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RESEARCH ARTICLE

L^1 Boundedness of a class of rough Fourier integral operators

Xiangrong ZHU, Yuchao MA

School of Mathematical Sciences, Zhejiang Normal University, Jinhua 321004, China

Abstract In this note, we consider a class of Fourier integral operators with rough amplitudes and rough phases. When the index of symbols in some range, we prove that they are bounded on L^1 and construct an example to show that this result is sharp in some cases. This result is a generalization of the corresponding theorems of Kenig-Staubach and Dos Santos Ferreira-Staubach.

Keywords Fourier integral operators, amplitude, phase
MSC2020 42B20, 35S30

1 Introduction

In this note we consider a Fourier integral operator defined by

$$T_{\phi,a}f(x)=\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\phi(x,\xi)}a(x,\xi)\hat{f}(\xi)\mathrm{d}\xi,$$

where a is called the symbol and ϕ is the phase. In particular, it is a pseudo-differential operator if $\phi(x,\xi)=x\cdot\xi$.

Fourier integral operators have been used widely in the theory of partial differential equations and micro-local analysis. For example, the solution of an initial value problem for a variable coefficient hyperbolic equation can be well approximated by an FIO of the initial value. A systematic study of these operators was initiated by Hörmander. One can also see [6, 10, 11].

For the symbols and phases in Fourier integral operators, some most important definitions are as follows.

a belongs to the Hörmander class $S^m_{\rho,\delta}(m\in\mathbb{R},0\leqslant\rho,\delta\leqslant1)$ if it satisfies

$$\sup_{x,\xi\in\mathbb{R}^n}(1+|\xi|)^{-m+\rho|\alpha|-\delta|\beta|}|\partial_x^\beta\partial_\xi^\alpha a(x,\xi)|<+\infty$$

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Corresponding author: Yuchao MA, E-mail: 15225808538@163.com

for all multi-indices α,β.

For the phase ϕ, if it is homogeneous of degree 1 in the frequency variable ξ and satisfies

$$\sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}}|\xi|^{-1+|\alpha|}|\partial_x^\beta\partial_\xi^\alpha\phi(x,\xi)|\leqslant C_{\alpha,\beta}$$

for all multi-indices α,β with $|\alpha|+|\beta|\geqslant k$, then we say that $\phi\in\Phi^k$. In general, we can assume that $k=2$.

Furthermore, a real-valued function $\phi\in C^2(\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\}))$ satisfies the strong non-degeneracy (SND) condition, if there exists a constant $\lambda>0$ such that

$$\left|\det\frac{\partial^2\phi(x,\xi)}{\partial x_j\partial\xi_k}\right|\geqslant\lambda.$$

For example, $\phi(x,\xi)=x\cdot\xi+|\xi|$ is in Φ^2 and satisfies the strong non-degeneracy condition. It matches the wave operator and is a typical Fourier integral operator.

The boundedness of Fourier integral operators on Lebesgue space has been extensively studied and there are numerous results, such as [1–4, 7–11, 13–16]. One can find more details in a survey of Dos Santos Ferreira and Staubach [5]. Especially, for the best result on L^2 so far, when $a\in S^m_{\rho,\delta}$ and $\phi\in\Phi^2$ satisfying the SND condition, if $m\leqslant\min\{0,\frac{n}{2}(\rho-\delta)\}$, $\delta<1$, or $m<\frac{n(\rho-1)}{2}$, $\delta=1$, the Fourier integral operator is bounded on L^2. One can see [5, Theorem 2.7]. The bound on m is sharp. Rodino in [17] constructed an $a\in S^{n(\rho-1)/2}_{\rho,1}$ such that the pseudo-differential operator is unbounded on L^2.

For the endpoint estimate, Seeger et al., in [18, 19] proved that when $a\in S^{\frac{1-n}{2}}_{1,0}$ and $\phi\in\Phi^2$ satisfying the SND condition, the Fourier integral operator is local H^1-L^1 bounded. And they say that when $m<\frac{1-n}{2}$, the corresponding operator is bounded on L^1. Their result is generalized to global boundedness. One can see [5]. Tao [20] subsequent proved that when $a\in S^{\frac{1-n}{2}}_{1,0}$ and $\phi\in\Phi^2$ satisfying the SND condition, the operator is also of weak type $(1,1)$. In the same paper, Tao pointed out the operator is not bounded on L^1.

Kenig and Staubach in [12] defined the rough Hörmander class $L^\infty S^m_\rho$ $(0\leqslant\rho\leqslant1)$. It consists all functions a that satisfy

$$\|\sup_{\xi\in\mathbb{R}^n}(1+|\xi|)^{k\rho-m}\nabla_\xi^k a(\cdot,\xi)\|_{L^\infty(\mathbb{R}^n)}=C_k<\infty, k=0,1,2,\ldots.$$

For $a\in L^\infty S^m_\rho$, Kenig–Staubach proved the pseudo-differential operator is bounded on L^1 if $m<n(\rho-1)$ and on L^∞ if $m<\frac{n(\rho-1)}{2}$.

In the maximal wave operator and maximal sphere average operator, the phase is not smooth in the spatial variable x. In these cases, $\phi(x,\xi)=x\cdot\xi+t(x)|\xi|, t\in L^\infty$. Dos Santos Ferreira and Staubach in [5] introduced the class $L^\infty\Phi^2$ and rough non-degeneracy condition. $\phi\in L^\infty\Phi^2$ if ϕ is homogeneous of degree 1 in the frequency variable ξ and satisfies

$$\sup_{\xi\in\mathbb{R}^n\setminus\{0\}}|\xi|^{-1+|\alpha|}\|\partial_\xi^\alpha\phi(\cdot,\xi)\|_{L^\infty(\mathbb{R}^n)}<+\infty$$

for all multi-indices α,β with $|\alpha|+|\beta|\geqslant 2$.

