

Boundedness of iterated spherical average

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Abstract The iterated spherical average $\Delta(A_1)^N$ is an important operator in harmonic analysis, and has very important applications in approximation theory and probability theory, where Δ is the Laplacian, A_1 is the unit spherical average and $(A_1)^N$ is its iteration. In this paper, we mainly study the sufficient and necessary conditions for the boundedness of this operator in Besov-Lipschitz space, and prove the boundedness of the operator in Triebel-Lizorkin space. Moreover, we use above conclusions to improve the existing results of the boundedness of this operator in L^p space.

Keywords Iterated spherical average, Besov-Lipschitz space, Triebel-Lizorkin space, L^p space

MSC2020 42B25, 42B35

1 Introduction

The average operator of functions f on the unit sphere is defined as

$$A_1(f)(x) = \int_{\mathbb{S}^{n-1}} f(x - y') d\sigma(y'),$$

where \mathbb{S}^{n-1} is the unit sphere in \mathbb{R}^n , $d\sigma$ is the normalized surface Lebesgue measure on \mathbb{R}^n . The generation of this operator can be traced back to the 1970s. It has a profound background and wide applications in harmonic analysis (see [10, 9]). Moreover, it is very important in the study of random walks in high dimensional spaces, which was originated by Pearson [8] about 120 years ago. The definition of random walks is the N -steps uniform walk in \mathbb{R}^n starts at the origin and consists of N independent steps of length 1, each of which is taken

into a uniformly random direction. By some calculations, the probability density function $p_N(\frac{n-2}{2}, x)$ of uniform walk in \mathbb{R}^n is the Fourier inverse of $(A_1)^N$ [2]. Besides, this operator also plays a significant role in the approximation theory [1, 4]. In order to obtain some equivalent forms of the K-functional in $L^p(\mathbb{R}^n)$ spaces, Belinsky, Dai and Ditzian studied the operator $(A_1)^N$ in [1] and obtained the following result.

Theorem A [1] *Let $1 \leq p \leq \infty$, $n \geq 2$ and $N > \frac{2(n+2)}{n-1}$. The inequality*

$$\|\Delta(A_1)^N(f)\|_{L^p(\mathbb{R}^n)} \preceq \|f\|_{L^p(\mathbb{R}^n)}$$

holds for all $f \in L^p(\mathbb{R}^n)$.

In the proof of Theorem A, they proposed a meaningful question: what was the smallest positive integer N to guarantee the inequality

$$\|\Delta(A_1)^N(f)\|_{L^1(\mathbb{R}^n)} \preceq \|f\|_{L^1(\mathbb{R}^n)}. \quad (1)$$

This question was completely solved by Fan, Lou and Wang. In [3], they obtained the following theorem.

Theorem B [3] *Let $n \neq 3, 5$, and N be positive integers. The inequality*

$$\|\Delta(A_1)^N(f)\|_{L^1(\mathbb{R}^n)} \preceq \|f\|_{L^1(\mathbb{R}^n)}$$

holds if and only if $N > \frac{n+3}{n-1}$. Let $n = 3, 5$, and N be positive integers. The inequality

$$\|\Delta(A_1)^N(f)\|_{L^1(\mathbb{R}^n)} \preceq \|f\|_{L^1(\mathbb{R}^n)}$$

holds if and only if $N \geq \frac{n+3}{n-1}$.

In the proof of Theorem B, they used the fact that iterate steps N is an positive integer. But, if we extend the iterate steps N to real number by the Fourier transforms, the conclusion of Theorem B is not sharp. On the other hand, there is no regularity in L^1 space. To study the boundedness of $(A_1)^N$ for $N \in \mathbb{R}$ and the influence of the regularity of function space on the boundedness of this operator, Huang [6] studied the boundedness of $(A_1)^N$ on modulation space and obtained the sufficient and necessary conditions of the boundedness of $\Delta(A_1)^N$ on the modulation space.

