Front. Math. China 2024, 19(4): 191–213
https://doi.org/10.3868/s140-DDD-024-0012-x

RESEARCH ARTICLE

Research on enumeration problem of pattern-avoiding permutations

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Abstract The problem of relevant enumeration with pattern-avoiding permutations is a significant topic in enumerative combinatorics and has wide applications in physics, chemistry, and computer science. This paper summarizes the relevant conclusions of the enumeration of pattern-avoiding permutations on the n-element symmetric group $\mathcal{S}_n$, alternating permutations, Dumont permutations, Ballot permutations, and inversion sequences. It also introduces relevant research results on avoiding vincular patterns and barred patterns in $\mathcal{S}_n$.

Keywords Pattern avoidance, combinatorial enumeration, combinatorial bijection
MSC2020 05-02, 05A05, 05A15

1 Research on pattern-avoiding enumeration on $\mathcal{S}_n$

Researches on pattern problems on permutation sets can be traced back to the 1880s, when MacMahon [47] used generating functions to study the distribution of permutations and inverse numbers in words. In 1985, Simion and Schmidt [62] first systematically studied the problem of pattern avoidance on permutations. Since then, the problem of patterns on permutation sets has attracted the attention of many combinatorial scholars. The problem of pattern avoidance on permutations with different constraints has become a hot topic in combinatorial mathematics, and relevant pattern-avoiding permutations can bijectively correspond with many combinatorial structures. This paper reviews relevant achievements in this field and introduces conclusions and conjectures on the n-element symmetric group S_n, alternating permutations, Dumont permutations, Ballot permutations, and pattern avoidance problems on inversion sequences. It also presents

Translated from *Advances in Mathematics* (*China*), 2024, 53(2): 225–242
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research results on avoiding vincular patterns and barred patterns in $\mathcal{S}_n$.

Denote the n-element symmetric group S_n as the set of all permutations on $[n]=\{1,2,\ldots,n\}$, such as $S_1=\{1\}$, $\mathcal{S}_2=\{12,21\}$, $\mathcal{S}_3=\{123,132,213,231,312,321\}$.

Definition 1.1 Given two permutations $\sigma=\sigma_1\sigma_2\ldots\sigma_n$ and $\pi=\pi_1\pi_2\ldots\pi_k$, if there exists $c_1<c_2<\cdots<c_k$, such that the subpermutation $\sigma_{c_1}\sigma_{c_2}\ldots\sigma_{c_k}$ and π are order isomorphic, i.e., $\pi_i<\pi_j$ if and only if $\sigma_{c_i}<\sigma_{c_j}$ $(1\leqslant i<j\leqslant k)$, then permutation σ is said to contain pattern σ. If there does not exist a subpermutation of σ that is order isomorphic to π, permutation σ is said to avoid pattern π. The set of permutations avoiding pattern π on S_n is denoted by $\mathcal{S}_n(\pi)$.

Definition 1.2 If $|\mathcal{S}_n(\pi)|=|\mathcal{S}_n(\tau)|$, we say the two patterns π and τ are Wilf equivalent in $\mathcal{S}_n$.

Knuth [40] first proposed the "pattern theory" and obtained that the enumeration result for avoiding any length-3 pattern is a Catalan number $C_n=\frac{1}{n+1}\binom{n}{2n}$ and provided the bijection between $\mathcal{S}_n(132)$ and Dyck paths. Later, Simion and Schmidt [62] systematically studied the problem of avoiding patterns on permutations and constructed the bijection between $S_n(123)$ and $\mathcal{S}_n(132)$. West [68] established the bijection between $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$ using a spanning tree. Reference [61] established the bijection between $\mathcal{S}_n(321)$ and $\mathcal{S}_n(132)$ with matrix transformation. Simion and Schmidt [62] also obtained all enumeration results for avoiding two and three length-3 patterns in S_n, as shown in the following theorems.

Theorem 1.1 (1) *For* $n\geqslant 1$, *any* $p\in\{(123,132),(123,213),(231,321),(132,213),(132,231),(132,312),(213,231),(213,312),(231,312),(312,321)\}$, *there exists*

$$|\mathcal{S}_n(p)|=2^{n-1}.$$

(2) *For* $n\geqslant 1$, *any* $p\in\{(123,231),(123,312),(132,321),(213,321)\}$, there exists

$$|\mathcal{S}_n(p)|=1+\binom{n}{2}.$$

(3) *For* $n\geqslant 5$, *there exists*

$$|\mathcal{S}_n(123,321)|=0.$$

Theorem 1.2 (1) *For* $n\geqslant 1$, *any* $p\in\{(123,132,213),(231,312,321)\}$, *there exists*

$$|\mathcal{S}_n(p)|=F_{n+1},$$

where F_n *is the* n*-th Fibonacci number.*

(2) *For* $n \geqslant 1$, *any* $p \in \{(123,132,231),(123,213,312),(123,231,312),(132,213,231),(132,213,312),(132,213,321),(132,213,321),(132,231,312),(132,231,321),(213,231,321),(213,312,321)\}$, *there exists*

$$|\mathcal{S}_n(p)| = n.$$

(3) *For* $n \geqslant 5$, *any* $p \in \{(123,132,321),(123,213,321),(123,231,321),(123,312,321)\}$, *there exists*

$$|\mathcal{S}_n(p)| = 0.$$

Simion and Schmidt [62] also obtained that the enumeration result for avoiding four and five length-3 patterns on $\mathcal{S}_n$ was a simple constant.

There are also abundant achievements in the research on avoiding length-4 patterns on $\mathcal{S}_n$. The permutations that avoid a single length-4 pattern can be divided into three Wilf equivalent classes. Bóna [7] and Regev [60] presented the generating functions for the enumeration of avoiding 1342 and 1234 pattern equivalence classes. Gessel [35] and Bóna [7] gave the enumeration results for avoiding these patterns. Conway and Guttmann [28] used a modified algorithm to estimate the enumeration result of $\mathcal{S}_n(1324)$ in the remaining Wilf equivalent classes. West [67] obtained the enumeration result for avoiding a length-3 and length-4 pattern at the same time on $\mathcal{S}_n$ with a spanning tree. Most of the results are related to Fibonacci number. Kremer and Shiu [42] detailed the enumeration problem of avoiding double length-4 patterns on $\mathcal{S}_n$. Mansour [50] gave the enumeration result of maximal partial permutation $\mathcal{S}_n(1324,2134,1243)$.

Among the studies of avoiding multiple-length patterns on $\mathcal{S}_n$, through constructing bijections, Backelin et al. [3] obtained $|\mathcal{S}_n(12\cdots k\tau)| = |\mathcal{S}_n(k(k-1)\cdots 1\tau)|$, where τ is a random permutation comprised of $\{k+1,\ldots,l\}$. Reference [61] got $|\mathcal{S}_n(\tau k(k-1))| = |\mathcal{S}_n(\tau(k-1)k)|$, where τ is a random permutation comprised of $\{1,\ldots,k-2\}$.

