for the third-order differential equation*

$$S_f = -2\gamma$$

to establish is that there exists a quadratically differentiable function $u, v : I \to \mathbb{R}$ satisfying that v cannot take 0 on I, and

$$u'' = \gamma u,\ v'' = \gamma v,\ f = \frac{u}{v}.$$

Specially, when $\gamma = 0$, the necessary and sufficient condition for $S_f = 0$ to hold is the existence of four constants $a, b, c, d \in \mathbb{R}$ satisfying $ad \neq bc$, $0 \notin cI + d$, such that

$$f(x) = \frac{ax + b}{cx + d}.$$

Lemma 3.2 [24] *Assume $f, g : I \to \mathbb{R}$ is third-order differentiable on interval I, and the first derivative is non-zero. The necessary and sufficient condition for $S_f = S_g$ to hold on interval I is that there exist $a, b, c, d \in \mathbb{R}$, where $ad \neq bc$, such that $cf + d$ is a positive function on interval I, and*

$$g(x) = \frac{af(x) + b}{cf(x) + d},\ x \in I.$$

The following introduces the relevant knowledge of probability measures. Assume that μ is a Borel probability measure on the interval [0, 1], the kth-order origin moment and kth-order central moment of μ are defined as

$$\widehat{\mu}_k := \int_0^1 t^k \mathrm{d}\mu(t),\ \mu_k := \int_0^1 (t - \widehat{\mu}_1)^k \mathrm{d}\mu(t),\ k \in \mathbb{N} \cup \{0\}.$$

Obviously, $\widehat{\mu}_0 = \mu_0 = 1$, $\mu_1 = 0$, and $\mu_{2k} \geqslant 0, k \in \mathbb{N}$. The necessary and sufficient condition for $\mu_{2k} = 0$ is that μ is a Dirac measure, namely $\mu = \delta_{\widehat{\mu}_1}$. It can be obtained from the binomial theorem

$$\mu_k = \int_0^1 \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} t^i \widehat{\mu}_1^{k-i} \mathrm{d}\mu(t) = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} \widehat{\mu}_i \widehat{\mu}_1^{k-i},\ k \in \mathbb{N};$$

$$\widehat{\mu}_k = \int_0^1 ((t - \widehat{\mu}_1) + \widehat{\mu}_1)^k \mathrm{d}\mu(t) = \sum_{i=0}^k \binom{k}{i} \mu_i \widehat{\mu}_1^{k-i},\ k \in \mathbb{N}.$$

If $\mu(A)=\mu((2\widehat{\mu}_1-A)\cap[0,1])$ holds for all Borel sets $A\subseteq[0,1]$, then we say μ is symmetric with respect to $\widehat{\mu}_1$. It is easy to prove that the sufficient and necessary condition for $\mu_{2k-1}=0$ to hold for all $k\in\mathbb{N}$ is that μ is symmetric with respect to $\widehat{\mu}_1$.

If $\mu^*(A)=\mu(1-A)$ holds for all Borel sets $A\subseteq[0,1]$, then we say μ^* is the conjugate measure of μ. If one measure satisfies to be self-conjugate, namely $\mu(A)=\mu(1-A), \mu^*=\mu$, then we say this measure is symmetric. Hence, $\mu^s=\frac{\mu+\mu^*}{2}$ is symmetric.

Lemma 3.3 [47] *Let μ be a Borel probability measure on interval* [0, 1]. *Then* $\widehat{\mu}_1+\widehat{\mu}_1^*=1$, *and*

$$\mu_n^*=(-1)^n\mu_n.$$

If μ is symmetric, then $\widehat{\mu}_1=\frac{1}{2}, \mu_{2k-1}=0, k\in\mathbb{N}$.

Reference [68] defined a kind of symmetric measure: given two positive real numbers l,p, define the probability measure $\pi:=\pi(l,p)$ satisfying

$$\pi_0:=1, \widehat{\pi}_1:=\frac{1}{2}, \pi_{2k-1}:=0, \pi_{2n}:=\frac{(2k)!}{k!}l^k p^{\langle k\rangle},\ k\in\mathbb{N}, \tag{3.8}$$

where

$$p^{\langle k\rangle}:=\prod_{i=0}^{k-1}\frac{p}{1+ip},\ k\in\mathbb{N}.$$

In particular, $\frac{1}{2}(\delta_0+\delta_1)=\pi\left(\frac{1}{16},2\right)$, and Lebesgue measure $\lambda=\pi\left(\frac{1}{16},\frac{2}{3}\right)$.

3.2 Equality problems of Bajraktarević means

Given $n\in\mathbb{N}, n\geqslant 2$, define the n-dimensional weight vector set Λ_n as

$$\Lambda_n:=\{(\lambda_1,\ldots,\lambda_n)\in\mathbb{R}^n|\lambda_1,\ldots,\lambda_n\geqslant 0, \lambda_1+\cdots+\lambda_n>0\}.$$

The n-variable weighted quasi-arithmetic mean $A_{\varphi,\lambda}: I^n\to\mathbb{R}$ is defined as follows:

$$A_{\varphi;\lambda}(x_1,\ldots,x_n):=\varphi^{-1}\left(\frac{\lambda_1\varphi(x_1)+\cdots+\lambda_n\varphi(x_n)}{\lambda_1+\cdots+\lambda_n}\right),$$

where $\varphi\in\mathcal{CM}(I)$ is a derived function, and the weight $\lambda=(\lambda_1,\ldots,\lambda_n)\in\Lambda_n$. Specially, when $\lambda_1=\cdots=\lambda_n=1$, we call it a quasi-arithmetic mean, denoted by A_φ.

Among many concept generalizations of weighted quasi-arithmetic mean, the most important one is the Bajraktarević mean. In 1958, Bajraktarević [6] defined the weighted quasi-arithmetic mean in an n-variable function (later known as Bajraktarević mean), which was defined as follows:

$$B_{f,g;\lambda}(x_1,\ldots,x_n):=\left(\frac{f}{g}\right)^{-1}\left(\frac{\lambda_1 f(x_1)+\cdots+\lambda_n f(x_n)}{\lambda_1 g(x_1)+\cdots+\lambda_n g(x_n)}\right),$$

where weight $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda_n$, $f, g : I \to \mathbb{R}$ satisfied $g \in \mathcal{CP}(I)$, $\frac{f}{g} \in \mathcal{CM}(I)$.

For all $n \geqslant 2$, the necessary and sufficient condition for two n-variable weighted arithmetic means to be equal is that the derived functions are equivalent. However, for the Bajraktarević mean, in 1958 Bajraktarević [7] proved that only when $n \geqslant 3$ and under second-order continuous differentiability, derived function pair equivalence is the necessary and sufficient condition for two Bajraktarević means to be equal. For the case of $n = 2$, the situation becomes completely different. If it is a bivariate asymmetric weighted Bajraktarević mean, it can be proven that the necessary and sufficient condition for two Bajraktarević means to be equal is that the derived functions are equivalent.

Theorem 3.1 [66] *Suppose $g, k \in \mathcal{CP}(I)$, $\frac{f}{g}, \frac{h}{k} \in \mathcal{CM}(I)$, and f, g, h, k are third-order continuously differentiable, $\lambda \in (1, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Then $B_{f,g;\lambda}(x, y) = B_{h,k;\lambda}(x, y)$. The necessary and sufficient condition for*

$$\left(\frac{f}{g}\right)^{-1} \left(\frac{\lambda f(x) + (1-\lambda) f(y)}{\lambda g(x) + (1-\lambda) g(y)}\right) = \left(\frac{h}{k}\right)^{-1} \left(\frac{\lambda h(x) + (1-\lambda) h(y)}{\lambda k(x) + (1-\lambda) k(y)}\right)$$

to hold is $(f, g) \sim (h, k)$.

For the equality problem of the Bajraktarević mean with bivariate symmetry, the situation is much more complex. We need to solve $f, g, h, k : I \to \mathbb{R}$, such that the equation below is established:

$$B_{f,g}(x, y) = B_{h,k}(x, y), \ x, y \in I, \tag{3.9}$$

that is

$$\left(\frac{f}{g}\right)^{-1} \left(\frac{f(x) + f(y)}{g(x) + g(y)}\right) = \left(\frac{h}{k}\right)^{-1} \left(\frac{h(x) + h(y)}{k(x) + k(y)}\right), \ x, y \in I.$$

It is easy to prove that if $(f, g) \sim (h, k)$, then Equation (3.9) holds. In fact, Bajraktarević mean

$$B_{f,g}(x, y) = \left(\frac{f}{g}\right)^{-1} \left(\frac{f(x) + f(y)}{g(x) + g(y)}\right), \ x, y \in I$$

is the unique solution to

$$\begin{vmatrix} f(z) & f(x) + f(y) \\ g(z) & g(x) + g(y) \end{vmatrix} = 0, \tag{3.10}$$

i.e., $z = B_{f,g}(x, y)$ in the above equation. It can be obtained from $(f, g) \sim (h, k)$ that

$$\begin{vmatrix} h(B_{h,k}(x, y)) & h(x) + h(y) \\ k(B_{h,k}(x, y)) & k(x) + k(y) \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} f(B_{h,k}(x, y)) & f(x) + f(y) \\ g(B_{h,k}(x, y)) & g(x) + g(y) \end{vmatrix} = 0,$$

and from the uniqueness of the solution to Equation (3.10), it can be obtained that $B_{f,g}(x, y) = B_{h,k}(x, y)$ holds. Therefore, $(f, g) \sim (h, k)$ is the sufficient condition but not necessarily the necessary condition for $B_{f,g}(x,y)=B_{h,k}(x,y)$ to be established. In 1999, Losonczi [43] obtained 32 possible solutions to Equation (3.9) under the sixth-order continuously differentiability with Maple V software.