Theorem C [6] *Let $\sigma = 2 - \frac{n-1}{2}N$, $1 \leq p_i, q_i \leq \infty$, $s_i \in \mathbb{R}$ ($i = 1, 2$). When $q_1 \leq q_2$, the iterated spherical average $\Delta(A_1)^N$ is bounded from $M_{p_1, q_1}^{s_1}(\mathbb{R}^n)$ to $M_{p_2, q_2}^{s_2}(\mathbb{R}^n)$ if and only if*

$$p_1 \leq p_2, \quad s_1 \geq s_2 + \sigma.$$

When $q_1 > q_2$, the iterated spherical average $\Delta(A_1)^N$ is bounded from $M_{p_1, q_1}^{s_1}(\mathbb{R}^n)$ to $M_{p_2, q_2}^{s_2}(\mathbb{R}^n)$ if and only if

$$p_1 \leq p_2, \quad s_1 + \frac{n}{q_1} \geq s_2 + \sigma + \frac{n}{q_2}.$$

On the other hand, Besov-Lipschitz space $B_{p,q}^s$ and Triebel-Lizorkin space $F_{p,q}^s$ are very important spaces in the theory of function space. They can not only characterize many function spaces, such as $L^p(\mathbb{R}^n)$ space, Sobolev space, Hardy space, but also have important applications in the fields of partial differential equations and time-frequency analysis [5, 7, 11, 12]. Thus, it is significant to study the boundedness of $\Delta(A_1)^N$ in Besov space and Triebel space. Besides, by the Littlewood-Paley theorem, when $s = 0, 1 < p < \infty, q = 2$, the Triebel space is equivalent to $L^p(\mathbb{R}^n)$ space, that is,

$$F_{p,2}^0(\mathbb{R}^n) \approx L^p(\mathbb{R}^n). \quad (2)$$

Therefore, the boundedness of $\Delta(A_1)^N$ in Triebel space can be used to study the boundedness of this operator in $L^p(\mathbb{R}^n)$ space. In this paper, we study the boundedness of $\Delta(A_1)^N$ in Besov space and Triebel space, and improve the boundedness of $\Delta(A_1)^N$ in $L^p(\mathbb{R}^n)$ space by the conclusion in Triebel space. The following theorems are our main results.

Theorem 1 *Let $\sigma_1 = 2 + \frac{n}{2} - \frac{n-1}{2}N, 1 \leq p_i, q \leq \infty, s_i \in \mathbb{R} (i = 1, 2)$. If*

$$p_1 \leq p_2, \quad s_1 - \frac{n}{p_1} \geq s_2 + \sigma_1 - \frac{n}{p_2},$$

then $\Delta(A_1)^N$ is bounded from $B_{p_1,q}^{s_1}(\mathbb{R}^n)$ to $B_{p_2,q}^{s_2}(\mathbb{R}^n)$.

Theorem 2 *Let $\sigma_2 = 2 - \frac{n}{2} - \frac{n-1}{2}N, 1 \leq p_i, q \leq \infty, s_i \in \mathbb{R} (i = 1, 2)$. If the iterated spherical average $\Delta(A_1)^N$ is bounded from $B_{p_1,q}^{s_1}(\mathbb{R}^n)$ to $B_{p_2,q}^{s_2}(\mathbb{R}^n)$, then the following conditions must be hold:*

$$p_1 \leq p_2, \quad s_1 \geq s_2 + \sigma_2.$$

Theorem 3 *Let $\sigma_1 = 2 + \frac{n}{2} - \frac{n-1}{2}N, 1 \leq p_i, q \leq \infty, s_i \in \mathbb{R} (i = 1, 2)$. If*

$$p_1 \leq p_2, \quad s_1 - \frac{n}{p_1} > s_2 + \sigma_1 + \frac{n}{p_2},$$

then $\Delta(A_1)^N$ is bounded from $F_{p_1,q}^{s_1}(\mathbb{R}^n)$ to $F_{p_2,q}^{s_2}(\mathbb{R}^n)$.

Corollary 1 *Let $1 < p < \infty$. When $N > \frac{n+4}{n-1}$, $\Delta(A_1)^N$ is bounded in $L^p(\mathbb{R}^n)$ space, that is*

$$\| \Delta(A_1)^N(f) \|_{L^p(\mathbb{R}^n)} \leq \| f \|_{L^p(\mathbb{R}^n)}.$$

Remark 1 We can find that the conclusion in Corollary 1 is better than Theorem A, since the smallest iterate steps N in Corollary 1 is smaller than Theorem A.