A real-valued function $\phi\in C^2(\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\}))$ satisfies the rough non-degeneracy condition (RND), if there exists a constant $\lambda>0$ such that

$$|\nabla_\xi\phi(x,\xi)-\nabla_\xi\phi(y,\xi)|\geqslant\lambda|x-y|$$

for any $x,y\in\mathbb{R}^n,\xi\in\mathbb{R}^n\setminus\{0\}$.

For $a\in L^\infty S^m_\rho,\phi\in L^\infty\Phi^2$ satisfying the RND condition, in [5], Dos Santos Ferreira and Staubach systematic studied the boundedness of Fourier integral operators on various Lebesgue spaces. For the boundedness on L^1, their main result can be stated as follows.

Theorem A *Suppose that $0\leqslant\rho\leqslant 1, a\in L^\infty S^m_\rho$ and $\phi\in L^\infty\Phi^2$ satisfying the RND condition. If $m<-\frac{n-1}{2}+n(\rho-1)$, then $T_{\phi,a}$ is bounded on L^1, i.e., there exists a constant $C>0$ such that*

$$\|T_{\phi,a}f\|_{L^1}\leqslant C\|f\|_{L^1}$$

for any $f\in L^1$.

In this note, we extend conditions on ϕ. We assume that there exist $C>0$ and $\delta\in(0,1]$, such that

$$|\{x:\nabla_\xi\phi(x,\xi)\in E\}|\leqslant C|E|; \tag{1}$$

$$|\nabla_\xi^k\phi(x,\xi)|\leqslant C|\xi|^{\delta-k},\ k\geqslant 2 \tag{2}$$

for any $x,\xi\in\mathbb{R}^n$ and $E\subset\mathbb{R}^n$.

Remark 1 At first, we show that the RND condition yields (1). For any small ball $B(x_0,r)$, since $|\nabla_\xi\phi(x,\xi)-\nabla_\xi\phi(y,\xi)|\geqslant\lambda|x-y|$, the diameter of the set $\{x:\nabla_\xi\phi(x,\xi)\in B(x_0,r)\}$ is no more than $\frac{2r}{\lambda}$. Therefore, we can get that $|\{x:\nabla_\xi\phi(x,\xi)\in B(x_0,r)\}|\leqslant(\frac{2}{\lambda})^n|B(x_0,r)|$. For any measurable set E, by using the definition of outer measure, one can derive (1).

Secondly, if $\phi\in C^\infty(\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\}))$, by using [5, Proposition 1.11], we know that the SND condition is equivalent to the RND condition. For any fixed ξ, set $F(x)=\nabla_\xi\phi(x,\xi)$. Then for any measurable set E, we have $E\subset F^{-1}(F(E))$. From (1) we can get that $|E|\leqslant|F^{-1}(F(E))|\leqslant CF(E)=C\int_E|\mathrm{det}F|\mathrm{d}x$. As E is arbitrary, by Lebesgue's differential theorem, the Jacobian determinant $|\mathrm{det}F|=|\mathrm{det}\frac{\partial^2\phi(x,\xi)}{\partial x\partial\xi}|$ is no less than $\frac{1}{C}$ almost everywhere. Thanks to the continuity of ϕ, we can derive the SND condition. So in this case, the SND condition, the RND condition and (1) are equivalent. At last, let $n=1$ and $\phi(x,\xi)=|x|\xi$. It is easy to see that $\nabla_\xi\phi(x,\xi)=|x|$ satisfies (1) but does not satisfy the RND condition. So (1) is a generalization of the RND condition.

In this note, we mainly prove the following result.

Theorem 1 *If* $a \in L^\infty S^m_\rho$ *and* ϕ *satisfies* (1) *and* (2), *then* $T_{\phi,a}$ *is bounded on* L^1 *when* $m < n\min\{\rho-1, -\frac{\delta}{2}\}$. *Furthermore, when* $\rho \leqslant 1-\frac{\delta}{2}$, *the bound on* m *is sharp.*

Remark 2 When it is a pseudo-differential operator, i.e., $\phi(x,\xi) = x\cdot\xi$. For any $m < n(\rho-1)$, we can find δ such that $0<\delta<-\frac{2m}{n}$. It is easy to see that $x\cdot\xi$ satisfies (1) and (2). By using this theorem, we immediately show that the pseudo-differential operator is bounded on L^1. Therefore, this theorem is a generalization of the result of L^1-boundedness of pseudo-differential operators proved by Kenig and Staubach [12].

Remark 3 We construct an example to show that the pseudo-differential operator is unbounded on L^1 when $a \in L^\infty S^{n(\rho-1)}_\rho$ in general. Therefore, the bound of the index m in the result of Kenig and Staubach is sharp. At the same time, it also shows that when $\rho \leqslant 1-\frac{\delta}{2}$, the bound on m in Theorem 1 is sharp.

Remark 4 When $\delta = 1$ and $m < n\min\{\rho-1, -\frac{1}{2}\}$, we prove that $T_{\phi,a}$ is bounded on L^1. When $\rho \leqslant \frac{1}{2}$, the range of m is sharp. In addition, when $\rho \leqslant 1-\frac{1}{2n}$, Theorem 1 is a generalization of Theorem A. Finally, when $1-\frac{1}{2n} < \rho \leqslant 1$, Theorem 1 is weaker than Theorem A, but here we do not assume that the phase is homogeneous of degree 1 in the frequency variable.

In Section 2 we prove the low frequency part. In the proof, we get an estimate of Fourier transform (Lemma 2) which has a separate meaning. In Section 3 we prove the high frequency part. At last we construct an example $a \in L^\infty S^{n(\rho-1)}_\rho$ such that the pseudo-differential operator is unbounded on L^1.

Throughout this note, we use C and c to denote some positive constants, which may vary from line to line, that depends only on ϕ, n, δ, λ and some quasi-norms of a.

2 Low frequency part

In the low frequency part, we need the following lemma.