When the denominator function of a Bajraktarević mean in Equation (3.9) is a non-zero constant function, the problem becomes an equality problem between the quasi-arithmetic mean and the Bajraktarević mean. In 2004, Daróczy et al. [14] studied this special case by solving F, G, Φ, such that the following equation holds:

$$\left(\frac{G}{F}\right)^{-1}\left(\frac{G\left(u\right)+G\left(v\right)}{F\left(u\right)+F\left(v\right)}\right)=\Phi^{-1}\left(\frac{\Phi\left(u\right)+\Phi\left(v\right)}{2}\right),\ u,v\in I.$$

Let $u=\Phi^{-1}\left(x\right)$, $v=\Phi^{-1}\left(y\right)$, $J=\Phi\left(I\right)$, and $g=G\circ\Phi^{-1}$, $f=F\circ\Phi^{-1}$, $\varphi=\frac{g}{f}$. Then the above equation is equivalent to

$$\varphi\left(\frac{x+y}{2}\right)\left(f\left(x\right)+f\left(y\right)\right)=\varphi\left(x\right)f\left(x\right)+\varphi\left(y\right)f\left(y\right),\ x,y\in J. \tag{3.11}$$

Reference [14] firstly proved the following regularity theorem.

Theorem 3.2 [14, Theorem 1] *Let $J\subset\mathbb{R}$ be a non-empty open interval, and $\varphi:J\to\mathbb{R}$ be a strictly monotonic, continuous function, $f:J\to(0,+\infty)$. If Equation* (3.11) *holds for all $x,y\in J$, then φ and f are arbitrarily differentiable finitely many times and for all $x\in J$, $\varphi'\left(x\right)\neq 0$ is established.*

Then, by combining [43, Theorems 3 and 4], the solution to Equation (3.11) can be obtained.

Theorem 3.3 [14, Theorem 2] *Assume that φ is strictly monotonic, continuous, and f is a positive function on the open interval J. The necessary and sufficient condition for (φ,f) to be the solution of Equation* (3.11) *is that one of the following conclusions holds*:

(1) $\varphi\left(x\right)=Ax+D, f\left(x\right)=E, x\in J$;

(2) $\varphi\left(x\right)=\frac{A}{x+C}, f\left(x\right)=E(x+C), x\in J$;

(3) $\varphi\left(x\right)=A\tanh\left(Bx+C\right)+D, f\left(x\right)=E\cosh(Bx+C), x\in J$;

(4) $\varphi\left(x\right)=A\coth\left(Bx+C\right)+D, f\left(x\right)=E\sinh(Bx+C)+D, x\in J$;

(5) $\varphi\left(x\right)=A\tan(Bx+C)+D, f\left(x\right)=E\cos(Bx+C), x\in J$;

(6) $\varphi\left(x\right)=A\exp(-2Bx)+D, f\left(x\right)=E\exp\left(Bx\right), x\in J$,

where $A,B,C,D,E\in\mathbb{R}$ are random constants that satisfy $ABE\neq 0$, such that $f\left(x\right)>0$, and $x\in J$ holds.

Finally, as an application, reference [14] provided a necessary and sufficient condition for a function weighted arithmetic mean (also known as a Beckenbach-Gini mean) to be a quasi-arithmetic mean, which could also be found in [58].

In 2018, Kiss and Páles [37] reconsidered Equation (3.11) with a new approach.

Lemma 3.4 [37, Lemma 7] *Suppose $\varphi, f: J \to \mathbb{R}$ satisfies that f is 0 nowhere, and function $g: J \to \mathbb{R}$ satisfies $g(x) = \varphi(x) f(x)$. Then Equation (3.11) is equivalent to*

$$\begin{vmatrix} f\left(\frac{x+y}{2}\right) & f(x) + f(y) \\ g\left(\frac{x+y}{2}\right) & g(x) + g(y) \end{vmatrix} = 0, \ x, y \in J.$$

With Lemma 3.4, according to the determinant differentiation rule, f, g is the solution to second-order homogeneous linear differential equation $Y'' = \alpha Y$. Therefore, it can be concluded that $(f, g) \sim (S_\alpha, C_\alpha)$. This method successfully avoids using the 32 results calculated by Maple V by Losonczi in [43] for verification, but follows a completely independent argument instead.

In 2020, Páles and Zakaria [67] considered the equality problem of weighted Bajraktarević mean and weighted quasi-arithmetic mean, and equivalently expressed the problem as a generalized form of Equation (3.11):

$$(tf(x) + (1-t) f(y)) \varphi(tx + (1-t) y) = tf(x) \varphi(x) + (1-t) f(y) \varphi(y), x, y \in I, \tag{3.12}$$

where $t \in (0,1), \varphi: I \to \mathbb{R}$ is strictly monotonic, and $f: I \to \mathbb{R}$ is a random unknown function.

On one hand, when $t = \frac{1}{2}$, with Equation (3.6) and Definition (3.2) of $\Phi_{f,g}, \Psi_{f,g}$ as well as the goal of relationship to be satisfied by two derived functions of a Bajraktarević mean. [67, Theorem 10] gives several equivalence conditions for a Bajraktarević mean to be a quasi-arithmetic mean.

Theorem 3.4 [67, Theorem 10] *Assume $f \in \mathcal{CP}(I)$, $\frac{g}{f} \in \mathcal{CM}(I)$, then the following propositions are equivalent.*

(1) There exists a continuous, strictly monotonic function $h: I \to \mathbb{R}$, such that

$$\left(\frac{g}{f}\right)^{-1}\left(\frac{g(x) + g(y)}{f(x) + f(y)}\right) = h^{-1}\left(\frac{x+y}{2}\right), \ x, y \in I.$$

(2) There exist real numbers α, β, γ, such that $\alpha f^2 + \beta fg + \gamma g^2 = 1$.

(3) If f, g is first-order continuously differentiable, and $f'g - fg'$ is not zero everywhere on interval I, then $h = \int W^{1,0}_{f,g}$.

(4) If f, g is second-order continuously differentiable, and $f'g - fg'$ is not zero everywhere on interval I, then there exists real number δ, such that

$$W^{2,1}_{f,g} = \delta (W^{1,0}_{f,g})^3.$$

(5) If f, g is second-order continuously differentiable, and $f'g - fg'$ is not zero everywhere on interval I, then $\Psi_{f,g}$ is differentiable, and

$$\Psi'_{f,g} = 2\Phi_{f,g}\Psi_{f,g}.$$

On the other hand, for asymmetric cases, i.e., $t\neq\frac{1}{2}$, [67, Theorem 11] provides an equivalent relationship between bivariate mean and multivariate mean.

Theorem 3.5 [67, Theorem 11] *Suppose $f\in\mathcal{CP}(I)$, $\frac{g}{f}\in\mathcal{CM}(I)$, then the following propositions are equivalent.*
(1) There exists $h\in\mathcal{CM}(I)$, such that

$$B_{g,f;\lambda}(x_1,\dots,x_n)=A_{h,\lambda}(x_1,\dots,x_n) \tag{3.13}$$

holds for all $n\in\mathbb{N}$, $(x_1,\dots,x_n)\in I^n$ and $\lambda=(\lambda_1,\dots,\lambda_n)\in\Lambda_n$.
(2) There exists $h\in\mathcal{CM}(I)$, such that

$$B_{g,f}(x_1,\dots,x_n)=A_h(x_1,\dots,x_n) \tag{3.14}$$

holds for all $n\in\mathbb{N}$ and $(x_1,\dots,x_n)\in I^n$.
(3) There exists $h\in\mathcal{CM}(I)$ and $n\geqslant 3$, such that Equation (3.14) holds for all $(x_1,\dots,x_n)\in I^n$.
(4) There exists $h\in\mathcal{CM}(I)$, such that Equation (3.13) holds for all $(x_1,x_2)\in I^2$ and $\lambda\in\Lambda_2$.
(5) There exist $t\in(0,\frac{1}{2})\cup(\frac{1}{2},1)$ and $h\in\mathcal{CM}(I)$, such that Equation (3.13) holds for all $(x_1,x_2)\in I^2$ and $\lambda=(t,1-t)$.
(6) There exist constants $a,b\in\mathbb{R}$, such that

$$af+bg=1.$$

(7) If f,g is second-order continuously differentiable, $\Psi_{f,g}=0$.

Recently, Losonczi, Páles, and Zakaria [48, 69] have re-examined the equality problem of the bivariate Bajraktarević mean. [48, Theorem 15] presented some equivalent conditions for Equation (3.9) to hold under the condition of sixth-order continuous differentiability.

Theorem 3.6 *Assume $f,g,h,k:I\to\mathbb{R}$ satisfy $g,k\in\mathcal{CP}(I)$, and $g,k\in\mathcal{CP}(I)$, $(\frac{f}{g})'$, $(\frac{h}{k})'$ cannot take 0 on interval I as a sixth-order continuously differentiable function. Then the following conditions are equivalent.*
(1) Means $B_{f,g}$ and $B_{h,k}$ satisfy Equation (3.9).
(2) $(f,g)\sim(h,k)$ holds, or there exist constants α,β,γ, such that

$$W_{f,g}^{(1,2)}=\alpha\left(W_{f,g}^{(0,1)}\right)^3, W_{h,k}^{(1,2)}=\beta\left(W_{h,k}^{(0,1)}\right)^3, W_{h,k}^{(0,1)}=\gamma W_{f,g}^{(0,1)}$$

holds on interval I.
(3) $(f,g)\sim(h,k)$ holds, or there exist real constants a,b,c,A,B,C,γ, such that

$$af^2+bfg+cg^2=1, Ah^2+Bhk+Ck^2=1, W_{h,k}^{(0,1)}=\gamma W_{f,g}^{(0,1)}$$

holds on interval I.
(4) $(f,g)\sim(h,k)$ holds, or there exist two real polynomials P and Q of degree at most two and positive function respectively on the range of $\frac{f}{g}$ and $\frac{h}{k}$, and still

there exist real constants γ, δ, such that

$$g = \frac{1}{\sqrt{P}} \circ \frac{f}{g}, k = \frac{1}{\sqrt{Q}} \circ \frac{h}{k}, \int \frac{1}{Q} \circ \frac{h}{k} = \gamma \left(\int \frac{1}{P} \right) \circ \frac{f}{g} + \delta.$$

(5) $(f,g) \sim (h,k)$ holds, or there exists a continuous, strictly monotonic function $\omega : I \to \mathbb{R}$ and real numbers α, β, such that

$$(f,g) \sim (S_\alpha \circ \omega, C_\alpha \circ \omega),\ (h,k) \sim (S_\beta \circ \omega, C_\beta \circ \omega).$$

(6) $(f,g) \sim (h,k)$ holds, or let $\omega := \int W^{(0,1)}_{f,g},\ B_{f,g} = B_{\omega,1} = B_{h,k}$ hold on I^2.
(7) $(f,g) \sim (h,k)$ holds, or there exists a continuous, strictly monotonic function $\omega : I \to \mathbb{R}$, such that $B_{f,g} = B_{\omega,1} = B_{h,k}$ holds on I^2.