Remark 2 We can obtain the necessary conditions of boundedness of

$\Delta(A_1)^N$ in Triebel space by a similar method, and the conclusion is also similar as Theorem 2. The proof follows the same pattern so that we leave the proof to the reader.

Throughout this paper, we use the inequality $A \preceq B$ to mean that there is a positive number C independent of all main variables such that $A \leq CB$, and use the notation $A \simeq B$ to mean $A \preceq B$ and $B \preceq A$.

2 Preliminaries and lemmas

In this section, we will give the definitions of Besov space and Triebel space, and discuss some basic properties of Besov space and Triebel space. Also, we will prove some estimates and lemmas which will be used in our proof.

Definition 1 [11] (Besov space) Let $\varphi_1(\xi)$ be a radial Schwartz function on \mathbb{R}^n which satisfy $\varphi_1(\xi) \in C_0^\infty(\mathbb{R}^n)$ and $\text{supp } \varphi_1 \subset \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$. When $\frac{2}{3} \leq |\xi| \leq \frac{3}{2}$, $\varphi_1(\xi) \equiv 1$. Assume $\varphi_0(\xi) \in C_0^\infty(\mathbb{R}^n)$, $\text{supp } \varphi_0 \subset \{\xi : |\xi| \leq 2\}$ and satisfy

$$\sum_{k=0}^{+\infty} \varphi_k \equiv 1,$$

where $\varphi_k(\xi) = \varphi(2^{-k}\xi)$, $k \in \mathbb{N}^+$. Define

$$\Delta_k f = (\varphi_k \widehat{f})^\vee.$$

Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$. The Besov space is defined as follows:

$$B_{p,q}^s(\mathbb{R}^n) := \left\{ f \in S' : \|f\|_{B_{p,q}^s(\mathbb{R}^n)} = \left(\sum_{k=0}^{\infty} 2^{ksq} \|\Delta_k f\|_{L^p}^q \right)^{\frac{1}{q}} < \infty \right\}.$$

Definition 2 [11] (Triebel space) Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$. The Triebel space is defined as follows:

$$F_{p,q}^s(\mathbb{R}^n) := \left\{ f \in S' : \|f\|_{F_{p,q}^s(\mathbb{R}^n)} = \left\| \left(\sum_{k=0}^{\infty} 2^{ksq} |\Delta_k f|^q \right)^{\frac{1}{q}} \right\|_{L^p} < \infty \right\}.$$

Proposition 1 [7] Let $0 < p_i, q \leq \infty$, $s_i \in \mathbb{R}$ ($i = 1, 2$). When

$$p_1 \leq p_2, \quad s_1 - \frac{n}{p_1} \geq s_2 - \frac{n}{p_2},$$

we have

$$B_{p_1,q}^{s_1} \subset B_{p_2,q}^{s_2}, \quad F_{p_1,q}^{s_1} \subset F_{p_2,q}^{s_2}. \quad (3)$$

Proposition 2 [7] *If $1 \leq p, q_i \leq \infty$ ($i = 1, 2$) and $s \in \mathbb{R}$, then $\forall \varepsilon > 0$, we have*

$$F_{p, q_1}^{s+\varepsilon} \subset F_{p, q_2}^s. \quad (4)$$

The Fourier multiplier $m(D)$ is a linear operator acting on the test function f , which is defined as:

$$\widehat{m(D)f}(\xi) = m(\xi)\hat{f}(\xi).$$

The function $m(\xi)$ is called the symbol or multiplier of $m(D)$. By Fourier transform, it is known that $m(D)$ is a convolution operator

$$m(D) = (m(\xi))^\vee * f.$$

By the Young inequality, we have

$$\|m(D)f\|_{L^p} \preceq \|(m(\xi))^\vee(x)\|_{L^1} \|f\|_{L^p}$$

for any $1 \leq p \leq \infty$. Thus, to obtain the boundedness of $m(D)$ in L^p space, we only need to estimate $\|(m(\xi))^\vee(x)\|_{L^1}$. For this purpose, we need the following Bernstein multiplier theorem to estimate $\|(m(\xi))^\vee(x)\|_{L^1}$.