Lemma 1 *If* u *is supported in* B_1 *and satisfies*

$$|\nabla^{n+1}u(x)| \leqslant A|x|^{\delta-1-n}\ln\frac{3}{|x|}$$

for some $\delta \in (0,1]$ *and* $A>0$, *then for any* $0<\mu<\delta$, *we have*

$$\left|\int_{B_1} \mathrm{e}^{-\mathrm{i}yx} u(x)\mathrm{d}x\right| \leqslant CA(1+|y|)^{-n-\mu},$$

where C depends only on n, δ, μ .

The proof of this lemma is almost identical to the proof of Lemma 1.17 in [5], with only slight differences. In this note we prove the following lemma which a generalization of Lemma 1.

Lemma 2 *If u is supported in B_1 and there exists some $0<\mu<1$ such that*

$$\int_{B_1} |\nabla^{n+1} u||x|^{1-\mu}\mathrm{d}x \leqslant A,$$

then we have

$$\left|\int_{B_1} \mathrm{e}^{-\mathrm{i}yx} u(x)\mathrm{d}x\right| \leqslant C_{n,\mu}A(1+|y|)^{-n-\mu},$$

where C depends only on n, μ .

In the proof we need the following lemma, a special case of the weighted Sobolev inequality. It may be found in some literature. For completeness, we give a simple proof here.

Lemma 3 *Suppose that u is supported in B_1 . For $k=1,2,\dots,n-1$ and $b>k-n$, we have*

$$\left(\int_{B_1} (|u(x)||x|^b)^{\frac{n}{n-k}}\mathrm{d}x\right)^{\frac{n-k}{n}} \leqslant C_{n,k,b}\int_{B_1} |\nabla^k u(x)||x|^b\mathrm{d}x. \tag{3}$$

Additionally, when $b>0$ we have

$$\sup_{x\in B_1} |u(x)||x|^b \leqslant C_{n,b}\int_{B_1} |\nabla^n u(x)||x|^b\mathrm{d}x. \tag{4}$$

Proof When $b>1-n$, as u is supported in B_1, by using simple computations, we can get that

$$\begin{aligned}\int_{B_1} |u(x)||x|^{b-1}\mathrm{d}x &= \int_{S^{n-1}}\int_0^1 |u(r\theta)|r^{b+n-2}\mathrm{d}r\mathrm{d}\theta \\ &\leqslant \int_{S^{n-1}}\int_0^1\int_r^1 |\partial_r u(t\theta)|\mathrm{d}t r^{b+n-2}\mathrm{d}r\mathrm{d}\theta \\ &= \frac{1}{b+n-1}\int_{S^{n-1}}\int_0^1 |\partial_r u(t\theta)|t^{b+n-1}\mathrm{d}t\mathrm{d}\theta \\ &\leqslant \frac{1}{b+n-1}\int_{B_1} |\nabla u(x)||x|^b\mathrm{d}x.\end{aligned}$$

By iterating, for $j\in\mathbb{N}$ and $b>j-n$, we have

$$\int_{B_1} |u(x)||x|^{b-j}\mathrm{d}x \leqslant C_{n,j,b}\int_{B_1} |\nabla^j u(x)||x|^b\mathrm{d}x. \tag{5}$$

For the term $|u(x)||x|^b$, by using (5) and the Sobolev embedding theorem $W^{k,1}\subset L^{\frac{n}{n-k}}(k<n)$, we can show that

$$\begin{aligned}\Big(\int_{B_1}(|u(x)||x|^b)^{\frac{n}{n-k}}\mathrm{d}x\Big)^{\frac{n-k}{n}} &\leqslant C_{n,k}\int_{B_1}|\nabla^k(|u(x)||x|^b)|\mathrm{d}x\\ &\leqslant C_{n,k}\sum_{j=0}^{k}\int_{B_1}|\nabla^{k-j}u(x)||x|^{b-j}\mathrm{d}x\\ &\leqslant C_{n,k,b}\int_{B_1}|\nabla^k u(x)||x|^b\mathrm{d}x.\end{aligned}$$

On the other hand, by using (5) and the Sobolev embedding theorem $W^{n,1}\subset C^0$, one can get that

$$\begin{aligned}\sup_{x\in B_1}|u(x)||x|^b &\leqslant C_n\int_{B_1}|\nabla^n(|u(x)||x|^b)|\mathrm{d}x\\ &\leqslant C_{n,b}\sum_{j=0}^{n}\int_{B_1}|\nabla^{n-j}u(x)||\nabla^j|x|^b|\mathrm{d}x\\ &\leqslant C_{n,b}\sum_{j=0}^{n}\int_{B_1}|\nabla^{n-j}u(x)||x|^{b-j}\mathrm{d}x\\ &\leqslant C_{n,b}\int_{B_1}|\nabla^n u(x)||x|^b\mathrm{d}x.\end{aligned}$$

This finishes the proof. □

Now we turn to prove Lemma 2.

Proof Firstly, as u is supported in B_1 and $0<\mu<1$, from (4) we get that

$$\begin{aligned}|\nabla u(x)| &\leqslant |x|^{\mu-1}\sup_{y\in B_1}|\nabla u(y)||y|^{1-\mu}\leqslant C_{n,\mu}|x|^{\mu-1}\int_{B_1}|\nabla^{n+1}u(y)||y|^{1-\mu}\mathrm{d}y\\ &\leqslant C_{n,\mu}A|x|^{\mu-1},\end{aligned}$$

which implies that

$$|u(r\theta)|\leqslant\int_r^1|\partial_r u(t\theta)|\mathrm{d}t\leqslant C_{n,\mu}A\int_r^1 t^{\mu-1}\mathrm{d}t\leqslant C_{n,\mu}A.$$

When $|y|\leqslant 3$, it is easy to see that

$$\Big|\int_{B_1}\mathrm{e}^{-\mathrm{i}yx}u(x)\mathrm{d}x\Big|\leqslant C\int_{B_1}|u(x)|\mathrm{d}x\leqslant C\leqslant C(1+|y|)^{-n-\mu}. \tag{6}$$