In reference [69], Equation (3.9) was first equivalently transformed into an equality problem between Bajraktarević mean and function weighted arithmetic mean through appropriate variable substitution (see [69, Lemma 2]), where the equivalent equation contains only three unknown functions. Then, it proved the regularity theorem of equivalent equations, which stated that continuously differentiable implied differentiable finitely many times. Finally, under the condition of continuous differentiability, Theorem 3.6(1), (3)–(7) were mutually equivalent to each other. Furthermore, under the condition of quadratic continuous differentiability, Theorem 3.6(2) was equivalent to all the rest.

In 2020, Grünwald and Páles [24] studied the equality problem of a class of weighted Bajraktarević mean

$$f^{-1}\left(\frac{p_1(x_1) f(x_1) + \cdots + p_n(x_n) f(x_n)}{p_1(x_1) + \cdots + p_n(x_n)}\right) = g^{-1}\left(\frac{q_1(x_1) g(x_1) + \cdots + q_n(x_n) g(x_n)}{q_1(x_1) + \cdots + q_n(x_n)}\right). \tag{3.15}$$

With Lemma 3.2 for the bivariate asymmetric Bajraktarević mean, that is $n = 2$, $p_1 \neq p_2$ in the above equation, under the third-order differentiability, it is obtained that the necessary and sufficient condition for (3.15) to hold is the existence of positive function $a, b, cd \in \mathbb{R}, ad \neq bc, cf + d$ on interval I, such that

$$g = \frac{af + b}{cf + d},\ q_1 = (cf + d)p_1,\ q_2 = (cf + d)p_2.$$

When $n \geqslant 3$, under the sixth-order differentiability, the necessary and sufficient condition for (3.15) to hold is the existence of positive function $a, b, cd \in \mathbb{R}$, $ad \neq bc, cf + d$ on interval I, such that

$$g = \frac{af + b}{cf + d},\ (q_1, \ldots, q_n) = (cf + d)(p_1, \ldots, p_n),$$

or existence of two polynomials P, Q of degree at most two and constants $\alpha, \beta \in \mathbb{R}$, such that

$$g=\left(\int\frac{1}{Q}\right)^{-1}\circ\left(\alpha\left(\int\frac{1}{P}\right)\circ f+\beta\right),\ p=P^{-\frac{1}{2}}\circ f,\ q=Q^{-\frac{1}{2}}\circ g.$$

3.3 Equality problems of Cauchy mean

To define the $n(n\geqslant 2)$-variate Cauchy mean, first review the definition of the difference quotient of function $f: I\to\mathbb{R}$ with respect to different points x_i on I as follows:

$$[x_1]_f := f(x_1),$$
$$[x_1,x_2]_f := \frac{f(x_1)-f(x_2)}{x_1-x_2},$$
$$[x_1,\ldots,x_n]_f := \frac{[x_1,\ldots,x_{n-1}]_f-[x_2,\ldots,x_n]_f}{x_1-x_n}, n\geqslant 3.$$

In 1984, Leach and Sholander [40] proposed the Cauchy mean value theorem for difference quotient.

Theorem 3.7 *Suppose $x_1\leqslant\cdots\leqslant x_n$, and there exist $f^{(n-1)}, g^{(n-1)}$ and $g^{(n-1)}(u)\neq 0, u\in[x_1,x_n]$. Then there are $t\in[x_1,x_n]$ (if $x_1<x_n$, then $t\in(x_1,x_n)$), such that*

$$\frac{[x_1,\ldots,x_n]_f}{[x_1,\ldots,x_n]_g}=\frac{f^{(n-1)}(t)}{g^{(n-1)}(t)}.$$

Assume $\frac{f^{(n-1)}}{g^{(n-1)}}$ is invertible, then the $n(n\geqslant 2)$-variate Cauchy mean can be defined as follows:

$$C_{f,g}(x_1,\ldots,x_n) := \left(\frac{f^{(n-1)}}{g^{(n-1)}}\right)^{-1}\left(\frac{[x_1,\ldots,x_n]_f}{[x_1,\ldots,x_n]_g}\right). \tag{3.16}$$

When $n\geqslant 3$, Losonczi [44] obtained the necessary and sufficient condition for two Cauchy means to be equal.

Theorem 3.8 *Assume $n\geqslant 3$, $f,g,F,G: I\to\mathbb{R}$ is the $(n+2)$ th-order continuously differentiable on interval I, functions $g^{(n-1)}$, $G^{(n-1)}$, $\left(\frac{f^{(n-1)}}{g^{(n-1)}}\right)'$, $\left(\frac{F^{(n-1)}}{G^{(n-1)}}\right)'$ are zero nowhere on interval I. Then the two Cauchy means are equal, that is the necessary and sufficient condition for*

$$C_{f,g}(x_1,\ldots,x_n)=C_{F,G}(x_1,\ldots,x_n)$$

to hold for all $x_1,\ldots,x_n\in I$ is

$$(f^{(n-1)},g^{(n-1)})\sim(F^{(n-1)},G^{(n-1)}).$$

For the case of $n=2$, due to the fact that the odd-order derivative of the equation such as first order, third order and fifth order cannot provide independent conclusions, only the conclusions corresponding to the second-order, fourth-order, and sixth-order derivatives can be used for solving. In 2003, Losonczi [45] studied this problem. Under the assumption of the seventh-order continuous differentiability, two equal equations of bivariate Cauchy means were obtained as follows:

$$C_{f,g}(x,y) = C_{F,G}(x,y),\ x,y\in I,$$

namely 33 types of solutions to

$$\left(\frac{f'}{g'}\right)^{-1}\left(\frac{f(x)-f(y)}{g(x)-g(y)}\right) = \left(\frac{F'}{G'}\right)^{-1}\left(\frac{F(x)-F(y)}{G(x)-G(y)}\right),\ x,y\in I. \tag{3.17}$$

Firstly, the equation is appropriately transformed into an equivalent equation containing only three unknown functions. Then, a sixth-order ordinary differential equation is obtained by using the second-order, fourth-order, and sixth-order partial derivatives of the equivalent equation on the diagonal line. The solution involves the Riccati equation

$$4\Phi' - 2\Phi = c(\Psi')^{\frac{4}{3}}, \tag{3.18}$$

where c is random constant, and Ψ is the solution to the nonlinear equation

$$9\frac{\Psi^{(4)}}{\Psi'} - 45\frac{\Psi'''}{\Psi'}\frac{\Psi''}{\Psi'} + 40\left(\frac{\Psi''}{\Psi'}\right)^3 = 0, \tag{3.19}$$

where

$$h = \frac{f'}{g'}\circ\left(\frac{F'}{G'}\right)^{-1},\ \Phi = \frac{h''}{h'},\ \Psi = G\circ\left(\frac{F'}{G'}\right)^{-1}.$$

The left side of Equation (3.18) is essentially 8 times the Schwarz derivative S_h (defined in (3.7)) of function h,

$$S(x) := 4\frac{h'''(x)}{h'(x)} - 6\left(\frac{h''(x)}{h'(x)}\right)^2 = 8S_h(x).$$

If $S(x)=0$ and Equation (3.17) holds, it can be obtained $(f',g')\sim(F',G')$. If $S(x)\neq 0$, the Riccati equation $4\Phi'-2\Phi=S$ can be transformed into a second-order homogeneous differential equation $z''+\frac{1}{8}Sz=0$ through appropriate variable substitution, and 32 types of solutions can be obtained by solving Equation (3.19).

Theorem 3.9 [45] *Suppose* $f,g,F,G: I\to\mathbb{R}$ *is a seventh-order differentiable function. Then the necessary and sufficient condition for Equation* (3.17) *to hold is*

$$(f',g')\sim(F',G'),$$

or f,g,F,G *contains* 32 *different solutions* (*see* [45, (41)–(72)] *for details*).

Matkowski subsequently reduced the seventh-order continuous differentiability in the condition of conclusion to first-order continuous differentiability [53]. In 2021, Losonczi et al. [48] provided a new representation for the equality problem of bivariate Cauchy means by a novel method.