The asymptotic enumeration problem of pattern-avoiding permutations on $\mathcal{S}_n$ has also attracted the attention of scholars. Marcus and Tardos [55] solved the famous Stanley-Wilf Conjecture, as shown in the following theorem.

Theorem 1.3 *For any positive integer* k *and any* $\tau \in S_k$, $\lim_{n\to\infty}(|\mathcal{S}_n(\tau)|)^{\frac{1}{n}}$ *exists and is finite.*

This is called the Stanley-Wilf limit for pattern τ, denoted by $L(\tau)$. When $\tau \in \mathcal{S}_3$, $L(\tau) = 4$ is proved. When $\tau \in \mathcal{S}_4$, there are three Wilf equivalent classes. Regev [60] proved $L(1234) = 9$, and Bóna [7] proved $L(1342) = 8$. The Stanley-Wilf limit of pattern 1324 in the remaining equivalent class remains an open issue. Conway et al. [29] obtained $10.271 < L(1324) < 13.5$. Mansour and Nassau [52] got $L(1324567\cdots k) \geqslant \left(k-4+\sqrt{9.47}\right)^2$.

2 Research on pattern-avoiding enumeration on alternating permutations

Definition 2.1 Given a permutation σ, if $\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \cdots$, then permutation σ is said to be an alternating permutation.

Denote $\mathcal{A}_n$ as a set comprised of all alternating permutations on $[n]$. The enumeration result of length-n alternating permutation is an Euler number E_n, with its exponential generating function satisfying

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sec x + \tan x.$$

Mansour [48] used the block decomposition method to obtain $|\mathcal{A}_{2n}| = C_n$. Later, Lewis [43] concluded that the enumeration results of alternating permutations avoiding any length-3 pattern are related to C_n. He established the bijection between $\mathcal{S}_n(132)$ and $\mathcal{A}_{2n+1}(132)$, and for the first time, he studied the enumeration problem of alternating permutations avoiding length-4 patterns [44]. Lewis [45] established the bijection between $\mathcal{A}_{2n}(1234)$, $\mathcal{A}_{2n}(2143)$ and standard Young tableaux of shape $\langle n, n, n\rangle$ by constructing a spanning tree, and obtained

$$|\mathcal{A}_{2n}(1234)| = |\mathcal{A}_{2n}(2143)| = \frac{2\cdot(3n)!}{n!\,(n+1)!\,(n+2)!}.$$

Meanwhile, Lewis [45] established the bijection between $\mathcal{A}_{2n+1}(2143)$ and standard Young Tableaux of shape $\langle n+2, n+1, n\rangle$, and obtained

$$|\mathcal{A}_{2n+1}(2143)| = \frac{2\,(3n+3)!}{n!\,(n+1)!\,(n+2)!\,(2n+1)\,(2n+2)\,(2n+3)}.$$

Besides, Lewis [45] also gave the following conjectures.

Conjecture 2.1 *For $n \geqslant 1$, any $p \in \{1243, 2134, 1432, 3214, 2341, 4123, 3421, 4312\}$, there exists*

$$|\mathcal{A}_{2n}(p)| = |\mathcal{A}_{2n}(1234)| = |\mathcal{A}_{2n}(2143)|.$$

Conjecture 2.2 *For $n \geqslant 0$, any $p \in \{2134, 4312, 3214, 4132\}$, there exists*

$$|\mathcal{A}_{2n+1}(p)| = |\mathcal{A}_{2n+1}(1234)|.$$

Conjecture 2.3 *For $n \geqslant 0$, any $p \in \{1243, 3421, 1432, 2341\}$, there exists*

$$|\mathcal{A}_{2n+1}(p)| = |\mathcal{A}_{2n+1}(2143)|.$$

Conjecture 2.4 *If permutations p and q are Wilf equivalent on any $\mathcal{A}_n$, then they are Wilf equivalent on any $\mathcal{S}_n$.*

The above Conjectures 2.1–2.3 have been fully resolved in recent years. Xu and Yan [69] established the bijection between $\mathcal{A}_{2n}(4123)$ and standard Young tableaux of shape $\langle n+2, n, n-2\rangle$, as well as the bijection between $\mathcal{A}_{2n+1}(4123)$ and standard Young tableaux of shape $\langle n+1, n, n-1\rangle$ by constructing a spanning tree, proving

$$\begin{aligned}&|\mathcal{A}_{2n}(4123)| = |\mathcal{A}_{2n}(1234)| = |\mathcal{A}_{2n}(1432)|,\\&|\mathcal{A}_{2n+1}(4123)| = |\mathcal{A}_{2n+1}(1234)|,\\&|\mathcal{A}_{2n+1}(1432)| = |\mathcal{A}_{2n+1}(2143)|.\end{aligned}$$

By constructing a spanning tree, Bóna [8, 9] proved

$$\begin{aligned}&|\mathcal{A}_{2n}(1234)| = |\mathcal{A}_{2n}(1243)|,\\&|\mathcal{A}_{2n}(12345)| = |\mathcal{A}_{2n}(12354)|,\\&|\mathcal{A}_{2n+1}(1234)| = |\mathcal{A}_{2n+1}(2134)|;\end{aligned}$$

and further obtained

$$\begin{aligned}&|\mathcal{A}_{2n}(12\ldots k)| = |\mathcal{A}_{2n}(12\ldots k(k-1))|,\\&|\mathcal{A}_{n}(12\ldots k)| = |\mathcal{A}_{n}(21\ldots(k-1)k)|.\end{aligned}$$

Chen et al. [21] proved by constructing a spanning tree

$$\begin{aligned}&|\mathcal{A}_{2n+1}(1243)| = |\mathcal{A}_{2n+1}(2143)|,\\&|\mathcal{A}_{2n}(4312)| = |\mathcal{A}_{2n}(1234)|.\end{aligned}$$

Lewis [45] also raised the following problems, which remain open to date.

Problem 2.1 Is there any other large pattern family that can be proved to be Wilf equivalent on alternating permutations?

Definition 2.2 Given a permutation σ, if for all $i \in \{k, 2k, \ldots\}$, $\sigma_i > \sigma_{i+1}$ is satisfied, that is, the descending set is $\{k, 2k, \ldots\}$, then the set comprised of permutations σ is $\mathrm{Des}_{n,k}$.

Problem 2.2 Can pattern avoidance be studied on $\mathrm{Des}_{n,k}$? On such a permutation set, can it be proved that there exists an equivalent pattern pair or family?

Definition 2.3 If alternating permutation σ satisfies $\sigma = \sigma^{-1}$, it is said to be an alternating involutory permutation.

Denote $\mathcal{AJ}_n$ as a set comprised of all alternating permutations on $[n]$. Barnabei et al. [4] studied the pattern-avoiding enumeration problem on alternating involutory permutations and yielded relevant results, as seen in Theorem 2.1.

Theorem 2.1 *For any permutation τ on $\{3, \ldots, m\}$, there exists*

$$|\mathcal{AJ}_n(12\tau)| = |\mathcal{AJ}_n(21\tau)|.$$

And for any permutation τ on $\{1,\ldots,k-2\}$, there exists

$$|\mathcal{AJ}_{2n}(\tau(k-1)k)| = |\mathcal{AJ}_{2n}(\tau k(k-1))|.$$

The results of avoiding single length-4 patterns on alternating involutory permutations are mostly related to the Motzkin number, while the results of avoiding double length-4 patterns are related to the Fibonacci number and power of 2. In addition, Barnabei et al. [4] proposed the following Conjectures 2.5 and 2.6.