Lemma 1 [12] (Bernstein multiplier theorem) *Assume $0 < p \leq 2$ and $\partial^\gamma m(\xi) \in L^2$ for all multi-indices γ with $|\gamma| \leq \left[n \left(\frac{1}{p} - \frac{1}{2} \right) \right] + 1$. We have*

$$\|(m(\xi))^\vee(x)\|_{L^p} \preceq \sum_{|\gamma| \leq \left[n \left(\frac{1}{p} - \frac{1}{2} \right) \right] + 1} \|\partial^\gamma m(\xi)\|_{L^2}. \quad (5)$$

By Fourier transform, ignoring constant factors, $m(D)$ is a Fourier multiplier as follows:

$$\Delta(\widehat{A_1})^N f = |\xi|^2 \left(V_{\frac{n-2}{2}}(|\xi|) \right)^N \hat{f}(\xi),$$

where

$$V_\delta(r) = \frac{J_\delta(r)}{r^\delta}$$

and $J_\delta(r)$ is the Bessel function of order δ which has following properties.

Lemma 2 [9] *When $\delta > -\frac{1}{2}$, we have*

$$V_\delta(r) = O(1), \quad \text{when } |r| \preceq 1, \quad (6)$$

$$\frac{dV_\delta(r)}{dr} = -rV_{\delta+1}(r). \quad (7)$$

Lemma 3 [9] *Let $r > 1$ and $\delta > -\frac{1}{2}$. For any $L \in \mathbb{Z}^+$ and $r \in [1, \infty)$, we have*

$$J_\delta(r) = \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{\delta\pi}{2} - \frac{\pi}{4}\right) + \sum_{j=1}^L a_j e^{ir} r^{-\frac{1}{2}-j} + \sum_{j=1}^L b_j e^{-ir} r^{-\frac{1}{2}-j} + E(r), \quad (8)$$

where a_j and b_j are constants for all j , and $E(r)$ is a C^∞ function satisfying

$$|E^{(k)}(r)| \preceq r^{-\frac{1}{2}-L-1}$$

for any $k = 0, 1, 2, \dots$.

3 Proof of main results

We firstly prove Theorem 1. By the definition of Besov space, we need to estimate $\|\Delta_k \Delta(A_1)^N f\|_{L^p}$. For this purpose, we obtain the following lemma.

Lemma 4 *Let $1 \leq p \leq \infty$, $\sigma_1 = 2 + \frac{2}{n} - \frac{n-1}{2}N$. Then we have the following estimate*

$$\|\Delta_k \Delta(A_1)^N f\|_{L^p(\mathbb{R}^n)} \preceq 2^{k\sigma_1} \|\Delta_k f\|_{L^p(\mathbb{R}^n)}.$$

Proof $\forall k \in \mathbb{N}$, $\Delta_k \Delta(A_1)^N f$ is a convolution operator $m_k(D)(f) = \Omega_k(x) * f$, where

$$\Omega_k(x) = \int_{\mathbb{R}^n} \varphi_k(\xi) |\xi|^2 \left(V_{\frac{n-2}{2}}(|\xi|) \right)^N e^{i\xi \cdot x} d\xi. \quad (9)$$

By the orthogonality of dyadic decomposition, when $|l - k| \geq 2$, we have $\varphi_l(\xi)\varphi_k(\xi) = 0$. Since