Theorem 3.10 [48] *Suppose $f, g, F, G : I \to \mathbb{R}$ is a seventh-order continuously differentiable function, then the following propositions are equivalent.*
(1) Two bivariate Cauchy means are equal, that is Equation (3.17) holds;
(2) $\Phi_{f',g'} = \Phi_{F,G}$ and there exist real numbers $a, b \in \mathbb{R}$, such that

$$\Psi_{f',g'} = a\big(W^{1,0}_{f'g'}\big)^{\frac{2}{3}} + \frac{1}{3}\Phi' - \frac{2}{9}\Phi^2, \Psi_{F',G'} = b\big(W^{1,0}_{f',g'}\big)^{\frac{2}{3}} + \frac{1}{3}\Phi' - \frac{2}{9}\Phi^2;$$

(3) $(f', g') \sim (F', G')$, or there exist real numbers a, b, c, A, B, C, γ, such that

$$af'^2 + bf'g' + cg'^2 = (W_{f',g'})^{\frac{2}{3}}, AF'^2 + BF'G' + CG'^2 = (W_{F',G'})^{\frac{2}{3}},$$

and $W^{1,0}_{F',G'} = \gamma W^{1,0}_{f',g'}$;
(4) $(f', g') \sim (F', G')$, or there exist at least two real polynomials P, Q to be positive functions on the range of $\frac{f'}{g'}$ and $\frac{F'}{G'}$, and two real constants γ, δ, such that

$$g' = \left(\frac{f'}{g'}\right)'\left(\frac{1}{\sqrt{P^3}} \circ \frac{f'}{g'}\right), G' = \left(\frac{F'}{G'}\right)'\left(\frac{1}{\sqrt{Q^3}} \circ \frac{F'}{G'}\right),$$
$$\left(\int \frac{1}{Q}\right) \circ \frac{F'}{G'} = \gamma^{\frac{1}{3}}\left(\int \frac{1}{P}\right) \circ \frac{f'}{g'} + \delta;$$

(5) $(f', g') \sim (F', G')$, or there exists a strictly monotonic, differentiable function $\varphi : I \to \mathbb{R}$ and real constants α and β, such that

$$(f', g') \sim (\varphi' \cdot S_\alpha \circ \varphi, \varphi' \cdot C_\alpha \circ \varphi), (F', G') \sim (\varphi' \cdot S_\alpha \circ \varphi, \varphi' \cdot C_\alpha \circ \varphi);$$

(6) $(f', g') \sim (F', G')$, or $C_{f,g} = A_\varphi = C_{F,G}$ holds on I^2, where $\varphi := \int \big(W^{1,0}_{f',g'}\big)^{\frac{1}{3}}$;
(7) $(f', g') \sim (F', G')$, or there exists a strictly monotonic function $\varphi : I \to \mathbb{R}$, such that $C_{f,g} = A_\alpha = C_{F,G}$ holds on I^2.

From (4) in Theorem 3.10, it can be concluded that P, Q are at most quadratic polynomials, so there are five possibilities, namely constant function, linear polynomial, and quadratic polynomial (single real root, double real root, and no real root). This indicates that there are $6 \times 6 = 36$ cases in Theorem 3.10 (4), which corresponds to the conclusion of [45] by Losonczi. From [48], the problem of equality of two Cauchy means has been completely solved.

Recently, there have been researches on the equation between Cauchy mean and other types of mean. In 2019, Kiss and Páles [37] investigated the equation between Cauchy mean and quasi-arithmetic mean.

$$\left(\frac{G'}{H'}\right)^{-1}\left(\frac{G(x)-G(y)}{H(x)-H(y)}\right)=\Phi^{-1}\left(\frac{\Phi(x)+\Phi(y)}{2}\right),\ x,y\in I. \tag{3.20}$$

Let

$$\varphi:=\frac{G'}{H'}\circ\Phi^{-1},\ f:=H\circ\Phi^{-1},\ F:=G\circ\Phi^{-1}.$$

The following equivalent equation

$$\varphi\left(\frac{x+y}{2}\right)(f(x)-f(y))=F(x)-F(y),\ x,y\in I \tag{3.21}$$

can be obtained.

Let $h>0$, function $f:I\to\mathbb{R}$. Define interval $I_h\subset I$ and function $\delta_h f: I_h\to\mathbb{R}$ to satisfy

$$I_h:=(I-h)\cap(I+h),\delta_h f(x):=f(x+h)-f(x-h),\ x\in I_h.$$

Then according to [38, Theorem 3], Equation (3.21) can be transformed into

$$\varphi\left(\frac{x+y}{2}\right)(\delta_h f(x)+\delta_h f(y))=\varphi(x)\,\delta_h f(x)+\varphi(y)\,\delta_h f(y),\ x,y\in I_h. \tag{3.22}$$

The above functional equation is consistent in form with the equivalent Equation (3.11) in the equality problem of Bajraktarević and the quasi-arithmetic mean. Therefore, the following conclusion can be drawn.

Theorem 3.11 [38] Suppose $\Phi\in\mathcal{CM}(I)$, and $G,H:I\to\mathbb{R}$ is first-order continuously differentiable. Then the necessary and sufficient condition for Equation (3.20) to hold is that h is a differentiable function with non-zero first derivative and there exists a constant $\alpha\in\mathbb{R}$, such that

$$(G',H')\sim(\Phi'\cdot S_\alpha\circ\Phi,\Phi'\cdot C_\alpha\circ\Phi),$$

where the definition of $S_\alpha,\ C_\alpha$ can be seen in (3.4) and (3.5).

With Theorem 2.1, if the bivariate mean satisfies symmetry, continuity, strict monotonicity, and bisymmetry, then the mean can only be a quasi-arithmetic mean. Based on (3.6) between trigonometric and hyperbolic functions, Lovas et al. [49] continued to study Equation (3.20) in 2020 and obtained the equivalence of Bajraktarević and Cauchy mean equal to the quasi-arithmetic mean.

Theorem 3.12 *Suppose $G,H:I\to\mathbb{R}$ is first-order continuously differentiable. The following propositions are equivalent.*

(1) There exists $\Phi\in\mathcal{CM}(I)$, such that Equation (3.20) holds.

(2) Cauchy mean $C_{G,H}$ is bisymmetric.

(3) There exist real numbers a,b,c,d,r,s satisfying $(a,b,c)\neq(0,0,0)$, such that

$$aG^2+bGH+cH^2+dG+rH+s=0.$$

(4) Assume G,H is second-order continuously differentiable, then there exist

constant real numbers a,b,c, such that

$$aG'^2+bG'H'+cH'^2=(W_{G,H}^{2,1})^{\frac{2}{3}}.$$

(5) Assume G,H is second-order continuously differentiable, then

$$B_{f,g}=A_\Phi,$$

where $f:=\frac{G'}{\Phi'}, g:=\frac{H'}{\Phi'}, \Phi:=\int\left(W_{G,H}^{2,1}\right)^{\frac{1}{3}}$.

(6) Assume G,H is second-order continuously differentiable, then

$$C_{G,H}=A_\Phi,$$

where $\Phi:=\int\left(W_{G,H}^{2,1}\right)^{\frac{1}{3}}$.

(7) Assume G,H is fourth-order continuously differentiable, then there exists a constant $c\in\mathbb{R}$, such that

$$\frac{3W_{G,H}^{4,1}+12W_{G,H}^{3,2}}{\left(W_{G,H}^{2,1}\right)^{\frac{5}{3}}}-\frac{5\left(W_{G,H}^{3,1}\right)^2}{\left(W_{G,H}^{2,1}\right)^{\frac{8}{3}}}=c.$$

4 Invariant equations of means

4.1 Invariant equations of Bajraktarević mean

This section will introduce the research results and progress of the invariant equation of Bajraktarević mean.

One form of Bajraktarević mean can be written as the weighted arithmetic mean of the following function:

$$B_s^\varphi(x,y):=\varphi^{-1}\left(\frac{s(x)}{s(x)+s(y)}\varphi(x)+\frac{s(y)}{s(x)+s(y)}\varphi(y)\right).$$

Firstly, in terms of the invariant equation of Bajraktarević mean concerning arithmetic mean, in 2010, Jarczyk [31] studied the solution of the weight function satisfying the harmonic oscillator equation by solving the following equation

$$B_s^\varphi(x,y)+B_t^\psi(x,y)=x+y, \tag{4.1}$$

where $\varphi,\psi:I\to\mathbb{R}$ is a continuous, strictly monotonic function, and function $s:I\to(0,\infty)$ satisfies the harmonic oscillator equation

$$s''(x)=ps(x). \tag{4.2}$$

Theorem 4.1 [31] *Assume $s:I\to(0,\infty)$ satisfies (4.2), and $\varphi,\psi:I\to\mathbb{R}$ is a strictly monotonic, fourth-order differentiable function. If invariant equation (4.1) holds, then there exist constants $a,b\in\mathbb{R}\setminus\{0\}$ and $c\in\mathbb{R}$, such that*

$$\varphi'(x) = \frac{a}{s(x)^2}\exp\left(c\int s(u)^{-\frac{4}{3}}\mathrm{d}u\right),\ x\in I;$$

$$\psi'(x) = \frac{b}{s(x)^2}\exp\left(c\int s(u)^{-\frac{4}{3}}\mathrm{d}u\right),\ x\in I.$$

Then, three special cases of function $s(x)$ are considered:
(1) $s(x) = x, x\in I\subset(0,\infty)$;
(2) $s(x) = \mathrm{e}^x, x\in I$;
(3) $s(x) = \cos x, x\in I\subset(0,\frac{\pi}{2})$.

Based on the characteristics of the invariant equation (4.1), all other cases where the function $s(x)$ satisfies (4.2) can be obtained by solving the above three cases.

Theorem 4.2 [31] *Suppose* $s: I\to(0,\infty)$, *and* $\varphi,\psi: I\to\mathbb{R}$ *is a strictly monotonic, fourth-order differentiable function, then the necessary and sufficient condition for invariant equation* (4.1) *to hold is: there exist constants* $a, c\in\mathbb{R}\setminus\{0\}$ *and* $b, d\in\mathbb{R}$, *such that*
(1) *if* $s(x) = x, x\in I\subset(0,\infty)$,

$$\varphi(x) = \frac{a}{x}+b, \psi(x) = \frac{c}{x}+d,\ x\in I;$$

(2) *if* $s(x) = \mathrm{e}^x, x\in I$,

$$\varphi(x) = a\mathrm{e}^{-2x}+b, \psi(x) = c\mathrm{e}^{-2x}+d,\ x\in I;$$

(3) *if* $s(x) = \cos x, x\in I\subset\left(0,\frac{\pi}{2}\right)$,

$$\varphi(x) = a\tan x+b, \psi(x) = c\tan x+d,\ x\in I.$$

In 2006, Domsta and Matkowski [16] studied the special case of $s(x) = x$, and their conclusion was consistent with Theorem 4.2 (1).