Conjecture 2.5 *For $n \geqslant 1$, there exists*

$$|\mathcal{AJ}_{2n}(1432)| = |\mathcal{AJ}_{2n+1}(1432)| = |\mathcal{AJ}_{2n}(3214)|,$$

and

$$|\mathcal{AJ}_{2n-1}(3214)| = M_n - M_{n-2},$$

where M_n is the nth Motzkin number.

Conjecture 2.6 *If τ is any arbitrary permutation of $\{4,\ldots,m\}$, where $m \geqslant 4$, there exists*

$$|\mathcal{AJ}_n(123\tau)| = |\mathcal{AJ}_n(321\tau)|.$$

Yan et al. [71] proved the above two conjectures and conducted Wilf equivalence classification of avoiding length-4 patterns on alternating involutory permutations.

3 Research on pattern-avoiding enumeration on dumount permutations

There are four kinds of Dumont permutations. The first and second kinds were proposed by Dumont [31]. Burstein and Stromquist [17] proved the Kitaev-Remmel [38, 39] Conjecture by establishing the bijection between $\mathcal{D}_{2n}^1$ and $\mathcal{D}_{2n}^3$, defining the relevant permutation set as the third kind of Dumont permutation. They used Foata's fundamental transformation to map the third kind of Dumont permutations onto the permutation set of the same order, hence the fourth kind of Dumont permutation. For convenience, we first give the following definitions.

Definition 3.1 The fixed point (excedance, deficiency) in permutation $\pi = \pi_1\pi_2\ldots\pi_n$ refers to location i, which satisfies the condition

$$\pi(i) = i \quad (\pi(i) > i,\ \pi(i) < i).$$

Definition 3.2 (1) Permutations where the locations of every even value are descending and the locations of every odd value are ascending or which end with an odd value are referred to as Dumont permutations of the first kind.

(2) Permutations where the locations of every even value are deficiencies, and the locations of every odd value are fixed points or exceedances are referred to as Dumont permutations of the second kind.

(3) Permutations where all descendances are from an even value to an even value are referred to as Dumont permutations of the third kind.

(4) Permutations where all deficiencies must be at locations of even values and correspond to even values are referred to Dumont permutations of the fourth kind.

For $1 \leqslant i \leqslant 4$, use $\mathcal{D}_{2n}^{i}$ to represent the set comprised of length-$2n$ Dumont permutations of ith kind. Use the G_{2n} to represent nth Genocchi number. Dumont, Burstein and Stromquist [17, 31] obtained

$$\left|\mathcal{D}_{2n}^{1}\right| = \left|\mathcal{D}_{2n}^{2}\right| = \left|\mathcal{D}_{2n}^{3}\right| = \left|\mathcal{D}_{2n}^{4}\right| = G_{2n+2},$$

where the exponential generating function of the nth Genocchi number satisfies

$$\sum_{n=1}^{\infty} G_{2n}\frac{x^{2n}}{(2n)!} = x\tan\left(\frac{x}{2}\right),\ \sum_{n=1}^{\infty}(-1)^{n}G_{2n}\frac{x^{2n}}{(2n)!} = -x\tanh\left(\frac{x}{2}\right).$$

3.1 Research on pattern-avoiding enumeration on $\mathcal{D}_{2n}^{1}$

Mansour [49] studied the enumeration problem of avoiding single length-3 patterns on $\mathcal{D}_{2n}^{1}$ for the first time, and obtained by recursion

$$\left|\mathcal{D}_{2n}^{1}(132)\right| = \left|\mathcal{D}_{2n}^{1}(231)\right| = \left|\mathcal{D}_{2n}^{1}(312)\right| = C_n,\ \left|\mathcal{D}_{2n}^{2}(321)\right| = C_n.$$

Burstein et al. [13] established the bijection between the permutations enumerated by C_n and Dyck paths. Besides, Mansour, Ofodile, and Burstein [16, 50, 58] enumerated the permutations with restricted inclusion of length-3 patterns one time on $\mathcal{D}_{2n}^{1}$. Burstein [12] studied the enumeration problem of avoiding single length-3 and two length-3 patterns on $\mathcal{D}_{2n}^{1}$, and obtained

$$\left|\mathcal{D}_{2n}^{1}(213)\right| = C_{n-1},\quad \left|\mathcal{D}_{2n}^{2}(231)\right| = 2^{n-1},$$

and the following theorems.

Theorem 3.1 *For $n \geqslant 2$, there exists*

$$\begin{aligned}\left|\mathcal{D}_{2n}^{1}(132,231)\right| &= \left|\mathcal{D}_{2n}^{1}(132,312)\right| = \left|\mathcal{D}_{2n}^{1}(213,312)\right| = \left|\mathcal{D}_{2n}^{1}(132,213)\right| \\ &= \left|\mathcal{D}_{2n}^{1}(231,312)\right| = \left|\mathcal{D}_{2n}^{1}(123,213)\right| = \left|\mathcal{D}_{2n}^{1}(231,321)\right| = 1.\end{aligned}$$

Burstein [12] further studied the enumeration problem of avoiding two length-4 patterns on $\mathcal{D}_{2n}^{1}$, and presented Theorem 3.2 and Theorem 3.3.

Theorem 3.2 *The ordinary generating function $G(x)$ for $\mathcal{D}_{2n}^{1}(1423, 4132)$*

enumeration sequence is

$$G\left(x\right)=\frac{1-3x-\left(1+x\right)\sqrt{1-4x}}{1-3x-2x^2-\left(1+x\right)\sqrt{1-4x}}.$$

Theorem 3.3 (1) *For $n \geqslant 0$, there exists*

$$\left|\mathcal{D}_{2n}^1\left(1342,1423\right)\right|=\left|\mathcal{D}_{2n}^1\left(2341,2413\right)\right|=\left|\mathcal{D}_{2n}^1\left(1342,2413\right)\right|=s_{n+1},$$
$$\left|\mathcal{D}_{2n}^1\left(2413,3142\right)\right|=\sum_{k=0}^{n-1}\frac{n-k}{n+k}\binom{n+k}{n}2^k+\delta_{n=0},$$

namely power of 2 *and convolution of Ballot numbers.*

$$\left|\mathcal{D}_{2n}^1\left(2413,4132\right)\right|=\left|\mathcal{D}_{2n}^1\left(1423,3142\right)\right|,$$

where s_n is the nth small Schröder number.

(2) *For $n \geqslant 1$, there exists $|\mathcal{D}_{2n}^1(1342,4213)|=2^{n-1}$.*

(3) *For $n \geqslant 3$, there exists $|\mathcal{D}_{2n}^1(2341,1423)|=b_n$, where b_n satisfies $b_n=3b_{n-1}+2b_{n-2}$ and $b_0=1$, $b_1=1$, $b_2=3$.*

Regarding avoiding single length-4 patterns on $\mathcal{D}_{2n}^1$, Burstein and Jones [14] gave the following Conjecture 3.1 and Conjecture 3.2, which remain open to date.