$$\sum_{k \in \mathbb{Z}^n} \Delta_k = I,$$

by Young's inequality and Minkowski's inequality, we have

$$\begin{aligned} \|\Delta_k \Delta(A_1)^N f\|_{L^p} &= \left\| \sum_{l \in \mathbb{N}} \Delta_l \Delta_k \Delta(A_1)^N f \right\|_{L^p} \\ &\preceq \sum_{l \in \mathbb{N}, |l-k| \leq 1} \left\| \Delta_l \Delta(A_1)^N \Delta_k f \right\|_{L^p} \\ &\preceq \sum_{l \in \mathbb{N}, |l-k| \leq 1} \left\| \left(\varphi_l(\xi) \cdot |\xi|^2 \left(V_{\frac{n-2}{2}}(|\xi|) \right)^N \right)^\vee * \Delta_k f \right\|_{L^p} \\ &\preceq \sum_{l \in \mathbb{N}, |l-k| \leq 1} \left\| \left(\varphi_l(\xi) \cdot |\xi|^2 \left(V_{\frac{n-2}{2}}(|\xi|) \right)^N \right)^\vee \right\|_{L^1} \|\Delta_k f\|_{L^p} \\ &= \sum_{l \in \mathbb{N}, |l-k| \leq 1} \|\Omega_l\|_{L^1} \|\Delta_k f\|_{L^p}. \end{aligned}$$

Thus, we only need to estimate

$$\sum_{l \in \mathbb{N}, |l-k| \leq 1} \|\Omega_l(x)\|_{L^1}.$$

For every $k \in \mathbb{N}$, the number of l which satisfy $l \in \mathbb{N} : |l - k| \leq 1$ is at most 3. So, we only need to estimate $\|\Omega_l(x)\|_{L^1}$ when $l \simeq k$. When $|k| < 100$, by condition (7), we have $|\Omega_l(x)| \leq 1$, when $|x| < 100$. If $|x| \geq 100$, without losing generality, assume $|x_1| \geq \frac{x}{n}$. By the derivative of $V_\delta(t)$ (see (8)) and taking integration by part on ξ_1 , we have

$$|\Omega_l(x)| \leq \frac{1}{|x_1|^{n+1}} \leq \frac{1}{|x|^{n+1}}.$$

So, when $|k| < 100$, $\|\Omega_l(x)\|_{L^1} \leq 1$, by the fact that $l \simeq k$. Then, we choose $L = 1$ in Lemma 3. We have the following asymptotic form of $V_\delta(r)$:

$$V_\delta(r) = r^{-\delta-\frac{1}{2}} \sqrt{\frac{2}{\pi}} \cos\left(r - \frac{\delta\pi}{2} - \frac{\pi}{4}\right) + O(r^{-\delta-\frac{3}{2}})$$

for $|r| < 1$. So, when $|k| > 100$, $l \simeq k$ and $\xi \in \text{supp } \varphi_l(\xi)$, we have

$$|V_\delta(|\xi|)|^N \leq |\xi|^{(-\delta-\frac{1}{2})N} \simeq (2^l)^{(-\delta-\frac{1}{2})N} \simeq 2^{-(\delta+\frac{1}{2})Nk}. \quad (10)$$

Then, by the chain rule and the derivative formula of $V_\delta(t)$, we obtain

$$\begin{aligned} \frac{\partial}{\partial \xi_i} (V_\delta(|\xi|))^N &= - (V_\delta(|\xi|))^{N-1} |\xi| \cdot V_{\delta+1}(|\xi|) \cdot \frac{\xi_i}{|\xi|} \\ &= - (V_\delta(|\xi|))^{N-1} V_{\delta+1}(|\xi|) \cdot \xi_i. \end{aligned}$$

By the asymptotic form of $V_\delta(r)$, we obtain that

$$\left| \frac{\partial}{\partial \xi_i} (V_\delta(|\xi|))^N \right| \leq |\xi|^{(-\delta-\frac{1}{2})(N-1)} \cdot |\xi|^{(-\delta-\frac{3}{2})} \cdot |\xi| \leq |\xi|^{(-\delta-\frac{1}{2})N} \simeq 2^{-(\delta+\frac{1}{2})Nk} \quad (11)$$

for $\xi \in \text{supp } \varphi_l(\xi)$. By (10) and (11), when $\delta > -\frac{1}{2}$ and $\xi \in \text{supp } \varphi_l(\xi)$, $V_\delta(|\xi|)^N$ and $\frac{\partial}{\partial \xi_i} (V_\delta(|\xi|))^N$ share the same upper bound which is $2^{-(\delta+\frac{1}{2})Nl}$. Since

$$\partial^\gamma (|\xi|^2) \begin{cases} \leq |\xi|^{2-|\gamma|}, & |\gamma| \leq 2; \\ = 0, & |\gamma| > 2, \end{cases}$$