In 2020, Jarczyk [34] considered the invariant equation of Bajraktarević mean with respect to the function-weighted arithmetic mean $A_{\lambda(x,y)}$, where

$$A_{\lambda(x,y)} = \lambda(x,y)\,x + (1-\lambda(x,y))\,y,\ \lambda(x,y) = \frac{r(x)}{r(x)+r(y)}.$$

So, the invariant equation is

$$r(x)\,B_s^{\varphi}(x,y) + r(y)\,B_t^{\psi}(x,y) = r(x)\,x + r(y)\,y. \tag{4.3}$$

According to the same calculation method mentioned in references [30, 31], the following conclusions can be drawn.

Theorem 4.3 *Assume* $r,s,t: I\to(0,+\infty)$ *is third-order differentiable, and* $r'(x)$ *cannot take 0 on interval I. If* $\varphi,\psi: I\to R$ *is a third-order differentiable function that satisfies the condition that its first-order derivative cannot take* 0, *the invariant equation* (4.3) *holds, and then there exist* $c, d\in\mathbb{R}\setminus\{0\}$, *such that*

$$\varphi'(x) = c\left(\frac{r(x)}{s(x)}\right)^2,\ \psi'(x) = d\left(\frac{r(x)}{t(x)}\right)^2,\ x\in I.$$

Reference [34] raised an open problem.

Open Problem 4.1 Can further necessary conditions be found for Equation (4.3) under higher-order differentiable conditions?

Regarding another form of Bajraktarević mean, Páles and Zakaria [65] made a surprising discovery in 2019: for asymmetric Bajraktarević means, the invariant equation with respect to arithmetic mean can be solved by considering the invariant equation

$$\left(\frac{f}{g}\right)^{-1}\left(\frac{tf(x)+sf(y)}{tg(x)+sg(y)}\right)+\left(\frac{h}{k}\right)^{-1}\left(\frac{sh(x)+th(y)}{sk(x)+tk(y)}\right)=x+y, \tag{4.4}$$

where $f, g, h, k : I \to \mathbb{R}$ are continuous, g, k are 0 nowhere on I, and ratio function $\frac{f}{g}$, $\frac{h}{k}$ is strictly monotonic on I, $t, s\in\mathbb{R}_+$, and $t{\neq}s$.

Firstly, equivalently rewrite the weighted Bajraktarević mean

$$B_{f,g}(x, y; t, s) = \left(\frac{f}{g}\right)^{-1}\left(\frac{tf(x)+sf(y)}{tg(x)+sg(y)}\right)$$

into the form of determinant

$$\begin{vmatrix} tf(x)+sf(y) & f(B_{f,g}(x, y; t, s)) \\ tk(x)+sk(y) & g(B_{f,g}(x, y; t, s)) \end{vmatrix} = 0, \tag{4.5}$$

and then take the partial derivative of one variable and let $y{=}x$ to obtain the first to fourth partial derivatives of the mean.

With Definition (3.2) of functions $\Phi_{f,g}$, $\Psi_{f,g}$, taking the first to fourth partial derivatives of the invariant equation (4.10), we can obtain the following lemma.

Lemma 4.1 [65] *Let* $t, s\in\mathbb{R}_+$, *and* $t\neq s$, $f, g, h, k : I\to\mathbb{R}$ *is a fourth-order continuously differentiable function. If the invariant equation* (4.10) *holds, then there exists the real number* p, *such that*

$$\Psi_{f,g} = \frac{1}{2}\Phi' - \frac{1}{4}\Phi^2 + p,\ \Psi_{h,k} = -\frac{1}{2}\Phi' - \frac{1}{4}\Phi^2 + p, \tag{4.6}$$

where $\Phi = \Phi_{f,g} = -\Phi_{h,k}$.

With Lemma 4.1, f, g and h, k are the solutions to second-order differential equations $Y'' = \Phi Y' + \left(\frac{1}{2}\Phi' - \frac{1}{4}\Phi^2 + p\right)Y$ and $Y'' = -\Phi Y' - \left(\frac{1}{2}\Phi' + \frac{1}{4}\Phi^2 - p\right)Y$, which fall into a kind of solvable Riccati equation. In fact, let

$$\varphi := \frac{1}{\sqrt{\left|W^{1,0}_{f,g}\right|}} = |W_{f,g}|^{-\frac{1}{2}}.$$

We can get

$$\frac{\varphi'}{\varphi}=-\frac{1}{2}\Phi,\ \frac{\varphi''}{\varphi}=-\frac{1}{2}\Phi'+\frac{1}{4}\Phi^2.$$

Then it can be obtained that $\Phi_{\varphi f,\varphi g}=0, \Phi_{\varphi f,\varphi g}=p$. Therefore, functions φf and φg are solutions to second-order homogeneous linear differential equation $y''=py$, and the following necessary and sufficient condition can be obtained.

Theorem 4.4 [65] *Suppose $p\in\mathbb{R}, \varphi: I\to\mathbb{R}_+$ is a positive continuous function, $f,g,h,k: I\to\mathbb{R}$ is continuous, g,k is 0 nowhere on I, and $\frac{f}{g}$, $\frac{h}{k}$ is a strictly monotonic function. If*

$$(f,g)\sim\left(\frac{S_p}{\varphi},\frac{C_\varphi}{\varphi}\right),\ (h,k)\sim(S_p\cdot\varphi, C_p\cdot\varphi)\,, \tag{4.7}$$

then for all $s,t\in\mathbb{R}_+$, invariant equation (4.10) can be established.

In addition, assume that functions f,g,h,k are fourth-order continuously differentiable, g,k, $\left(\frac{f}{g}\right)'$, $\left(\frac{h}{k}\right)'$ are 0 nowhere on I, $s,t\in\mathbb{R}_+$ and $s\neq t$, and the invariant equation (4.10) holds, then there exists a positive fourth-order continuously differentiable function $\varphi: I\to\mathbb{R}_+$ and real parameter $p\in\mathbb{R}$, such that the equivalence relationship (4.7) can be established.

Then, one class of generalized Bajraktarević mean will be discussed.

$$A_{f,p}(x,y):=f^{-1}\left(\frac{p_1(x)f(x)+p_2(y)f(y)}{p_1(x)+p_2(y)}\right),\ x,y\in I,$$

where $f: I\to\mathbb{R}$ is a strictly monotonic, continuous function, and $(p_1,p_2): I\to\mathbb{R}_+^2$ is a positive function pair. In particular, when $p_1{=}p_2$, the mean becomes a symmetric Bajraktarević mean (2.3). In 2022, Günwald and Páles [25] studied the invariant equation of this asymmetric generalized Bajraktarević mean with respect to arithmetic mean, that is

$$f^{-1}\left(\frac{p_1(x)f(x)+p_2(y)f(y)}{p_1(x)+p_2(y)}\right)+g^{-1}\left(\frac{p_1(x)f(x)+p_2(y)f(y)}{q_1(x)+q_2(y)}\right)=x+y, x,y\in I, \tag{4.8}$$

where $f,g: I\to\mathbb{R}$ is a strictly monotonic, continuous function, $p_1,p_2,q_1,q_2: I\to\mathbb{R}_+^2$ are all unknown functions, and $p_1{\neq}p_2$, $q_1{\neq}p_2$. With auxiliary function constructed by Schwarz derivative and Lemma 3.1, the following equivalence conditions for invariant equation (4.8) to hold can be obtained.

Theorem 4.5 [25] *Assume $f,g: I\to\mathbb{R}$ satisfies the condition that its first derivative is non-zero on interval I and it is fourth-order continuously differentiable, and $(p_1,p_2): I\to\mathbb{R}_+^2$ is second-order continuously differentiable, $(q_1,q_2): I\to\mathbb{R}_+^2$. Suppose the set $P:=\{x\in I\mid p_1(x)=p_2(x)\}$ is nowhere dense on interval I, the following propositions are equivalent to each other.*

(1) The invariant equation (4.8) holds for all $(x, y) \in I^2$.
(2) There exists an open set $U \in I^2$ containing I^2 diagonal line, such that the invariant equation (4.8) holds for all $(x,y) \in U$.
(3) $(q_1, q_2) : I \to \mathbb{R}_+^2$ is second-order continuously differentiable, and the following equations hold on the I^2 diagonal line.

$$\partial_1 A_{f,p} + \partial_1 A_{g,p} = 1,\ \partial_1\partial_2 A_{f,p} + \partial_1\partial_2 A_{g,p} = 0,$$
$$\partial_1^2\partial_2 A_{f,p} + \partial_1^2\partial_2 A_{g,p} = 0,\ \partial_1^2\partial_2^2 A_{f,p} + \partial_1^2\partial_2^2 A_{g,p} = 0.$$

(4) There exists a real constant $\gamma \in \mathbb{R}$ and solution $u, v, w, z : I \to \mathbb{R}$ to second-order linear differential function $F'' = \gamma F$ satisfying the linear independence among $v(x), z(x) > 0, x \in I, \{u, v\}$ and $\{w, z\}$, and functions f, g, p_1, p_2, q_1, q_2 satisfy

$$f = \frac{u}{v},\ g = \frac{w}{z},\ p_1 q_1 = p_2 q_2 = vz. \tag{4.9}$$

People have also studied other invariant equation problems related to Bajraktarević mean, among which Matkowski [57] examined the invariant equation of quasi-arithmetic mean pair (A_f, A_g) with respect to Bajraktarević mean $B_{f,g}$, and obtained the necessary and sufficient condition for Bajraktarević mean $B_{f,g}$ to be (A_f, A_g)-invariant under the assumption of no regularity. In other words, Bajraktarević mean is quasi-arithmetic mean or quasi-geometric mean.