Conjecture 3.1 *For $n \geqslant 0$, there exists*

$$\left|\mathcal{D}_{2n}^1\left(2143\right)\right|=\left|\mathcal{D}_{2n}^1\left(3421\right)\right|.$$

Definition 3.3

$$a_k=|\{\,\pi\in\mathcal{D}_{2n}^1\left(2143\right)|(2\underline{31})\pi=k\}|,$$
$$b_k=|\{\,\pi\in\mathcal{D}_{2n}^1\left(3421\right)|(\,\underline{13}2)\pi=k\}|,$$

where $(2\underline{31})\pi$ $((\underline{13}2)\pi)$ is the occurrence of vincular pattern $2\underline{31}$ $(\underline{13}2)$ appearing in π.

Conjecture 3.2 *For $n \geqslant 0$, there exists*

$$\begin{cases} a_k=b_k=1, & \text{when } k=\binom{n}{2};\\ a_k=b_k=0, & \text{when } k>\binom{n}{2};\\ \sum_{k=0}^{m}a_k\geqslant\sum_{k=0}^{m}b_k, & \text{when } 0\leqslant m\leqslant\binom{n}{2}. \end{cases}$$

The condition for the last formula to hold is $m=\binom{n}{2}$.

3.2 Research on pattern-avoiding enumeration on $\mathcal{D}_{2n}^2$

The foregoing has provided the enumeration results for avoiding patterns 321, 312 and 231 on $\mathcal{D}_{2n}^2$. According to the definition, $\mathcal{D}_{2n}^2$ contains patterns 123, 132, and 213, so the enumeration result for avoiding the three patterns is naturally zero. Hence, we have obtained all enumeration results for avoiding single length-3 patterns on $\mathcal{D}_{2n}^2$. Among the research on avoiding single length-4 patterns on $\mathcal{D}_{2n}^2$, Burstein [12] obtained $|\mathcal{D}_{2n}^2(3142)| = C_n$, and Elizalde and Mansour [13] found $|\mathcal{D}_{2n}^2(4132)| = |\mathcal{D}_{2n}^2(321)| = C_n$, along with the following Theorem 3.4.

Theorem 3.4 *For $n \geqslant 0$, there exists*

$$\left|\mathcal{D}_{2n}^2(2143)\right| = a_n a_{n+1},$$

where

$$a_{2m} = \frac{1}{2m+1}\binom{3m}{m}, \quad a_{2m+1} = \frac{1}{m+1}\binom{3m+1}{m}.$$

Use $\mathcal{D}_{2n}^2(\tau;1)$ to indicate a permutation set with restricted inclusion of length-4 pattern τ once in $\mathcal{D}_{2n}^2$. For $\tau \in \mathcal{D}_4^2$, Ofodile [58] obtained the enumeration result. Burstein and Ofodile [16] obtained the enumeration result of $\mathcal{D}_{2n}^2(2143;1)$, as shown in Theorems 3.5 and 3.6.

Theorem 3.5 *For $n \geqslant 0$, there exists*

$$\left|\mathcal{D}_{2n}^2(3142;1)\right| = \binom{2n-1}{n-1}.$$

Theorem 3.6 *For $n \geqslant 0$, there exists*

$$\left|\mathcal{D}_{2n}^2(2143;1)\right| = a_n b_{n+1} + b_n a_{n+1} + a_{n-1}a_n,$$

where

$$a_{2k} = \frac{1}{2k+1}\binom{3k}{k}, \ b_{2k} = \binom{3k-3}{k-2}.$$

3.3 Research on pattern-avoiding enumeration on $\mathcal{D}_{2n}^3$

Burstein et al. [15] obtained the following enumeration results of avoiding patterns 231 and 312 on $\mathcal{D}_{2n}^3$.

Theorem 3.7 *For $n \geqslant 0$, there exists*

$$\left|\mathcal{D}_{2n}^3(231)\right| = \left|\mathcal{D}_{2n}^3(312)\right| = C_n.$$

Genocchi's Seidel triangle is a Pascal triangular array of integers $(g_{i,j})_{i,j\geqslant 1}$, a subdivision of the Genocchi number, which satisfies the following recursion:

$$\begin{cases} g_{2i+1,j} = g_{2i+1,j-1} + g_{2i,j}, & \text{for } j = 1, 2, \ldots, i+1, \\ g_{2i,j} = g_{2i,j+1} + g_{2i-1,j}, & \text{for } j = i, i-1, \ldots, 1, \end{cases}$$

where $g_{1,1}=1$ and $g_{i,j}=0$. If $j\leqslant 0, i\leqslant 0$, or $i>\lceil \frac{j}{2}\rceil$. Then

$$g_{2n-1,n-1}=g_{2n-1,n}=g_{2n,n}=G_{2n+2},$$
$$g_{2n-1,1}=g_{2n-2,1}=H_{2n-1},$$

where G_{2n} is the nth Genocchi number and H_{2n-1} is the nth central Genocchi number. For $\mathcal{D}_{2n}^3$ and $\mathcal{D}_{2n}^4$, Burstein et al. [15] gave the following conjectures, which remain open to date.

Conjecture 3.3 *In $\mathcal{D}_{2n}^3$, the number of permutations where $2k$ follows $2n$ is $g_{2n,k}$, and the number of permutations where $2k$ is in front of 2 is $g_{2n,n-k+1}$.*

Conjecture 3.4 *In $\mathcal{D}_{2n}^4$, the number of permutations ending with $2k$ is $g_{2n,k}$, and the number of permutations where 2 is at the location of $2k$ is $g_{2n,n-k+1}$.*

Conjecture 3.5 *In $\mathcal{D}_{2n}^4$, the number of permutations $\pi=\pi_1\pi_2\dots\pi_n$ where $\pi_{2n}=2$ and $m\in[n-1]$ satisfies the condition that the minimal integer of $\pi_{2m+1}=2m+2$ is $g_{2n-1,m}$, and the number of permutations π where $\pi_{2n}=2$ without such $m\in[n-1]$ that $\pi_{2m+1}=2m+2$ is $g_{2n-1,n-1}$.*

3.4 Research on pattern-avoiding enumeration on $\mathcal{D}_{2n}^4$

Burstein and Jones [14] studied the pattern avoidance on $\mathcal{D}_{2n}^4$. When $n\geqslant 4$, it can be seen from the definition that $\mathcal{D}_{2n}^4$ contains pattern 1234 in $\mathcal{D}_4^4$, so the enumeration result of avoiding this pattern is naturally zero. The enumeration result of avoiding other patterns in $\mathcal{D}_4^4$ is as shown in the following Theorem 3.8.