we can obtain that

$$\begin{aligned} \|\Omega_l(x)\|_{L^1} &= \sum_{|\gamma| \leq [\frac{n}{2}]+1} \left\| \partial^\gamma (\varphi_l(\xi) \cdot |\xi|^2 \left(V_{\frac{n-2}{2}}(|\xi|) \right)^N) \right\|_{L^2} \\ &\leq \sum_{|\gamma| \leq [\frac{n}{2}]+1} \sum_{\gamma_1+\gamma_2+\gamma_3=\gamma} \left\| \partial^{\gamma_1} \varphi_l(\xi) \cdot \partial^{\gamma_2} |\xi|^2 \cdot \partial^{\gamma_3} \left(V_{\frac{n-2}{2}}(|\xi|) \right)^N \right\|_{L^2(\text{supp } \varphi_l(\xi))} \\ &\leq 2^{\frac{kn}{2}} \cdot 2^{2k} \cdot 2^{-\frac{n-1}{2}Nk} \\ &= 2^{(2+\frac{n}{2}-\frac{n-1}{2}N)k}. \end{aligned} \quad (12)$$

By the definition of Besov space and above estimate, we have

$$\begin{aligned}
\|\Delta(A_1)^N f\|_{B_{p_2,q}^{s_2}} &= \left(\sum_{k=0}^{+\infty} 2^{ks_2q} \left\| \Delta_k \Delta(A_1^N) f \right\|_{p_2}^q \right)^{\frac{1}{q}} \\
&= \left(\sum_{k=0}^{100} 2^{ks_2q} \left\| \Delta_k \Delta(A_1^N) f \right\|_{p_2}^q + \sum_{k>100} 2^{ks_2q} \left\| \Delta_k \Delta(A_1^N) f \right\|_{p_2}^q \right)^{\frac{1}{q}} \\
&\preceq \left(\sum_{k=0}^{100} 2^{ks_2q} \left\| \Delta_k \Delta(A_1^N) f \right\|_{p_2}^q \right)^{\frac{1}{q}} + \left(\sum_{k>100} 2^{ks_2q} \left\| \Delta_k \Delta(A_1^N) f \right\|_{p_2}^q \right)^{\frac{1}{q}} \\
&\preceq \left(\sum_{k=0}^{100} 2^{(s_2+2+\frac{n}{2}-\frac{n-1}{2}N)qk} \left\| \Delta_k \Delta(A_1^N) f \right\|_{p_2}^q \right)^{\frac{1}{q}} \\
&\quad + \left(\sum_{k>100} 2^{(s_2+2+\frac{n}{2}-\frac{n-1}{2}N)qk} \left\| \Delta_k \Delta(A_1^N) f \right\|_{p_2}^q \right)^{\frac{1}{q}} \\
&\preceq \|f\|_{B_{p_2,q}^{s_2+2+\frac{n}{2}-\frac{n-1}{2}N}}.
\end{aligned}$$

By the embedding property (3) of Besov space, when $p_1 \leq p_2$, $s_1 - \frac{n}{p_1} \geq s_2 + \sigma_1 - \frac{n}{p_2}$, we can easily obtain that

$$\|\Delta(A_1)^N f\|_{B_{p_2,q}^{s_2}} \preceq \|f\|_{B_{p_2,q}^{s_2+\sigma_1}} \preceq \|f\|_{B_{p_2,q}^{s_1}}.$$

Theorem 1 is proved.

Next, we will prove Theorem 2. From Lemma 3, we can find that the main term in asymptotic expansion of Bessel function is a trigonometric function which has zero points in every semiperiod. Thus, we can't estimate the dual operator of the iterated spherical average at these zero points. To solve this problem, we need the following lemmas.

Lemma 5 *Assume $i \in \mathbb{Z}^+$ which is large enough. Then there exists a constant $\varepsilon_0 > 0$, such that*

$$V_{\frac{n-1}{2}}(|\xi|) \geq \varepsilon_0 |\xi|^{-\frac{n-1}{2}} \quad (13)$$

for $|\xi| \in [i\pi + \frac{n\pi}{4} - \frac{3}{4}\pi + \frac{\pi}{20}, (i+1)\pi + \frac{n\pi}{4} - \frac{3}{4}\pi - \frac{\pi}{20}]$.