When $|y|>3$, choose a cut function $\eta\in C_c^\infty(B(0,2|y|^{-1}))$ such that

$$0 \leqslant \eta(x) \leqslant 1, \quad \eta(x) = 1 \text{ for } |x| < |y|^{-1}, \quad |\nabla \eta(x)| \leqslant C|y|.$$

Without loss of generality, we can assume that $|y_1| \geqslant \frac{|y|}{n}$. By using integration by parts and (5), we have

$$\begin{aligned}
&\left|\int_{B_1} \mathrm{e}^{-\mathrm{i}yx} u(x) \mathrm{d}x\right| \\
&= C_n |y_1|^{-n} \left|\int_{B_1} \mathrm{e}^{-\mathrm{i}yx} \partial_1^n u(x) \mathrm{d}x\right| \\
&= C_n |y_1|^{-n} \left|\int_{B_1} \mathrm{e}^{-\mathrm{i}yx} \partial_1^n u(x)(\eta(x) + 1 - \eta(x)) \mathrm{d}x\right| \\
&\leqslant C_n |y|^{-n} \left(\int_{B_1} |\nabla^n u(x)| \eta(x) \mathrm{d}x + \left|\int_{B_1} \mathrm{e}^{-\mathrm{i}yx} \partial_1^n u(x)(1 - \eta(x)) \mathrm{d}x\right|\right) \\
&\leqslant C_n |y|^{-n} \left(\int_{|x|<2|y|^{-1}} |\nabla^n u(x)| \mathrm{d}x + |y_1|^{-1} \int_{B_1} |\partial_1 [\partial_1^n u(x)(1 - \eta(x))]| \mathrm{d}x\right) \\
&\leqslant C_n |y|^{-n} \Bigg(\int_{|x|<2|y|^{-1}} |\nabla^n u(x)| \mathrm{d}x + |y|^{-1} \int_{B_1} |\partial_1^{n+1} u(x)|(1 - \eta(x)) \mathrm{d}x \\
&\quad + |y|^{-1} \int |\partial_1^n u(x) \partial_1 \eta(x)| \mathrm{d}x\Bigg) \\
&\leqslant C_n |y|^{-n} \Bigg(\int_{|x|<2|y|^{-1}} |\nabla^n u(x)| \mathrm{d}x + |y|^{-1} \int_{|y|^{-1}<|x|<1} |\nabla^{n+1} u(x)| \mathrm{d}x \\
&\quad + |y|^{-1} \int_{|x|<2|y|^{-1}} |\nabla^n u(x)||y| \mathrm{d}x\Bigg) \\
&\leqslant C_n |y|^{-n} \left(\int_{B_1} |\nabla^n u(x)|(|x||y|)^{-\mu} \mathrm{d}x + |y|^{-1} \int_{B_1} |\nabla^{n+1} u(x)|(|x||y|)^{1-\mu} \mathrm{d}x\right) \\
&\leqslant C_n |y|^{-n-\mu} \left(\int_{B_1} |\nabla^n u(x)||x|^{-\mu} \mathrm{d}x + \int_{B_1} |\nabla^{n+1} u(x)||x|^{1-\mu} \mathrm{d}x\right) \\
&\leqslant C_{n,\mu} |y|^{-n-\mu} \int_{B_1} |\nabla^{n+1} u(x)||x|^{1-\mu} \mathrm{d}x \\
&\leqslant C_{n,\mu} A |y|^{-n-\mu}.
\end{aligned} \tag{7}$$

In the penultimate inequality we use Lemma 3. In conclusion, for all $y \in \mathbb{R}^n$, we complete the proof of Lemma 2. □

Below we give the boundedness of L^1 for the low frequency part of the Fourier integral operator.

Theorem 2 *If* $a \in L^\infty S_\rho^m$ *and* ϕ *satisfies* (1) *and* (2), *then for any*

$\eta \in C_c^\infty(B_1)$, the operator

$$T_{0,\phi,a}f(x) = \int_{R^n} \mathrm{e}^{\mathrm{i}\phi(x,\xi)} a(x,\xi)\eta(\xi)\hat{f}(\xi)\mathrm{d}\xi$$

is bounded on L^1.

Proof Take any point $\xi_0 \in S^{n-1}$. Set $w(x,\xi) = \phi(x,\xi) - \nabla_\xi\phi(x,\xi_0)\cdot\xi$. By using some simple calculations, one have

$$\begin{aligned}
&T_{0,\phi,a}f(x)\\
&= \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\phi(x,\xi)} a(x,\xi)\eta(\xi)\hat{f}(\xi)\mathrm{d}\xi\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(\phi(x,\xi)-y\cdot\xi)} a(x,\xi)\eta(\xi)f(y)\mathrm{d}y\mathrm{d}\xi\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(\nabla_\xi\phi(x,\xi_0)-y)\cdot\xi}\mathrm{e}^{\mathrm{i}w(x,\xi)} a(x,\xi)\eta(\xi)f(y)\mathrm{d}\xi\mathrm{d}y\\
&= \int_{\mathbb{R}^n} k_0(x,y)f(y)\mathrm{d}y,
\end{aligned}$$

where $k_0(x,y) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(\nabla_\xi\phi(x,\xi_0)-y)\cdot\xi}\mathrm{e}^{\mathrm{i}w(x,\xi)} a(x,\xi)\eta(\xi)\mathrm{d}\xi$.