Theorem 3.8 *For $n\geqslant 1$, there exists*

$$\left|\mathcal{D}_{2n}^4(1342)\right|=2^{n-1}.$$

For $n\geqslant 0$, there exists

$$\left|\mathcal{D}_{2n}^4(1432)\right|=\left|\mathcal{D}_{2n}^4(321)\right|=C_n.$$

The partial enumeration result for avoiding length-4 patterns except $\mathcal{D}_4^4$ and containing length-3 pattern 321 once on $\mathcal{D}_{2n}^4$ is as follows:

Theorem 3.9 *For $n\geqslant 0$, there exists*

$$\left|\mathcal{D}_{2n}^4(1324)\right|=\left|\mathcal{D}_{2n}^4(1243)\right|=\left|\mathcal{D}_{2n}^4(213)\right|=2\binom{n}{2}+1=n^2-n+1,$$
$$\left|\mathcal{D}_2^4\ (1423)\right|-\left|\mathcal{D}_2^4\ (312)\right|.$$

([14] also provided the ordinary generating function of $|\mathcal{D}_{2n}^4(1423)|$ in the form of a continued fraction, and obtained the corresponding OEIS sequence for the $\mathcal{D}_{2n}^4(312)$ counting sequence.)

Theorem 3.10 *For $n\geqslant 1$, there exists*

$$\left|\mathcal{D}_{2n}^{4}(321;1)\right| = \frac{3}{n}\binom{2n}{n-3} + \frac{3}{n+1}\binom{2n+2}{n-2}.$$

4 Research on pattern-avoiding enumeration on Ballot permutations

Definition 4.1 The prefix of a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ refers to a continuous initial subsequence $\sigma_1\sigma_2\cdots\sigma_p (1 \leqslant p \leqslant n)$.

Definition 4.2 Given a permutation $\sigma \in S_n$, if $\sigma_i > \sigma_{i+1}$, then $i \in [n-1]$ is said to be a descent of permutation σ. If $\sigma_i < \sigma_{i+1}$, then $i \in [n-1]$ is said to be an ascent of permutation σ.

Definition 4.3 Given a permutation σ, if the number of descents in any prefix of σ is no less than the number of ascents, then the permutation σ is said to be a Ballot permutation.

For example, permutation 14325 is not a Ballot permutation because the number of descents (2) is greater than the number of ascents (1) in the prefix 1432.

Definition 4.4 A Gessel walk is constrained to the 1/4 $\mathbb{N}^2$ plane, with each step consisting of a $\{\uparrow, \downarrow, \nearrow, \swarrow\}$ lattice path.

Definition 4.5 For $i, j \in \mathbb{N}$, denote $\mathcal{G}(n; 0, j)$ as the set comprised of n th Gessel walks from $(0, i)$ to $(0, j)$.

Denote $\mathcal{B}_n$ as the set of all Ballot permutations on $[n]$. Lin et al. [46] first studied the enumeration of avoiding length-3 patterns on Ballot permutations and obtained $|\mathcal{B}_n(123)| = C\left(\lceil \frac{n}{2} \rceil\right)$ and $|\mathcal{B}_n(321)| = \frac{3}{n+1}\binom{2n-2}{n-2}$.

In addition, they concluded that patterns 213 and 312 as well as patterns 132 and 231 are Wilf-equivalent on $\mathcal{B}_n$. They established the bijection between $\mathcal{B}_n(213)$ and Gessel walks with endpoints on the y-axis, and summarized the following two theorems.

Theorem 4.1 *For $n \geqslant 1$, there exists*

$$|\mathcal{B}_n(213)| = |\mathcal{B}_n(312)| = \sum_{j=0}^{n} |\mathcal{G}(n; 0, j)|.$$

Theorem 4.2 *For $n \geqslant 0$, there exists*

$$|\mathcal{B}_n(132)| = |\mathcal{B}_n(231)| = C\left(\left\lfloor \frac{n}{2} \right\rfloor\right) C\left(\left\lfloor n + \frac{1}{2} \right\rfloor\right).$$

They also raised the following problems.

Definition 4.6 For $\boldsymbol{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$, denote $\mathcal{G}_m$ as the set of multiple permutations comprised of $\{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$. For an element π in $\mathcal{G}_m$, if for every $i, 1 \leqslant i \leqslant \sum_{i=1}^{n} m_n$, there exists

$$|\{j \in [i] : \pi_j < \pi_{j+1}\}| \geqslant |\{j \in [i] : \pi_j > \pi_{j+1}\}|.$$

Then we call this element a Ballot multiple permutation.

Problem 4.1 In $\mathcal{G}_m$, can Ballot multiple permutations be counted for fixed $m \in \mathbb{N}^n$?

Sun [65] further obtained all enumeration results of avoiding two length-3 patterns and three length-3 patterns, concluded the Wilf equivalence of corresponding patterns to $\mathcal{B}_n(132,213)$, $\mathcal{B}_n(213,231)$, $\mathcal{B}_n(231,312)$, $\mathcal{B}_n(132,312)$, established the bijection between $\mathcal{B}_n(132,312)$ and the left factor of the Dyck path, and raised the following problems.

Problem 4.2 Can the enumeration result of Ballot permutations avoiding length-4 patterns be obtained?

Problem 4.3 Can the enumeration result of Ballot multiple permutations avoiding patterns in $\mathcal{S}_n$ be obtained?

Problem 4.4 Can the enumeration result of Ballot permutations, whose number of ascents is at least λ times over the number of descents, be obtained for avoiding a single lengh-3 pattern or two length-3 patterns?

5 Research on pattern-avoiding enumeration on inversion sequences

The pattern-avoiding enumeration of inversion sequences is a problem related to word enumeration. However, its research methods and results are mostly related to permutation enumeration, which has attracted the attention of scholars in recent years. Therefore, this paper will introduce it.

Definition 5.1 Given a length-n integer sequence $e = e_0 e_1 \cdots e_n$, if any $0 \leqslant i \leqslant n$ satisfies $0 \leqslant e_i \leqslant i$, then it is said to be an inversion sequence.

Definition 5.2 Given any length-k word τ on $[k]$, if there is a length-k subsequence q in the inversion sequence e that is order isomorphic to the pattern τ, then the inversion sequence e is said to contain the pattern τ; otherwise, the inversion sequence e avoids the pattern τ. Denote $\mathcal{J}_n$ as the set composed of all inversion sequences on $[n]$.

For example, $e = 01021321 \in \mathcal{J}_7$ contains patterns 120 and 000, corresponding to subsequence 231 and 111 in e, but e avoids pattern 201.

The concept of pattern avoidance in inversion sequences was proposed by Corteel et al. [30] and Mansour and Shattuck [53]. Mansou and Shattuck [53] conducted a systematic study for the first time on the problem of avoiding length-3 patterns without repeating letters in inversion sequences. They presented the enumeration results of avoiding patterns 012, 021, 102, 201, and 210 in inversion sequences and obtained the enumeration results of avoiding pattern 120 in inversion sequences under certain conditions. In addition, Corteel et al. [30] proved that pattern 201 and pattern 210 are Wilf equivalent on $\mathcal{J}_n$, obtained the recursive relationship of $\mathcal{J}_n(201)$ and $\mathcal{J}_n(210)$ enumeration results, and linked the enumeration results with classical combinatorial numbers, as shown in Theorems 5.1 and 5.2.