Theorem 4.6 [57] *Assume function $f, g : I \to \mathbb{R}$ is a one-to-one continuous function, and $f(x) g(x) \neq 0, x \in I$ and $\frac{f}{g}$ is one-to-one. Then invariant equation*

$$B_{f,g}(A_f, A_g) = B_{f,g}$$

holds if and only if one of the following conditions is established.
(1) *$f \sim g$ and $B_{f,g} = A_g = A_f$;*
(2) *There exists a non-zero real number c, such that $f(x) g(x) = c, x \in I$ and*

$$B_{f,g}(x, y) = g^{-1}\left(\sqrt{g(x) g(y)}\right), A_f(x, y) = g^{-1}\left(\frac{2g(x) g(y)}{g(x) + g(y)}\right),\ x, y \in I.$$

For a more general case, the invariant equation $B_{f,g}(A_{f,p}, A_{g,r}) = B_{f,g}$ of the weighted quasi-arithmetic mean with respect to Bajraktarević mean, Matkowski gave a similar conclusion under second-order differentiability in [57, Theorem 2].

In addition, Matkowski [59] studied the invariant equation of Beckenbach-Gini mean with respect to Bajraktarević mean

$$B_{\varphi,\psi}(B_\varphi, B_\psi) = B_{\varphi,\psi},$$

where the definition of Beckenbach-Gini mean B_φ is given in (2.4), and it is found that the necessary and sufficient condition for the invariant equation to hold under third-order differentiability is the existence of $p \in \mathbb{R}, p \neq 0$, such that

$$\varphi(x) = ae^{px},\ \psi(x) = be^{-px},\ x \in I.$$

The above conclusions are all based on the bivariate Bajraktarević mean, while there are relatively few researches on the ternary and above definition as well as their invariant equations. Matkowski [56] proposed a generalized form of $n(n \geqslant 2)$-variate Bajraktarević mean and considered its invariant equation with respect to the quasi-arithmetic mean. This paper proposes a definition of generalized Bajraktarević mean using the transition mapping of the independent variable sequence, which is more significant than the invariant equation itself. Given a positive integer k, $\boldsymbol{\sigma}_k$ is defined as the transition of the set $\{1, \ldots, k\}\ (\mathrm{mod} k)$, as is shown below:

$$\sigma_k(j) = \begin{cases} j+1, & j \in \{1, \ldots, k-1\}, \\ 1, & j = k. \end{cases}$$

Define the ith iteration of mapping $\boldsymbol{\sigma}_k$ as follows:

$$\sigma_k^i(j) = \begin{cases} j+i, & j \in \{1, \ldots, k-i\}, \\ j+1-k, & j \in \{k-i+1, \ldots, k\}, \end{cases}$$

where $\sigma_k^1 = \sigma_k, \sigma_k^0 = \mathrm{id}$.

Suppose $k \geqslant 2$, is continuous, $f : I \to \mathbb{R}$ is a strictly monotonic function, and $g_1, \ldots, g_k : I \to (0, \infty)$ is continuous, then the definition of generalized k-variate Bajraktarević mean is as follows:

$$B_{f,g_1,\ldots,g_k;\sigma_k^i}(x_1, \ldots, x_k) := f^{-1}\left(\frac{\sum_{j=1}^{k} f\left(x_{\sigma_k^i(j)}\right) g_j(x_j)}{\sum_{j=1}^{k} g_j(x_j)}\right), \tag{4.10}$$

where $i \in \{0, \ldots, k-1\}, x_1, \ldots, x_k \in I$. It can be proved that $B_{f,g_1,\ldots,g_k;\sigma_k^i}(x_1, \ldots, x_k)$ is all strict means on I for every $i = 0, \ldots, k-1$.

In particular, when $i = 0$, $g_1 = \cdots = g_k = g$, (4.10) becomes a Bajraktarević mean:

$$B_{f,g_1,\ldots,g_k;\sigma_k^4}(x_1, \ldots, x_k) = B_{f,g}(x_1, \ldots, x_k) = f^{-1}\left(\frac{\sum_{j=1}^{k} f(x_j) g(x_j)}{\sum_{j=1}^{k} g(x_j)}\right).$$

But when $k \geqslant 3$ and $g_1 = \cdots = g_k = g$, $i \in \{1, \ldots, k-1\}$, $B_{f,g_1,\ldots,g_k;\sigma_k^i}(x_1, \ldots, x_k)$ does not necessarily fall into the class of Bajraktarević mean (refer to [56, Note 2]). And the case is different when $k = 2$. If $g_1 = g_2 = g$, $B_{f,g;\sigma_2^0} = B_{f,g}$ and $B_{f,g;\sigma_2^1} = B_{f,\frac{1}{g}}$ both fall into the class of Bajraktarević mean.

The invariant equation for this class of generalized Bajraktarević mean can be concluded as follows.

Theorem 4.7 [56] Assume $f \in CM(I)$, $g_1, \ldots, g_k : I \to \mathbb{R}_+$ then

$$A_f \circ \left(B_{f,g_1,\ldots,g_k;\sigma_k^0}, B_{f,g_1,\ldots,g_k;\sigma_k^{k-1}}\right) = A_f,$$

i.e., the arithmetic mean A_f is invariant with respect to $(B_{f,g_1,\ldots,g_k;\sigma_k^0}, B_{f,g_1,\ldots,g_k;\sigma_k^{k-1}})$.

In summary, for the invariant equation of symmetric Bajraktarević mean with respect to arithmetic mean, because the odd-order partial derivatives on the diagonal line are invalid information when solving, it places higher requirements on differentiability to solve this problem by relying solely on the information of even-order partial derivatives. As a result, an open problem is left here due to huge computational complexity and complex formula.

Open Problem 4.2 Under what circumstances does the sum of two symmetric Bajraktarević means equal twice the arithmetic mean? Solve Equation (4.1) or solve the following equivalent equation

$$\left(\frac{f}{g}\right)^{-1}\left(\frac{f(x)+f(y)}{g(x)+g(y)}\right)+\left(\frac{h}{k}\right)^{-1}\left(\frac{h(x)+h(y)}{k(x)+k(y)}\right)=x+y. \tag{4.11}$$

4.2 Invariant equations of Cauchy mean

The Lagrange mean is a special class of Cauchy mean. Before introducing the relevant research conclusions on the invariant equation of Cauchy mean, let's first look at the invariant equation problem of Lagrange mean. In 2005, Matkowski [54] studied the invariant equation problem of Lagrange mean with respect to arithmetic mean. Firstly, he proved that the functional equation satisfied the continuous, differentiability-implying regularity theorem, and then obtained the necessary and sufficient condition for this problem without differentiability.

Theorem 4.8 *Suppose $f,g\in\mathcal{CM}(I)$. Then the necessary and sufficient condition for the invariant equation of Lagrange mean with respect to arithmetic mean*

$$f^{-1}\left(\int_0^1 f(tx+(1-t)y)\,\mathrm{d}t\right)+g^{-1}\left(\int_0^1 g(tx+(1-t)y)\,\mathrm{d}t\right)=x+y,\ x,y\in I \tag{4.12}$$

to hold is the existence of constants $a,c,p\in\mathbb{R}\setminus\{0\}$, $b,d\in\mathbb{R}$, such that

$$f(x)=a\mathrm{e}^{px}+b, g(x)=c\mathrm{e}^{-px}+d,\ x\in I;$$

or

$$f(x)=ax+b, g(x)=cx+d,\ x\in I.$$

For the invariant equation of Lagrange mean with respect to geometric mean, solve the equation

$$(f')^{-1}\left(\frac{f(x)-f(y)}{x-y}\right)(g')^{-1}\left(\frac{g(x)-g(y)}{x-y}\right)=xy,\ x,y\in I,\ x\neq y, \tag{4.13}$$

In 2007, Glazowska and Matkowski [23] gave the necessary condition for it to be established.

Theorem 4.9 *Assume $f, g: I \to \mathbb{R}$ is first-order continuously differentiable. If the invariant equation* (4.13) *of Lagrange mean with respect to geometric mean holds, then there exist constants $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}, a_1 a_2 \neq 0$, such that*

$$f(x) = a_1 x^{-1} + b_1 x + c_1,\ g(x) = a_2 x^{-1} + b_2 x + c_2,\ x \in I; \tag{4.14}$$

or there exist constants $a, c_1, c_2 \in \mathbb{R}, c_1 c_2 \neq 0$, such that

$$f''(x) = c_1 x^{-3} \exp\left(a x^{-\frac{4}{9}}\right),\ g''(x) = c_2 x^{-3} \exp\left(-a x^{-\frac{4}{9}}\right),\ x \in I. \tag{4.15}$$

In 2011, Glazowska [19], using Mathematica 4.1 software to perform calculation obtained that substituting condition (4.15) into the invariant equation (4.13) could lead to contradictions. Therefore, condition (4.15) is excluded, and (4.14) is equivalent to two Lagrange means that are both geometric means, that is, $L_f = L_g = G$. Therefore, (4.14) is a sufficient and necessary condition for the invariant equation (4.13) to hold. As a result, both types of problems of Lagrange mean with respect to the arithmetic mean and geometric mean have been completely solved. Regarding the invariant equation of the Cauchy mean, due to the difficulty of solving four unknown functions, it has not been completely solved yet, and existing conclusions have only considered certain special cases

In 2011, Glazowska [20] considered the invariance of Cauchy mean with respect to geometric mean when the denominator derived function is a power function, i.e.,

$$\left(\frac{f'}{g'}\right)^{-1}\left(\frac{f(x) - f(y)}{g(x) - g(y)}\right)\left(\frac{h'}{k'}\right)^{-1}\left(\frac{h(x) - h(y)}{k(x) - k(y)}\right) = xy,\ x, y \in I,\ x \neq y, \tag{4.16}$$

where $f, h: I \to \mathbb{R}$ is a differentiable function, and $\frac{f'}{g'}, \frac{h'}{k'}$ is injective on I,

$$g(x) = x^p,\ k(x) = x^r,\ p, r \in \mathbb{R} \setminus \{0\},$$

obtaining the necessary and sufficient condition for invariant equation (4.16) to hold in the case of $p + r = 0$.