Proof Choosing $L = 1$ in Lemma 3, we have

$$\begin{aligned}
V_{\frac{n-2}{2}}(r) &= r^{-\frac{n-1}{2}} \sqrt{\frac{2}{\pi}} \cos\left(r - \frac{n\pi}{4} + \frac{\pi}{4}\right) + O(r^{-\frac{n+1}{2}}) \\
&= r^{-\frac{n-1}{2}} \sqrt{\frac{2}{\pi}} \sin\left(r - \frac{n\pi}{4} + \frac{3\pi}{4}\right) + O(r^{-\frac{n+1}{2}}).
\end{aligned}$$

Define

$$u(r) = \sin\left(r - \frac{n\pi}{4} + \frac{3\pi}{4}\right)$$

and $\varepsilon_0 = \frac{1}{2} \sin \frac{\pi}{20}$. When $r - \frac{n\pi}{4} + \frac{3\pi}{4} \in [i\pi + \frac{\pi}{20}, (i+1)\pi - \frac{\pi}{20}]$, we have

$$|u(r)| \geq 2\varepsilon_0.$$

Moreover, when r is large enough, it is easy to obtain

$$O(r^{-\frac{n+1}{2}}) \leq \varepsilon_0 r^{-\frac{n-1}{2}}.$$

Thus, when $|\xi| \in [i\pi + \frac{n\pi}{4} - \frac{3\pi}{4} + \frac{\pi}{20}, (i+1)\pi + \frac{n\pi}{4} - \frac{3\pi}{4} - \frac{\pi}{20}]$ and i is large enough, we have

$$|V_{\frac{n-2}{2}}(|\xi|)| \geq \varepsilon_0 |\xi|^{-\frac{n-1}{2}}.$$

Lemma 6 *Let $1 \leq p \leq \infty$, $\sigma_2 = 2 - \frac{n}{2} - \frac{n-1}{2}N$. For $k \geq 2$, if the smooth function $\{f_k\}$ satisfy*

$$\text{supp} \hat{f}_k(\xi) \subset \left\{ \xi : |\xi| \in \left[i_k\pi + \frac{n\pi}{4} - \frac{3}{4}\pi + \frac{\pi}{20}, (i_k+1)\pi + \frac{n\pi}{4} - \frac{3}{4}\pi - \frac{\pi}{20} \right] \right\},$$

where $i_k \in \mathbb{Z}^+$, and $[i_k\pi + \frac{n\pi}{4} - \frac{3}{4}\pi, (i_k+1)\pi + \frac{n\pi}{4} - \frac{3}{4}\pi] \subseteq (\frac{2}{3} \cdot 2^k, \frac{3}{2} \cdot 2^k)$, then we can obtain

$$2^{k\sigma_2} \|f_k\|_{L^p} \preceq \|\Delta_k \Delta(A_1)^N f_k\|_{L^p}.$$

Proof By the assumption of $\{f_k\}$, it is obvious that $|\xi| \simeq 2^k$ when $\xi \in \text{supp} \hat{f}_k(\xi)$. Thus, by (12) and Lemma 5, when $\xi \in \text{supp} \hat{f}_k(\xi)$, we have

$$\varepsilon_0 |\xi|^{-\frac{n-1}{2}} \leq \left| \left(V_{\frac{n-2}{2}}(|\xi|) \right) \right| \preceq |\xi|^{-\frac{n-1}{2}}, \quad (14)$$

that is, $|V_{\frac{n-2}{2}}(|\xi|)| \simeq 2^{-\frac{n-1}{2}k}$. Then, by the chain rule and the derivative formula of $V_\delta(t)$, we can obtain

$$\begin{aligned} \left| \frac{\partial}{\partial \xi_i} \left(V_{\frac{n-2}{2}}(|\xi|) \right)^{-N} \right| &= \left| - \left(V_{\frac{n-2}{2}}(|\xi|) \right)^{-(N+1)} V_{\frac{n-2}{2}}(|\xi|) \cdot \xi_i \right| \\ &\leq |\xi|^{(-\frac{n-1}{2})(-N-1)} |\xi|^{-\frac{n+1}{2}} |\xi| \\ &\simeq 2^{\frac{n-1}{2}Nk}. \end{aligned} \quad (15)$$