Theorem 5.1

(1) *For* $n \geqslant 1$, *there exists*

$$|\mathcal{J}_n\left(012\right)| = F_{2n-1}.$$

(2) *For* $n \geqslant 1$, *there exists*

$$|\mathcal{J}_n\left(021\right)| = r_{n-1}.$$

(3) *For* $n \geqslant 0$, *there exists*

$$|\mathcal{J}_n\left(102\right)| = \sum_{m=\lceil \frac{n}{2} \rceil}^{n} \frac{(-1)^{n-m}}{2m+1}\binom{3m}{m}\binom{m}{n-m}.$$

Additionally, the generating function of $|\mathcal{J}_n\left(012\right)|$ is

$$\sum_{n\geqslant 1}|\mathcal{J}_n\left(012\right)|x^n = \frac{x}{1-x} + \frac{x^2}{(1-x)\left(1-3x+x^2\right)} = \sum_{n\geqslant 1} F_{2n-1}x^n,$$

the generating function of $|\mathcal{J}_n\left(021\right)|$ is

$$\sum_{n\geqslant 1}|\mathcal{J}_n\left(021\right)|y^n = \frac{1-y-\sqrt{1-6y+y^2}}{2} = \sum_{n\geqslant 1} r_{n-1}y^n,$$

where F_n is the nth Fibonacci number, and r_n is the nth large Schröder number.

Definition 5.3 For $e \in \mathcal{J}_n$, Let $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_t$ be a weak maximum sequence of e from left to right, and assume $b_1 < b_2 < \cdots < b_{n-t}$ is the sequence of remaining indices in $[n]$. Define $e^{\text{top}} = (e_{a_1}, e_{a_2}, \ldots, e_{a_t})$ and top $(e) = e_{a_t}$, define $e^{\text{bottom}} = \left(e_{b_1}, e_{b_2}, \ldots, e_{b_{n-t}}\right)$ and bottom $(e) = e_{b_{n-t}}$. If every item of e is a weak maximum from left to right, then e^{bottom} is empty, and here let bottom $(e) = -1$.

Theorem 5.2 *For* $n \geqslant 0$, *there exists*

$$|\mathcal{J}_n\left(201\right)| = |\mathcal{J}_n\left(210\right)| = \sum_{a=0}^{n-1}\sum_{b=-1}^{a-1}\mathcal{T}_{n,a,b} = \frac{1}{n+1}\binom{2n}{n} + \sum_{a=0}^{n-1}\sum_{b=0}^{a-1}\mathcal{T}_{n,a,b},$$

where $\mathcal{T}_{n,a,b}$ *is the number of* $e \in \mathcal{J}_n(201)$, *and* e *satisfies* $tan\left(e\right) = a$, *bottom*

$(e) = b$.

Corteel et al. [30] also studied the problem of avoiding length-3 patterns with repeated letters in inversion sequences, obtained the counting results of avoiding patterns 000, 001, and 011, and proved $|\mathcal{J}_n(101)| = |\mathcal{J}_n(110)|$, as shown in Theorems 5.3 and 5.4.

Theorem 5.3 *For $n \geqslant 1$, there exists*

$$|\mathcal{J}_n(000)| = |E_{n+1}|,$$
$$|\mathcal{J}_n(001)| = 2^{n-1},$$
$$|\mathcal{J}_n(011)| = B_n,$$

where E_n is the nth Euler number, and B_n is the nth Bell number, namely the number of all divisions of $[n]$.

Theorem 5.4 *For $n \geqslant 1$, there exists*

$$|\mathcal{J}_n(101)| = |\mathcal{J}_n(110)|.$$

Conjecture 5.1 *For $n \geqslant 1, 0 \leqslant k \leqslant n-1$, denote $d_{n,k}$ as the number of words with the last element being $k-1$ in $\mathcal{J}_n(000)$. Then $d_{n,k}$ satisfies the recursive relationship*

$$d_{n,k} = d_{n,k-1} + d_{n-1,n-k}.$$

Later, Kotsiias et al. [41] used a spanning tree-based algorithm to obtain the function equation of $\mathcal{J}_n(100)$ and $\mathcal{J}_n(201)$ generating functions. Testart [66] obtained the counting results of $\mathcal{J}_n(010)$ by decomposing the inversion sequence.

Martinez and Savage [56] linked the problem of avoiding patterns in inversion sequences with classical combinatorial structures and other pattern-avoiding permutations, providing simpler explanations for some combinatorial sequences than before. For the first time, they redefined length-3 patterns as triples of binary relations and obtained the counting results of inversion sequences avoiding legnth-3 patterns and conjectures about the counting results of $\mathcal{J}_n(021, 120)$, namely Theorem 5.5 and Conjecture 5.2.

Theorem 5.5 *For $n \geqslant 1$, there exists*

$$|\mathcal{J}_n(201, 100)| = \mathrm{SB}_n,$$

where SB_n is the nth semi-Baxter number.

Definition 5.4 $\mathcal{J}_n(\rho_1, \rho_2, \rho_3)$ is a set comprised of $e \in \mathcal{J}_n$ satisfying the following conditions: there is no such $i < j < k$ that $e_i \rho_1 e_j$, $e_j \rho_2 e_k$, $e_i \rho_3 e_k$.

For example, $|\mathcal{J}_n(<, >, <)| = |\mathcal{J}_n(021)|$.

Conjecture 5.2 *For $n \geqslant 1$, there exists*

$$|\mathcal{J}_n(>, \neq, -)| = |\mathcal{J}_n(<, >, \neq)|.$$

Recently, Yan and Lin [70] obtained numerous counting results for inversion sequences avoiding double length-3 patterns, which were related to Fibonacci numbers, powers of 2, Schröder numbers and Bell numbers. They also solved the conjecture by Martinez and Savage [56] about the effect of inversion sequence pattern avoidance on the counting results. Subsequently, Kotsiias et al. [41] used a spanning tree-based algorithm to obtain the spanning tree of pattern pairs $\mathcal{J}_n(000, 021)$, $\mathcal{J}_n(100, 021)$, $\mathcal{J}_n(110, 021)$, $\mathcal{J}_n(102, 021)$, $\mathcal{J}_n(100, 012)$, $\mathcal{J}_n(011, 201)$, $\mathcal{J}_n(011, 210)$, and $\mathcal{J}_n(120, 210)$, and used the kernel function method to get the corresponding counting results' generating functions and counting sequences.

Hong and Li [36] studied the Wilf equivalence class problem for inversion sequences avoiding length-4 patterns. Kotsireans et al. [41, 54] studied the counting problem of inversion sequences while avoiding pattern 021 and length-4 patterns using the spanning tree method. Lin and Ma [70] proposed the following conjecture on the counting results of the inversion sequence avoiding pattern 0012.

Conjecture 5.3 *For $n \geqslant 1$, there exists*

$$|\mathcal{J}_n(0012)| = 1 + \sum_{i=1}^{n-1} \binom{2i}{i-1}.$$

Later, Chern [23] proved the conjecture. Hong and Li [36] proposed the following conjecture on the counting results of the inversion sequence avoiding pattern 0012.