Theorem 4.10 *Suppose $f, h: I \to \mathbb{R}$ is a differentiable function and $\frac{f'}{g'}, \frac{h'}{k'}$ is injective on I, $g(x) = x^p, k(x) = x^{-p}, p \in \mathbb{R} \setminus \{0\}$. Then the necessary and sufficient condition for invariant equation* (4.16) *to hold is there exists $q \in \mathbb{R} \setminus \{0, p\}$, such that*

$$f(x) = A_1 x^q + B_1 x^p + C_1,\ h(x) = A_2 x^{-q} + B_2 x^{-p} + C_2;$$

or there exists $q \in \{0, p\}$, such that

$$f(x) = A_1 x^q \ln x + B_1 x^p + C_1,\ h(x) = A_2 x^{-q} \ln x + B_2 x^{-p} + C_2,$$

where $A_i, B_i, C_i \in \mathbb{R}, A_i \neq 0, i \in \{1, 2\}$.

The invariant equation of Cauchy mean with respect to arithmetic mean is even more complex. In 2021, reference [79] studied this equation and obtained the necessary condition for the equation to hold when p, r satisfied certain conditions.

Theorem 4.11 *Assume* $g, k : I \to \mathbb{R}$ *satisfies*

$$g(x) = x^p,\ k(x) = x^r,\ x \in I,$$

where $p, r \in \mathbb{R} \setminus \{0\}$ *satisfies*

$$4p^3 - 4p^2r - 4pr^2 + 4r^3 + 31p^2 + 46pr + 31r^2 - 162p - 162r + 216 = 0. \tag{4.17}$$

Suppose $f, h : I \to \mathbb{R}$ is a differentiable function, and $\frac{f'}{g'}, \frac{h'}{k'}$ is injective on I. If invariant equation

$$\left(\frac{f'}{g'}\right)^{-1}\left(\frac{f(x) - f(y)}{g(x) - g(y)}\right) + \left(\frac{h'}{k'}\right)^{-1}\left(\frac{h(x) - h(y)}{k(x) - k(y)}\right) = x + y,\ x \neq y \tag{4.18}$$

holds, then the following conclusions can be reached.
(1) When $5p + r = 9$, there exist $A_i, B_i, C_i \in \mathbb{R}, A_i \neq 0, i \in \{1, 2\}$, such that

$$f(x) = A_1 x^p \ln x + B_1 x^p + C_1, h(x) = A_2 x^t + B_2 x^r + C_2,$$

where $t = 6 - 2p - r$.
(2) When $5p + r \neq 9$, there exist $A_i, B_i, C_i \in \mathbb{R}, A_i \neq 0, i \in \{1, 2, 3, 4, 5\}$, such that

$$f(x) = \begin{cases} A_1 x^q + B_1 x^p + C_1, & 2p + r \neq 9, \\ A_2 ln x + B_2 x^p + C_2, & 2p + r = 9; \end{cases} \tag{4.19}$$

$$h(x) = \begin{cases} A_3 x^s + B_3 x^r + C_3, & p + 2r \neq 9, p + 5r \neq 9, \\ A_4 ln x + B_4 x^r + C_4, & p + 2r = 9, p + 5r \neq 9, \\ A_5 x^r ln x + B_5 x^r + C_5, & p + 2r \neq 9, p + 5r = 9, \end{cases} \tag{4.20}$$

where $q = \frac{1}{3}(9 - 2p - r), s = \frac{1}{3}(9 - p - 2r)$.
(3) When $2p + 2r = 13$, there exist $A_i, B_i \in \mathbb{R}, A_i \neq 0, i \in \{1, 2\}$, such that

$$f(x) = A_1\left(x^p \int x^{B_1 - \frac{c-ab}{2}\cdot \ln x} \mathrm{d}x - \int x^{p + B_1 - \frac{c-ab}{2}\cdot \ln x} \mathrm{d}x\right),$$
$$h(x) = A_2\left(x^r \int x^{B_2 + \frac{c-ab}{2}\cdot \ln x} \mathrm{d}x - \int x^{r + B_2 + \frac{c-ab}{2}\cdot \ln x} \mathrm{d}x\right),$$

where

$$a = \frac{5p + r - 6}{3}, b = \frac{2(p + r - 2)}{9}, c = \frac{6p^2 + 8pr + 2r^2 - 33p - 17r + 34}{9}. \tag{4.21}$$

(4) When $2p + 2r \neq 13$, there exist $A_i, B \in \mathbb{R}, A_i \neq 0, i \in \{1, 2\}$, such that

$$f(x) = A_1\left(x^p \int x^{\alpha} \exp\left(B x^{-b+1}\right) \mathrm{d}x - \int x^{p+\alpha} \exp\left(B x^{-b+1}\right) \mathrm{d}x\right),$$

$$h\left(x\right)=A_2\left(x^r\int x^{\beta}\exp\left(-Bx^{-b+1}\right)\mathrm{d}x-\int x^{r+\beta}\exp\left(-Bx^{-b+1}\right)\mathrm{d}x\right),$$

where $\alpha=\frac{ab-c}{b-1},\beta=\frac{c-ab}{b-1}-2\left(p+r-2\right)$, and for a,b,c, follow (4.21).

In summary, currently there are two open problems with the invariance of the Cauchy mean.

Open Problem 4.3 Under what circumstances does the product of two Cauchy means equal the square of the geometric mean, i.e., solving the invariant equation (4.16) without special requirements for the denominator function?

Open Problem 4.4 Under what circumstances does the sum of two Cauchy means equal twice the arithmetic mean, i.e., solving the invariant equation (4.18) without special requirements for the denominator function?

4.3 Invariant equations of generalized quasi-arithmetic means

In 2008, Losonczi and Páles [46] defined a new class of generalized quasi-arithmetic means (also known as generalized Bajraktarević means).

$$M_{f,g;\mu}\left(x,y\right):=\left(\frac{f}{g}\right)^{-1}\left(\frac{\int_0^1 f\left(tx+\left(1-t\right)y\right)\mathrm{d}\mu\left(t\right)}{\int_0^1 g\left(tx+\left(1-t\right)y\right)\mathrm{d}\mu\left(t\right)}\right),\ x,y\in I, \tag{4.22}$$

where $f,g:I\to\mathbb{R}$ is a continuous function on a given interval I, g is monotonic positive function, $\frac{f}{g}$ is a monotonic continuous function, and μ is the Borel probability measure on [0,1]. This mean includes the Bajraktarević mean and the Cauchy mean, and has been widely studied in recent years. Losonczi and Páles [46] discussed inequality problems corresponding to different measures.

$$M_{f,g;\mu}\left(x,y\right)\leqslant M_{h,k;\nu}\left(x,y\right),\ x,y\in I.$$

According to Lemma 3.3, the odd-order central moments of symmetric measures are all 0. In 2011, based on this property, Losonczi and Páles [47] considered the equality problem of two generalized quasi-arithmetic means when the measure satisfied symmetry or antisymmetry.

$$M_{f,g;\mu}=M_{h,k;\nu}. \tag{4.23}$$

In 2021, Losonczi et al. [48] studied the equality problem when two derivative measures were the same

$$M_{f,g;\mu}=M_{h,k;\mu}.$$

Certain necessary conditions for the problem under the assumption of sixth-order differentiability were obtained and applied to get the necessary and sufficient conditions for two Bajraktarević means and two Cauchy means to be equal. In 2022, Páles and Zakaria [68] restudied Problem (4.23), obtained the necessary and sufficient conditions under the eighth-order differentiability, and applied them to gain the intersection of Bajraktarević means and Cauchy means as the quasi-arithmetic mean.

Recently, Zhang and Li [75] studied the invariant equation of the quasi-arithmetic mean with respect to the arithmetic mean for two derived measures that were simultaneously generalized

$$\left(\frac{f}{g}\right)^{-1}\left(\frac{\int_0^1 f\left(tx+(1-t)y\right)\mathrm{d}\mu(t)}{\int_0^1 g\left(tx+(1-t)y\right)\mathrm{d}\mu(t)}\right) + \left(\frac{h}{k}\right)^{-1}\left(\frac{\int_0^1 h\left(tx+(1-t)y\right)\mathrm{d}\mu(t)}{\int_0^1 k\left(tx+(1-t)y\right)\mathrm{d}\mu(t)}\right) = x+y, \tag{4.24}$$

where $x,y\in I$, $f,g,h,k:I\to\mathbb{R}$ is a continuous function satisfying that g, k are positive functions, and $\frac{f}{g},\frac{h}{k}$ is strictly monotonic, and μ is the Borel probability measure on interval [0,1]. Reference [75] obtained some necessary conditions for the denominator functions to satisfy second-order constant coefficient homogeneous differential equation

$$y''=py, \tag{4.25}$$

and for derived measures to satisfy special conditions.

Theorem 4.12 *Suppose μ is the Borel probability measure on interval* [0,1], *which satisfies that $\mu_2\neq 0, f,g,h,k:I\to\mathbb{R}$ is fourth-order continuously differentiable, and k=g satisfies* (4.25). *If the invariant equation* (4.24) *holds, then there exists $r\in\mathbb{R}$, such that*

$$f=g\int\frac{1}{g^2}\exp\left(\int rg^\beta\right),\ h=g\int\frac{1}{g^2}\exp\left(\int -rg^\beta\right),$$

where $\beta:=\frac{2(\mu_4-3\mu_2^2)}{3\mu_4}$.