From (2), when $\xi \in B_1$ and $0 < \delta \leqslant 1$, it is easy to check that

$$\begin{aligned}
|\nabla_\xi w(x,\xi)| &= |\nabla_\xi\phi(x,\xi) - \nabla_\xi\phi(x,\xi_0)|\\
&\leqslant \left|\nabla_\xi\phi(x,\xi) - \nabla_\xi\phi\left(x,\frac{\xi}{|\xi|}\right)\right| + \left|\nabla_\xi\phi\left(x,\frac{\xi}{|\xi|}\right) - \nabla_\xi\phi(x,\xi_0)\right|\\
&\leqslant \int_{|\xi|}^1 |\nabla_\xi^2\phi\left(x,t\frac{\xi}{|\xi|}\right)|dt + \left|\xi_0 - \frac{\xi}{|\xi|}\right| \sup_{\zeta\in S^{n-1}} |\nabla_\xi^2\phi(x,\zeta)|\\
&\leqslant C\left(\int_{|\xi|}^1 r^{\delta-2}dr + 1\right)\\
&\leqslant C|\xi|^{\delta-1}\ln\frac{3}{|\xi|}.
\end{aligned}$$

Here the term $\ln\frac{3}{|\xi|}$ is used when $\delta = 1$.

On the other hand, when $k \geqslant 2$, we have

$$|\nabla_\xi^k w(x,\xi)| = |\nabla_\xi^k\phi(x,\xi)| \leqslant C_k|\xi|^{\delta-k}.$$

So, for any $\xi \in B_1$ and $k \in \mathbb{N}$, we can get that

$$|\nabla_\xi^k w(x,\xi)| \leqslant C|\xi|^{\delta-k}\ln\frac{3}{|\xi|}.$$

As $|\xi| < 1$, by using direct computations we get that

$$
\begin{aligned}
&|\nabla_\xi^{n+1}[\mathrm{e}^{\mathrm{i}w(x,\xi)}a(x,\xi)\eta(\xi)]| \\
&\quad \leqslant C\sum_{k_0=0}^{n+1}\sum_{t=1}^{n+1-k_0}|\nabla_\xi^{k_0}[a(x,\xi)\eta(\xi)]|\sum_{k_1+\cdots+k_t=n+1-k_0,k_s>0}\prod_{s=1}^{t}|\nabla_\xi^{k_s}w(x,\xi)| \\
&\quad \leqslant C\sum_{k_0=0}^{n+1}\sum_{t=1}^{n+1-k_0}\sum_{k_1+\cdots+k_t=n+1-k_0,k_s>0}\prod_{s=1}^{t}\left(|\xi|^{\delta-k_s}\ln\frac{3}{|\xi|}\right) \\
&\quad \leqslant C\sum_{k_0=0}^{n+1}\sum_{t=1}^{n+1-k_0}|\xi|^{t\delta+k_0-n-1}\ln^t\frac{3}{|\xi|} \\
&\quad \leqslant C\sum_{k_0=0}^{n+1}\sum_{t=1}^{n+1-k_0}\left(|\xi|^{\delta}\ln\frac{3}{|\xi|}\right)^t|\xi|^{k_0-n-1} \\
&\quad \leqslant C|\xi|^{\delta}\ln\frac{3}{|\xi|}|\xi|^{-n-1}=C|\xi|^{\delta-n-1}\ln\frac{3}{|\xi|}.
\end{aligned}
$$

In the second inequality, we use the fact that $|\nabla_\xi^{k_0}[a(x,\xi)\eta(\xi)]|\leqslant C$ when $|\xi|<1$, while in the last inequality, we use the fact that $|\xi|^{\delta}\ln\frac{3}{|\xi|}\leqslant C_\delta$ when $|\xi|<1$.

For $0<\mu<\delta$, we have

$$
\int_{B_1}|\nabla_\xi^{n+1}[\mathrm{e}^{\mathrm{i}w(x,\xi)}a(x,\xi)\eta(\xi)]||\xi|^{1-\mu}\mathrm{d}\xi\leqslant C\int_{B_1}|\xi|^{\delta-n-\mu}\ln\frac{3}{|\xi|}\mathrm{d}\xi<\infty.
$$

By using Lemma 2, we get that

$$
\begin{aligned}
|k_0(x,y)|&=\left|\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(\nabla_\xi\phi(x,\xi_0)-y)\cdot\xi}\mathrm{e}^{\mathrm{i}w(x,\xi)}a(x,\xi)\eta(\xi)\mathrm{d}\xi\right| \\
&\leqslant C(1+|\nabla_\xi\phi(x,\xi_0)-y|)^{-n-\mu}. \qquad (8)
\end{aligned}
$$

On the other hand, for $y\in\mathbb{R}^n$, we set $E=B(y,r)$ in (1) and show that

$$
|\{x:|\nabla_\xi\phi(x,\xi_0)-y|<r\}|\leqslant Cr^n, \qquad (9)
$$

where C is independent of r,y,ξ_0.

For any $y\in\mathbb{R}^n$, from (8) and (9) we can get that

$$
\begin{aligned}
\int_{\mathbb{R}^n}|k_0(x,y)|\mathrm{d}x&\leqslant C\int_{\mathbb{R}^n}(1+|\nabla_\xi\phi(x,\xi_0)-y|)^{-n-\mu}\mathrm{d}x \\
&\leqslant C\Bigg(\int_{|\nabla_\xi\phi(x,\xi_0)-y|<1}(1+|\nabla_\xi\phi(x,\xi_0)-y|)^{-n-\mu}\mathrm{d}x \\
&\quad+\sum_{j=1}^{\infty}\int_{2^{j-1}\leqslant|\nabla_\xi\phi(x,\xi_0)-y|<2^j}(1+2^{j-1})^{-n-\mu}\mathrm{d}x\Bigg) \\
&\leqslant C(|\{x:|\nabla_\xi\phi(x,\xi_0)-y|<1\}|
\end{aligned}
$$

$$+\sum_{j=1}^{\infty} 2^{-j(\mu+n)}|\{x:|\nabla_\xi\phi(x,\xi_0)-y|<2^j\}|)$$
$$\leqslant C\left(1+\sum_{j=1}^{\infty}2^{-j\mu}\right)<\infty.$$

The constant C is independent of y. So we can immediately show that $T_{0,\phi,a}$ is bounded on L^1. □

3 The high frequency part

In this section, we consider the high frequency part of $T_{\phi,a}$.