Conjecture 5.4 *Let $A_n = |\mathcal{J}_n(0021)|$ and $A(x) = \sum_n \geqslant 1 A_n x^n$, then $A(x)$ satisfies*

$$\frac{1}{(1 - A(x))(1 + A(x))^2} = 1 - x.$$

This conjecture has been proved by Chern et al. [24] and Mansour [51] at the same time.

6 Research on enumeration of avoiding vincular patterns on $\mathcal{S}_n$

Definition 6.1 Given the pattern $\tau = \tau_1 \tau_2 \cdots \tau_k$, some consecutive digits in τ are underlined as follows:

$$\underline{\tau_i \tau_{i+1} \cdots \tau_j}.$$

If permutation σ contains a pattern τ, the corresponding letters appearing in permutation σ must be adjacent, while there is no adjacent condition for consecutive letters without an underline. This pattern τ is called a vincular pattern.

For example, permutation $\sigma = 541326$ contains pattern $\tau = 1\underline{23}$ one time, corresponding to the subsequence 126 in σ.

The vincular pattern had already appeared in other references before it was formally studied in reference [2]. Simon and Stanton [63] found that patterns 231, 213, 132, and 312 were related to a set of orthogonal polynomials of a generalized Laguerre polynomial. In addition, many permutation statistics could also be described by vincular patterns, such as valley (213 or 312), peak (231 or 132), continuous ascendance ($\underline{123}$), and continuous descendance (321). More generally, the continuous ascendance and descendance of length k are $12\cdots k$ and $k\,(k-1)\cdots 1$, respectively.

6.1 Research on enumeration of permutations avoiding length-3 vincular patterns

Claesson [25] first studied the enumeration problem of avoiding length-3 vincular patterns on $\mathcal{S}_n$, and found that the partial counting results of avoiding two length-3 vincular patterns were related to the Motzkin number, Bessel number, and involution number. At the same time, Claesson obtained the relationship between the partial counting results of avoiding length-3 vincular patterns on S_n and other combinatorial structures.

The counting result of $\mathcal{S}_n\,(123)$ is a Bell number, indicating that the Stanley-Wilf Conjecture doesn't hold for the case of vincular patterns, nor does the conjecture by Noonan and Zeilberger [57]. That is, the counting result of avoiding a vincular pattern may not necessarily be polynomially recursive.

Claesson and Mansour extended the results of [25] in [27], providing complete counting results for the two vincular patterns of type (1, 2) or (2, 1), and conjectured the counting results for avoiding three or more vincular patterns of type (1, 2) or (2, 1) on $\mathcal{S}_n$. Bernini et al. [5] proved the partial counting conjecture for avoiding three vincular patterns on $\mathcal{S}_n$. Bernini and Pergola [6] proved the partial counting conjecture for avoiding four, five, and six vincular patterns on $\mathcal{S}_n$. These counting results were related to the Motzkin number, Fibonacci number, binomial coefficient, and power of 2.

6.2 Research on enumeration of permutations avoiding length-4 vincular patterns

Steingrímsson [64] proposed that there were 48 symmetric classes for length-4 vincular patterns on $\mathcal{S}_n$, and through computer experiments, it was speculated that there were at least 24 Wilf equivalent classes. For the counting problem of avoiding length-4 vincular patterns on S_n, the counting results of 7 Wilf classes were known, as described below.

Elizalde and Noy [34] provided exponential generating functions for three consecutive length-4 patterns $\underline{1234}$, $\underline{1243}$, $\underline{1342}$ in seven Wilf equivalence classes.

Kitaev [37] and Elizalde [32] studied patterns $\underline{1234}$ and $\underline{1324}$ in two Wilf equivalence classes based on the counting results of avoiding consecutive length-4 patterns on S_n by Elizalde and Noy, and proved that their generation functions were respectively

$$\exp\left(\frac{\sqrt{3}}{2}\int_0^x \frac{\mathrm{e}^{t/2}\mathrm{d}t}{\cos\left(\frac{\sqrt{3}}{2}+\frac{\pi}{6}\right)}\right), \quad \exp\left(\int_0^x \frac{\mathrm{d}t}{1-\int_0^x \mathrm{e}^{-u^2/2}\mathrm{d}u}\right).$$

Callan [18] proved $|\mathcal{S}_n(3\bar{5}241)| = |\mathcal{S}_n(3142)|$, and gave two recursion formulas for $|\mathcal{S}_n(3142)|$ [19]. One recursion formula of $a_n = |\mathcal{S}_n(3142)|$ is as follows:

$$a_n = \sum_{i=0}^{n-1} a_i c_{n-i}, n \geqslant 1;$$

$$c_n = \sum_{i=0}^{n-1} i a_{n-1,i}, n \geqslant 2;$$

$$a_{n,k} = \begin{cases} \sum_{i=0}^{k-1} a_i \sum_{j=k-i}^{n-1-i} a_{n-1-i,j}, & 1 \leqslant k \leqslant n-1; \\ a_{n-1}, & k = n. \end{cases}$$

Callan [20] also obtained:

$$|\mathcal{S}_n(1\underline{23}4)| = \sum_{k=1}^{n} u(n,k),$$

where $u(n,k)$ has the following recursive relationship:

$$u(n,k) = u(n-1,k-1) + k\sum_{j=k}^{n-1} u(n-1,j)\ (n \geqslant 1,\ u(0,0)=0,\ u(n,0)=0).$$

This result turns out an improvement to the $|\mathcal{S}_n(1234)|$ complex recursion.

Asinowski et al. [1] proved there was a relationship between Baxter permutation set $\mathcal{S}_n(2413, 3142)$ and set $\mathcal{S}_n(2143, 3412)$, and obtained the counting results and generating function for $\mathcal{S}_n(2\underline{14}3, 3\underline{412})$, as shown in Theorem 6.1.

Theorem 6.1 *The generating function of* $\mathcal{S}_n(2\underline{14}3, 3\underline{412})$ is

$$\sum_{n\geqslant 1} x^{n-1}(1-x)^n b_n,$$

where

$$b_n = |\mathcal{S}_n(2\underline{413}, 3\underline{142})| = \sum_{m=0}^{n} \frac{2}{n(n+1)^2}\binom{n+1}{m}\binom{n+1}{m+1}\binom{n+1}{m+2}$$

is the counting result of length-n Baxter permutation. So,

$$|\mathcal{S}_n\,(2\underline{14}3, 3\underline{41}2)| = \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} (-1)^i \binom{n+1-i}{i} b_{n+1-i}.$$

Elizalde [33] obtained the following theorem.