For the case where $\mu_3\neq 0$, the following two conclusions were obtained in [75].

Theorem 4.13 *Assume μ is the Borel probability measure on interval* [0,1], *$\mu_3\neq 0,\ \mu_4=3\mu_2^2,\ \mu_5\neq 10\mu_2\mu_3,$ and $f,g,h,k:I\to\mathbb{R}$ is fifth-order continuously differentiable. If invariant equation* (4.24) *holds, then one of the following conclusions can be reached.*

(1) When $\mu_5=2\mu_2\mu_3$, there exists $p,q\in\mathbb{R}$, such that

$$(f,g)\sim\left(\mathrm{e}^{\frac{p}{2}x}\cdot S_{\frac{p^2}{4}},\mathrm{e}^{\frac{p}{2}x}\cdot C_{\frac{p^2}{4}}\right),(h,k)\sim\left(\mathrm{e}^{-\frac{p}{2}x}\cdot S_{-\frac{7p^2}{4}},\mathrm{e}^{-\frac{p}{2}x}\cdot C_{-\frac{7p^2}{4}}\right).$$

(2) When $\mu_5\neq 2\mu_2\mu_3$, there exists $p,q\in\mathbb{R}$, such that *f*,*g* and *h*,*k* are solutions to

the following differential equation:

$$y'' = py' + \frac{2p^2 q e^{p\kappa x}}{1 - qe^{p\kappa x}} y,\ y'' = -py' + \frac{-2p^2}{1 - qe^{p\kappa x}} y, \tag{4.26}$$

where

$$\kappa = -\frac{2(\mu_5 - 10\mu_3\mu_2)}{\mu_5 - 2\mu_3\mu_2}.$$

Theorem 4.14 *Suppose μ is the Borel probability measure on the interval* [0,1], $\mu_3 \neq 0,\ \mu_4 \neq 3\mu_2^2,\ \mu_5 \neq 10\mu_2\mu_3$ *, and $f, g, h, k : I \to \mathbb{R}$ is fifth-order continuously differentiable. If invariant equation* (4.24) *holds, then one of the following conclusions can be established.*
(1) Function pair $(f, g), (h, k)$ satisfies

$$(f, g) \sim (1, x), (h, k) \sim (1, x). \tag{4.27}$$

(2) There exists $c \in \mathbb{R}$, such that f, g and h, k are solutions to the following differential equation:

$$y'' = \Phi(x) y' + \left(\alpha\Phi'(x) - \Phi^2(x)\right) y, y'' = -\Phi(x) y' + (-\alpha\Phi'(x) - \Phi^2(x)) y, \tag{4.28}$$

where $\alpha := \frac{\mu_4}{2(\mu_4 - 3\mu_2^2)}$, and function Φ satisfies

$$\Phi(x) = \pm\frac{1}{\sqrt{\frac{r}{q}x + c}},\ x \in I,\ \text{where } \mu_5 = \frac{10\mu_2\mu_3\mu_4}{\mu_4 + 12\mu_2^2},$$

or

$$\Phi' = \sqrt{c\Phi^{-\frac{2q}{p}} + \frac{r}{2p+q}\Phi^4},\ \text{where } \mu_5 \neq \frac{10\mu_2\mu_3\mu_4}{\mu_4 + 12\mu_2^2},$$

and p, q, r are determined by measure μ.

Note 4.1 When $\mu_3 \neq 0, \mu_5 = 10\mu_2\mu_3$ or $\mu_3 = 0$, since the invariant equation (4.24) odd-order partial derivative cannot provide effective information, it is difficult to solve the problem. Reference [75] did not settle these two cases.

The overall situation concerning the generalized arithmetic mean is difficult to solve. Here, two special cases of the open problem can be considered.

Open Problem 4.5 When two measures are the same, consider the invariance of the generalized quasi-arithmetic mean with respect to the arithmetic mean when μ is a symmetric measure and satisfies (3.8), that is, solve Equation (4.24).

Open Problem 4.6 When two measures are different, consider the invariant equation of the generalized quasi-arithmetic mean with respect to the arithmetic mean in the case of conjugate measures, that is, solve

$$\left(\frac{f}{g}\right)^{-1}\left(\frac{\int_0^1 f\left(tx+(1-t)y\right)\mathrm{d}\mu\left(t\right)}{\int_0^1 g\left(tx+(1-t)y\right)\mathrm{d}\mu\left(t\right)}\right)+\left(\frac{h}{k}\right)^{-1}\left(\frac{\int_0^1 h\left(tx+(1-t)y\right)\mathrm{d}\nu\left(t\right)}{\int_0^1 k\left(tx+(1-t)y\right)\mathrm{d}\nu\left(t\right)}\right)=x+y,$$

where ν is the conjugate measure of μ, i.e., $\nu=\mu^*$.

5 Applications

The researches on mean and mean equation have gone a long way and been widely applied in the fields such as economics, operations research, and number theory currently. Below are some specific research results and open problems on the application of mean in these fields.

In economics and social welfare, as early as 1976, Kolm [39] proposed a series of problems to stimulate the study of inequality measurement. He focused on the relationship between inequality and social welfare using relevant axioms of mean to characterize their relationship. In 1997, Ebert [17] used the quasi-arithmetic mean to give the concept of linear inequality, and then used the Atkinson-Kolm-Sen index and a new dual index to characterize the inequality for evaluation of people's living standards. The concept of linear inequality and the related issues of ethical inequality ordering have been fully answered. However, there are certain problems in characterizing the concept of linear inequality based on empirical evidence of inequality. To solve this problem, Ebert used nonlinear inequality and turned to mathematical expert Aczél for help. Aczél described the problem in mathematical language, which was actually solving functional equation

$$\frac{1}{n}\sum_{i=1}^{n}t\left(x_i\right)=h\left(\frac{1}{n}\sum_{i=1}^{n}x_i,f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}f\left(x_i\right)\right)\right), \tag{5.1}$$

where known function f defined a quasi-arithmetic mean, and t,h were unknown functions.

Járai et al. [28] answered this question in 2000 and drawn the following conclusions.

Theorem 5.1 (1) *Assume* $n\geqslant 4$ *. If* t *is the continuous solution to Equation* (5.1), *then there exist constants* c_1,c_2,c_3 *, such that*

$$t\left(x\right)=c_1 f\left(x\right)+c_2 x+c_3.$$

(2) *Assume* $n\geqslant 3$ *. Suppose* f *is a continuously differentiable function which satisfies that the first-order derivative cannot take* 0, *and it is not an affine function on any subinterval of* I, *then for each function* $t: I\to\mathbb{R}$ *, the necessary and sufficient condition for Equation* (5.1) *to hold under the existence of a continuously differentiable function* $h: I\times I\to\mathbb{R}$ *is there exist constants* c_1,c_2,c_3, *such that*

$$t(x) = c_1 f(x) + c_2 x + c_3.$$

(3) *Assume* $n = 2$. *If* f *is strictly convex or concave on interval* I, *then for any function there exists a unique function* $h : I \times I \to \mathbb{R}$, *such that Equation* (5.1) *holds.*

In operations research, Páles and Pasteczka [63] applied the generalized Bajraktarević mean to this field. Motivated by attempting to use the mean method to study farm structure optimization problems [74], they discovered a generalized construction method for decision functions and effort functions when studying the so-called weighted Bajraktarević mean. This method provides a nonlinear approach to optimization problems and useful new methods for game theory and decision problems.

In number theory and numerical solutions, the Gauss iteration of the mean can be applied to the calculation of irrational numbers and numerical calculations of elliptic integrals. The research on Gauss arithmetic-geometric mean $A \otimes G$ has had a long history and is of extraordinary significance. In other words, this mean can be applied to calculate the irrational number π. Borwein wrote a treatise [8] introducing the relationship between the two. The mean can be represented by an elliptic integral of the first kind, so the Gauss arithmetic-geometric mean can be used to estimate it. The other class of mean, the Toader mean [71], is related to an elliptic integral of the second kind and defined as follows:

$$\mathrm{TD}(a, b) = \int_0^{\frac{\pi}{2}} \sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}\,\mathrm{d}\theta,\ a, b \in \mathbb{R}_+, \tag{5.2}$$

which is equivalent to

$$\mathrm{TD}(a, b) = \begin{cases} \dfrac{2a}{\pi}\varepsilon\left(\sqrt{1 - \left(\dfrac{b}{a}\right)^2}\right), & a \geqslant b, \\ \dfrac{2b}{\pi}\varepsilon\left(\sqrt{1 - \left(\dfrac{a}{b}\right)^2}\right), & a < b, \end{cases}$$

where $a, b \in \mathbb{R}_+$, and

$$\varepsilon(r) = \int_0^{\frac{\pi}{2}} \sqrt{1 - r^2 \sin^2\theta}\,\mathrm{d}\theta,\ r \in (0, 1). \tag{5.3}$$

For elliptic integral of the second kind, reference [78] provided the optimal bound estimate by studying the Toader mean inequality. In the recent reference [60], Matkowski presented some new results on Gauss arithmetic-geometric mean, represented $A \otimes G$ by the Gaussian combination of harmonic mean and geometric mean, and proposed two open problems for the ternary case.

Open Problem 5.1 Can an explicit expression be given for the invariant mean of ternary geometric, arithmetic, and harmonic mean?

Open Problem 5.2 Does the explicit expression of the invariant mean of ternary geometric, arithmetic, and harmonic mean have a form?

$$K(x,y,z)=\left(\frac{\pi}{2}\right)^2\left(\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi\mathrm{d}\psi}{\sqrt{x^2\cos^2\varphi\cos^2\psi+y^2\sin^2\varphi\cos^2\psi+z^2\sin^2\psi}}\right)^{-1}?$$

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