Set $\lambda=\min\{\rho,1-\frac{\delta}{2}\}$. As $\delta>0$, we get that $0\leqslant\lambda<1$. For $j\in\mathbb{N}$, consider the ball $B=B(\xi_B,2^{j\lambda-1})$ with $2^j<|\xi_B|<2^{j+1}$. It is easy to see that B is contained in the ring $\{\xi:2^{j-1}<|\xi|<2^{j+2}\}$. For any $\eta_B\in C_c^\infty(B)$ satisfying $|\nabla^k\eta_B|\leqslant C_k 2^{-k\lambda}$, $k\in\mathbb{N}$, we define

$$T_{B,\phi,a}f(x)=\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\phi(x,\xi)}a(x,\xi)\eta_B(\xi)\hat{f}(\xi)\mathrm{d}\xi.$$

The main estimate in this section is given below.

Theorem 3 *If $a\in L^\infty S^m_\rho$ and ϕ satisfies* (1) *and* (2), *then we have*

$$\|T_{B,\phi,a}f\|_1\leqslant C2^{jm}\|f\|_1.$$

Proof Set $w_B(x,\xi)=\phi(x,\xi)-\nabla_\xi\phi(x,\xi_B)\cdot\xi$. By using simple computations we show that

$$\begin{aligned}
&T_{B,\phi,a}f(x)\\
&=\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\phi(x,\xi)}a(x,\xi)\eta_B(\xi)\hat{f}(\xi)\mathrm{d}\xi\\
&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(\phi(x,\xi)-y\cdot\xi)}a(x,\xi)\eta_B(\xi)f(y)\mathrm{d}y\mathrm{d}\xi\\
&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(\nabla_\xi\phi(x,\xi_B)-y)\cdot\xi}\mathrm{e}^{\mathrm{i}w_B(x,\xi)}a(x,\xi)\eta_B(\xi)f(y)\mathrm{d}\xi\mathrm{d}y\\
&=\int_{\mathbb{R}^n}k_B(x,y)f(y)\mathrm{d}y,
\end{aligned}$$

where $k_B(x,y)=\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(\nabla_\xi\phi(x,\xi_B)-y)\cdot\xi}\mathrm{e}^{\mathrm{i}w_B(x,\xi)}a(x,\xi)\eta_B(\xi)\mathrm{d}\xi$.

Define the operator $L=1-2^{2j\lambda}\nabla_\xi^2$. It is obvious a self-adjoint operator. By using simple computations, one can get that

$$L(\mathrm{e}^{\mathrm{i}(\nabla_\xi\phi(x,\xi_B)-y)\cdot\xi})=(1+2^{2j\lambda}|\nabla_\xi\phi(x,\xi_B)-y|^2)\mathrm{e}^{\mathrm{i}(\nabla_\xi\phi(x,\xi_B)-y)\cdot\xi}.$$

Since $\lambda=\min\{\rho,1-\frac{\delta}{2}\}\leqslant\rho$, for any $k=0,1,\ldots$, we have

$$|\nabla_\xi^k[a(x,\xi)\eta_B(\xi)]| \leqslant C \sum_{k_1+k_2=k} |\nabla_\xi^{k_1} a \nabla_\xi^{k_2}\eta_B|$$
$$\leqslant C \sum_{k_1+k_2=k} C2^{j(m-k_1\rho)}2^{-jk_2\lambda} \leqslant C2^{j(m-k\lambda)}. \tag{10}$$

Also due to $\lambda = \min\{\rho, 1-\frac{\delta}{2}\} \leqslant 1-\frac{\delta}{2}$, mean value theorem and (2), we can obtain that

$$|\nabla_\xi w_B(x,\xi)| = |\nabla_\xi\phi(x,\xi) - \nabla_\xi\phi(x,\xi_B)| \leqslant |\xi-\xi_B| \sup_{\zeta\in B}|\nabla_\zeta^2\phi(x,\zeta)|$$
$$\leqslant C2^{j(\lambda+\delta-2)} \leqslant C2^{-j\lambda}.$$

On the other hand, for any $k \geqslant 2$, from (2), one have

$$|\nabla_\xi^k w_B(x,\xi)| = |\nabla_\xi^k\phi(x,\xi)| \leqslant C2^{j(\delta-k)} = C2^{jk(\frac{\delta}{k}-1)} \leqslant C2^{-jk\lambda}.$$