Theorem 6.2

$$\sum_{n\geqslant 0} |\mathcal{S}_n(\,1\underline{23}, 3\underline{12}, \underline{3421}\,)|\, x^n = -1 + \sum_{k\geqslant 0} \frac{k(1+kx)x^2}{(1-(k+1)x)\prod_{j=1}^{k-1}(1-jx)},$$

$$\sum_{n\geqslant 0} |\mathcal{S}_n(\,1\underline{23}, \underline{3421}\,)|\, x^n = \sum_{k\geqslant 0} \frac{(1+kx)x^{k+1}}{(1+x)^k(1-kx)(1-(k+1)x)}.$$

6.3 Research on enumeration of including vincular pattern times

Claesson and Mansour [26] studied the counting problem of $\mathcal{S}_n$ that includes vincular patterns 123 and 132 exactly one time, and obtained the following theorem, where $\mathcal{S}_n^1\,(\tau)$ represents a permutation set containing pattern τ exactly one time in $\mathcal{S}_n$.

Theorem 6.3 *Assume B_n is the n-th Bell number, for $n \geqslant -1$, there exists*

$$\left|\mathcal{S}_{n+2}^1\,(1\underline{23})\right| = 2\left|\mathcal{S}_{n+1}^1\,(1\underline{23})\right| + \sum_{k=0}^{n-1}\binom{n}{k}\left[\left|\mathcal{S}_{k+1}^1\,(1\underline{23})\right| + B_{k+1}\right] \quad \left(\left|\mathcal{S}_0^1\,(1\underline{23})\right| = 0\right).$$

Theorem 6.4 *For $n \geqslant 0$, there exists*

$$\left|\mathcal{S}_{n+1}^1\,(1\underline{32})\right| = \left|\mathcal{S}_n^1\,(1\underline{32})\right| + \sum_{k=1}^{n-1}\left[\binom{n}{k}\left|\mathcal{S}_k^1\,(1\underline{32})\right| + \binom{n-1}{k-1}B_k\right] \quad \left(\left|\mathcal{S}_0^1\,(1\underline{32})\right| = 0\right).$$

7 Research on enumeration of permutations avoiding barred patterns on $\mathcal{S}_n$

Definition 7.1 A length-r barred pattern is a permutation where some digits are underlined. We use $\overline{\mathcal{S}}_r$ to represent the set of all barred patterns with a length of r.

For example, $\overline{\mathcal{S}}_1 = \{1, \overline{1}\}$, $\overline{\mathcal{S}}_2 = \{12, \overline{1}2, 1\overline{2}, \overline{12}, 21, \overline{2}1, 2\overline{1}, \overline{21}\}$. 24153 and 31524 are barred patterns in $\overline{\mathcal{S}}_5$.

Definition 7.2 For $\overline{\tau} \in \overline{\mathcal{S}}_r$, where τ is the pattern obtained by removing all the overlines in $\overline{\tau}$. Define τ' as the pattern obtained by removing all underlined digits and simplifying them.

For example, if $\overline{\tau}$ =53214, then τ =53214 and τ' =312.

Definition 7.3 Given a permutation π and pattern $\overline{\tau}$, if each occurrence of

τ' in π is a part of the occurrence of τ in π, then permutation π is called pattern $\overline{\tau}$ avoiding.

For example, permutation $\pi = 1625374$ avoids pattern $\overline{\tau} = 4\overline{1}32$, because the corresponding subsequences of $\tau' = 321$ in π are 635 and 645, which are part of subsequences 6253 and 6254, corresponding to the pattern $\tau = 4132$ in π.

The counting result of $\mathcal{S}_n$ avoiding length-1 and length-2 barred patterns is trivial, and the counting result of avoiding length-3 barred patterns with 3 digits all overlined is $n! - \frac{1}{n+1}\binom{2n}{n}$. In addition, the counting result of avoiding other length-3 barred patterns with 1 digit overlined is 1 and $(n-1)!$.

Callan [18] and Pudwell [59] studied all length-4 patterns with 1 digit overlined, and obtained the following theorems.

Theorem 7.1 *For* $p \in \{1\overline{4}23, 134\overline{2}, 23\overline{1}4, \overline{2}431, \overline{3}124, 32\overline{4}1, 4\overline{1}32, 421\overline{3}, 2\overline{4}13, 24\overline{1}3, 24\overline{1}3, 241\overline{3}, \overline{3}142, 3\overline{1}42, 31\overline{4}2, 314\overline{2}\}$, *there exists*

$$|\mathcal{S}_n(p)| = B_n,$$

where B_n *is the nth Bell number.*

Theorem 7.2 *For* $p \in \{\overline{1}432, 234\overline{1}, 23\overline{1}4, \overline{4}123\}$, *there exists*

$$|\mathcal{S}_n(p)| = (n-1)! + \sum_{j=2}^{n} \frac{(n-2)!}{(j-2)!} + \sum_{i=3}^{n} \sum_{j=2}^{n-i+2} \sum_{l=j+i-2}^{n} \frac{(n-i)!\,(l-j-1)!}{(l-i)!\,(i-3)!}.$$

Theorem 7.3 *For* $p \in \{\overline{1}432, \overline{1}342, 231\overline{4}, 2\overline{4}31, 312\overline{4}, 324\overline{1}, \overline{4}132, \overline{4}213, \overline{1}324, 132\overline{4}, 423\overline{1}, \overline{4}231\}$, *there exists*

$$|\mathcal{S}_n(p)| = \sum_{i=1}^{n} (n-i)!\,|\mathcal{S}_{i-1}(p)|.$$

Elizalde [33] studied the counting problem of simultaneously avoiding vincular patterns and barred patterns and obtained the following theorem.

Theorem 7.4 *For* $n \geqslant 1$, *there exists*

$$|\mathcal{S}_n(213, \overline{2}\underline{31})| = M_{n-1},$$

where M_{n-1} *is the* $(n-1)$ *-th Motzkin number.*

In addition, Pudwell [59] obtained $|\mathcal{S}_n(1\overline{4}23)| = |\mathcal{S}_n(123)|$, Chen et al. [22] got $\mathcal{S}_n(321, 3\overline{1}42)| = M_n$, Bousquet-Mélou and Butler [10] reached $|\mathcal{S}_n(4231, 45\overline{3}12)| = |\mathcal{S}_n(4231, 3412)|$, and combined with $|\mathcal{S}_n(2413)| = |\mathcal{S}_n(3412)|$ to obtain the following theorem.

Theorem 7.5 *For* $n \geqslant 1$, *there exists*

$$|\mathcal{S}_n(2\underline{41}3)| = |\mathcal{S}_n(\underline{34}12)| = \left|\mathcal{S}_n(21\overline{3}54)\right|.$$

Bousquet-Mélou and Claesson [11] also got the counting result of $\mathcal{S}_n$ avoiding $3\bar{1}52\bar{4}$, and obtained the following theorem.

Theorem 7.6 *For $n \geqslant 1$, there exists*

$$\left|\mathcal{S}_n\left(3\bar{1}52\bar{4}\right)\right| = \sum_{k=1}^{n} \binom{\binom{k+1}{2} + n - k - 1}{n-k}.$$

The kth item of the above sum contains k counting results of minimum permutations from right to left, hence solving Pudwell's conjecture [59]. At the same time, the generating function corresponding to the above result is

$$\sum_{n \geqslant 0} \left|\mathcal{S}_n\left(3\bar{1}52\bar{4}\right)\right| x^n = \sum_{k \geqslant 1} \frac{x^k}{(1-x)^{\binom{k+1}{2}}}.$$

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