So, for any $k = 0, 1, \ldots$, we can get that

$$|\nabla_\xi^k w_B(x,\xi)| \leqslant C2^{-jk\lambda}. \tag{11}$$

From (10) and (11), for any $M \in \mathbb{N}$, we have

$$\begin{aligned}
&|k_B(x,y)| \\
&= \left|\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(\nabla_\xi\phi(x,\xi_B)-y)\cdot\xi}\mathrm{e}^{\mathrm{i}w_B(x,\xi)}a(x,\xi)\eta_B(\xi)\mathrm{d}\xi\right| \\
&= (1+2^{2j\lambda}|\nabla_\xi\phi(x,\xi_B)-y|^2)^{-M}\left|\int_{\mathbb{R}^n} L^M(\mathrm{e}^{\mathrm{i}(\nabla_\xi\phi(x,\xi_B)-y)\cdot\xi})\mathrm{e}^{\mathrm{i}w_B(x,\xi)}a(x,\xi)\eta_B(\xi)\mathrm{d}\xi\right| \\
&= (1+2^{2j\lambda}|\nabla_\xi\phi(x,\xi_B)-y|^2)^{-M}\left|\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(\nabla_\xi\phi(x,\xi_B)-y)\cdot\xi}L^M[\mathrm{e}^{\mathrm{i}w_B(x,\xi)}a(x,\xi)\eta_B(\xi)]\mathrm{d}\xi\right| \\
&\leqslant C(1+2^{2j\lambda}|\nabla_\xi\phi(x,\xi_B)-y|^2)^{-M}\int_{\mathbb{R}^n}\sum_{k_0=0}^{2M}\sum_{t=0}^{2M-k_0} 2^{jk_0\lambda}|\nabla_\xi^{k_0}[a(x,\xi)\eta_B(\xi)]| \\
&\quad\cdot \sum_{k_1+\cdots+k_t\leqslant 2M-k_0, k_s>0}\prod_{s=1}^{t}(2^{jk_s\lambda}|\nabla_\xi^{k_s}w_B(x,\xi)|)\mathrm{d}\xi \\
&\leqslant C(1+2^{2j\lambda}|\nabla_\xi\phi(x,\xi_B)-y|^2)^{-M}\int_B\sum_{k_0=0}^{2M}\sum_{t=0}^{2M-k_0} 2^{jk_0\lambda}2^{j(m-k_0\lambda)} \\
&\quad\cdot \sum_{k_1,\ldots,k_t}\prod_{s=1}^{t}(2^{jk_s\lambda}2^{-jk_s\lambda})\mathrm{d}\xi \\
&\leqslant C(1+2^{2j\lambda}|\nabla_\xi\phi(x,\xi_B)-y|^2)^{-M}|B|2^{jm} \\
&\leqslant C2^{j(m+n\lambda)}(1+2^{2j\lambda}|\nabla_\xi\phi(x,\xi_B)-y|^2)^{-M}.
\end{aligned}$$

By using (9) and some similar calculations in the first part, for any $y \in \mathbb{R}^n$ and $M > \frac{n}{2}$, we get that

$$\begin{aligned}
&\int_{\mathbb{R}^n}|k_B(x,y)|\mathrm{d}x\\
&\leqslant C2^{j(m+n\lambda)}\int_{\mathbb{R}^n}(1+2^{2j\lambda}|\nabla_\xi\phi(x,\xi_B)-y|^2)^{-M}\mathrm{d}x\\
&\leqslant C2^{j(m+n\lambda)}\Bigg(\int_{|\nabla_\xi\phi(x,\xi_B)-y|<2^{-j\lambda}}(1+2^{2j\lambda}|\nabla_\xi\phi(x,\xi_B)-y|^2)^{-M}\mathrm{d}x\\
&\quad+\sum_{s=1}^{\infty}\int_{2^{s-1-j\lambda}<|\nabla_\xi\phi(x,\xi_B)-y|<2^{s-j\lambda}}(1+2^{2j\lambda}|\nabla_\xi\phi(x,\xi_B)-y|^2)^{-M}\mathrm{d}x\Bigg)\\
&\leqslant C2^{j(m+n\lambda)}\Bigg(|\{x:|\nabla_\xi\phi(x,\xi_B)-y|<2^{-j\lambda}\}|\\
&\quad+\sum_{s=1}^{\infty}2^{-2Ms}|\{x:|\nabla_\xi\phi(x,\xi_B)-y|<2^{s-j\lambda}\}|\Bigg)\\
&\leqslant C2^{j(m+n\lambda)}\left(2^{-jn\lambda}+\sum_{s=1}^{\infty}2^{-2Ms}2^{n(s-j\lambda)}\right)\\
&\leqslant C2^{jm}.
\end{aligned}$$

This finishes the proof of this theorem. □

Finally, for $j\geqslant 1$, one can easily check that $\{\xi:2^j<|\xi|\leqslant 2^{j+1}\}$ can be covered by no more than $C2^{jn(1-\lambda)}$ balls $B_j^v=B(\xi_j^v,2^{j\lambda-1})$ with $2^j<|\xi_j^v|\leqslant 2^{j+1}$. On the other hand, it is easy to find functions $\eta_0\in C_c^\infty(B_2),\eta_j^v\in C_c^\infty(B_j^v)$ such that

$$\eta_0+\sum_{j=1}^{\infty}\sum_{v=1}\eta_j^v=1,\ |\nabla^k\eta_0|\leqslant C_k,\ |\nabla^k\eta_j^v|\leqslant C_k2^{-jk\lambda},\ k>0.$$

So the operator $T_{\phi,a}$ can be decomposed into

$$\begin{aligned}
T_{\phi,a}f(x)&=\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\phi(x,\xi)}a(x,\xi)\hat f(\xi)\mathrm{d}\xi\\
&=\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\phi(x,\xi)}a(x,\xi)\left(\eta_0+\sum_{j=1}^{\infty}\sum_{v=1}\eta_j^v\right)\hat f(\xi)\mathrm{d}\xi\\
&=\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\phi(x,\xi)}a(x,\xi)\eta_0\hat f(\xi)\mathrm{d}\xi+\sum_{j=1}^{\infty}\sum_{v=1}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\phi(x,\xi)}a(x,\xi)\eta_j^v\hat f(\xi)\mathrm{d}\xi\\
&=T_{0,\phi,a}f(x)+\sum_{j=1}^{\infty}\sum_{v=1}T_{B_j^v,\phi,a}f(x).
\end{aligned}$$

From Theorem 2 and Theorem 3, when $m<n(\lambda-1)=n\min\{\rho-1,-\frac{\delta}{2}\}$, we can show that

$$\begin{aligned}\|T_{\phi,a}f\|_1 &\leqslant \|T_{0,\phi,a}f\|_1 + \sum_{j=1}^{\infty}\sum_{v=1} \|T_{B_j^v,\phi,a}f\|_1 \\ &\leqslant C\left(1+\sum_{j=1}^{\infty}\sum_{v=1} 2^{jm}\right)\|f\|_1 \\ &\leqslant C\left(1+\sum_{j=1}^{\infty} 2^{j(m+n(1-\lambda))}\right)\|f\|_1 \\ &= C\|f\|_1.\end{aligned}$$

So, the first half of Theorem 1 is done.

4 A counterexample

In this section, we construct an example $a \in L^\infty S_\rho^{n(\rho-1)}$ such that T_a is unbounded on L^1.

At first, for any $j \in \mathbb{N}$, we use A_j to denote the set of all lattice points (each coordinate component is an integer) in the ring $\{z : \frac{3}{4}4^{j(1-\rho)} < |z| < \frac{5}{4}4^{j(1-\rho)}\}$. It is obvious that the number of points in A_j is about $O(4^{jn(1-\rho)})$.

In addition, we take φ such that

$$\hat{\varphi} \in C_c^\infty\left(B\left(0,\frac{1}{3}\right)\right);\ 0 \leqslant \hat{\varphi} \leqslant 1;\ \hat{\varphi}(z)=1, z\in B\left(0,\frac{1}{4}\right);\ |\nabla^k \hat{\varphi}| \leqslant 20^k.$$

Now we set

$$a(x,\xi) = \sum_{j=1}^{\infty}\sum_{\alpha\in A_j} 4^{jn(\rho-1)} \mathrm{e}^{-\mathrm{i}4^{j\rho}x\cdot\alpha}\hat{\varphi}(4^{-j\rho}(\xi - 4^{j\rho}\alpha)).$$

As $\hat{\varphi} \in C_c^\infty(B(0,\frac{1}{3}))$, for any ξ, there is at most one $j \in \mathbb{N}$ and one point $\alpha \in A_j$ such that $\hat{\varphi}(4^{-j\rho}(\xi - 4^{j\rho}\alpha)) \neq 0$ (Note that the distance of two different lattice points is no less than 1). In this case we have $\frac{5}{12}4^j < |\xi| < \frac{19}{12}4^j$. It is easy to see that

$$|\xi|^{k\rho+n(1-\rho)}|\partial_\xi^\alpha a(x,\xi)| \leqslant |\xi|^{k\rho+n(1-\rho)}4^{jn(\rho-1)}4^{-jk\rho}20^k \leqslant \left(\frac{19}{12}\right)^{k\rho+n(1-\rho)} 20^k \leqslant 2^n 40^k$$

for any k, x, ξ. Thus $a \in L^\infty S_\rho^{n(\rho-1)}$. In the above analysis, we can also see that $a(x,\cdot)$ is supported in the set $\bigcup_{j=1}^{\infty}\{\xi : \frac{5}{12}4^j < |\xi| < \frac{19}{12}4^j\}$.

Similar to φ, we choose ψ such that

$$\hat{\psi} \in C_c^\infty\left(B\left(0,\frac{5}{3}\right)\right);\ \hat{\psi}(z)=1, z\in B\left(0,\frac{19}{12}\right).$$

For any $N \in \mathbb{N}$, set

$$a_N(x,\xi)=a(x,\xi)\hat{\psi}(4^{-N}\xi).$$

It is obvious that $\|T_{a_N}\|_{L^1-L^1}\leqslant\|T_a\|_{L^1-L^1}$.

By using some simple calculations, the kernel of T_{a_N} is

$$\begin{aligned}k_N(x,y)&=\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(x-y)\xi}a_N(x,\xi)\mathrm{d}\xi\\&=\sum_{j=1}^{N}\sum_{\alpha\in A_j}4^{jn(\rho-1)}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(x-y)\xi}\mathrm{e}^{-\mathrm{i}4^{j\rho}x\cdot\alpha}\hat{\varphi}(4^{-j\rho}(\xi-4^{j\rho}\alpha))\mathrm{d}\xi\\&=\sum_{j=1}^{N}\sum_{\alpha\in A_j}4^{jn(\rho-1)}\mathrm{e}^{-\mathrm{i}4^{j\rho}y\cdot\alpha}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(x-y)(\xi-4^{j\rho}\alpha)}\hat{\varphi}(4^{-j\rho}(\xi-4^{j\rho}\alpha))\mathrm{d}\xi\\&=\sum_{j=1}^{N}\sum_{\alpha\in A_j}4^{jn(\rho-1)}\mathrm{e}^{-\mathrm{i}4^{j\rho}y\cdot\alpha}4^{jn\rho}\varphi(4^{j\rho}(x-y)).\end{aligned}$$

As $\hat{\varphi}(0)=1$, by using some simple calculations, for any y we have

$$\begin{aligned}\int_{\mathbb{R}^n}k_N(x,y)\mathrm{d}x&=\sum_{j=1}^{N}\sum_{\alpha\in A_j}4^{jn(\rho-1)}\mathrm{e}^{-\mathrm{i}4^{j\rho}y\cdot\alpha}\int_{\mathbb{R}^n}4^{jn\rho}\varphi(4^{j\rho}(x-y))\mathrm{d}x\\&=\sum_{j=1}^{N}\sum_{\alpha\in A_j}4^{jn(\rho-1)}\mathrm{e}^{-\mathrm{i}4^{j\rho}y\cdot\alpha}.\end{aligned}$$

When $|y|<4^{-N-2}$, since $\alpha\in A_j, j\leqslant N$, we get that $|4^{j\rho}y\cdot\alpha|\leqslant\frac{5}{16}$. Additionally, as the number of points in A_j is about $O(4^{jn(1-\rho)})$, when $|y|<4^{-N-2}$ we have

$$\left|\int_{\mathbb{R}^n}k_N(x,y)\mathrm{d}x\right|=\left|\sum_{j=1}^{N}\sum_{\alpha\in A_j}4^{jn(\rho-1)}\mathrm{e}^{-\mathrm{i}4^{j\rho}y\cdot\alpha}\right|\geqslant\left|\sum_{j=1}^{N}\sum_{\alpha\in A_j}4^{jn(\rho-1)}\cos\frac{5}{16}\right|\geqslant c_nN,$$

which implies that $\|T_{a_N}\|_{L^1-L^1}>c_nN$. Since N is arbitrary, T_a is not bounded by L^1. By synthesizing all the proofs, we get the main theorem of this note. □

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