Therefore, when $\xi \in \text{supp} \hat{f}_k(\xi)$, $|V_{\frac{n-2}{2}}(|\xi|)|^{-N}$ and $|\frac{\partial}{\partial \xi_i} (V_{\frac{n-2}{2}}(|\xi|))^{-N}|$ share the same upper bound which is $2^{\frac{n-1}{2}Nk}$.

By Definition 1, $\varphi_k(\xi) \equiv 1$ when $\frac{2}{3} \cdot 2^k \leq |\xi| \leq \frac{3}{2} \cdot 2^k$. So, when k is large enough, we can choose some suitable $i_k \in \mathbb{Z}^+$ such that $[i_k\pi + \frac{n\pi}{4} - \frac{3}{4}\pi, (i_k+1)\pi + \frac{n\pi}{4} - \frac{3}{4}\pi] \subseteq (\frac{2}{3} \cdot 2^k, \frac{3}{2} \cdot 2^k)$. So, we have

$$\varphi_k(\xi) \hat{f}_k(\xi) = \hat{f}_k(\xi),$$

when $\text{supp} \hat{f}_k(\xi) \subset \{\xi : |\xi| \in [i_k\pi + \frac{n\pi}{4} - \frac{3}{4}\pi + \frac{\pi}{20}, (i_k+1)\pi + \frac{n\pi}{4} - \frac{3}{4}\pi - \frac{\pi}{20}]\}$. Thus, by (13), (14), Young's inequality and Bernsteins Multiplier Theorem, we can obtain

$$\begin{aligned}
\|f_k\|_{L^p} &= \left\| (\widehat{f}_k)^\vee \right\|_{L^p} \\
&= \left\| (\varphi_k(\xi)\varphi_k(\xi) \cdot |\xi|^{-2} \left(V_{\frac{n-2}{2}}(|\xi|)\right)^{-N} \cdot |\xi|^2 \left(V_{\frac{n-2}{2}}(|\xi|)\right)^N \widehat{f}_k)^\vee \right\|_{L^p} \\
&= \left\| (\varphi_k(\xi)|\xi|^{-2} \left(V_{\frac{n-2}{2}}(|\xi|)\right)^{-N} \cdot \varphi_k(\xi)|\xi|^2 \left(V_{\frac{n-2}{2}}(|\xi|)\right)^N \widehat{f}_k)^\vee \right\|_{L^p} \\
&\preceq \left\| (\varphi_k(\xi)|\xi|^{-2} \left(V_{\frac{n-2}{2}}(|\xi|)\right)^{-N})^\vee \right\|_{L^1} \left\| \Delta_k \Delta(A_1)^N f_k \right\|_{L^p} \\
&\preceq \sum_{|\gamma| \leq [\frac{n}{2}] + 1} \left\| \partial^\gamma (\varphi_k(\xi)|\xi|^{-2} \left(V_{\frac{n-2}{2}}(|\xi|)\right)^{-N}) \right\|_{L^2} \left\| \Delta_k \Delta(A_1)^N f_k \right\|_{L^p} \\
&\preceq \sum_{\substack{|\gamma| \leq [\frac{n}{2}] + 1 \\ \gamma_1 + \gamma_2 + \gamma_3 = \gamma}} \left\| \partial^{\gamma_1} \varphi_k(\xi) \cdot \partial^{\gamma_2} |\xi|^{-2} \cdot \partial^{\gamma_3} \left(V_{\frac{n-2}{2}}(|\xi|)\right)^{-N} \right\|_{L^2} \\
&\quad \cdot \left\| \Delta_k \Delta(A_1)^N f_k \right\|_{L^p} \\
&\preceq \sum_{\substack{|\gamma| \leq [\frac{n}{2}] + 1 \\ \gamma_1 + \gamma_2 = \gamma}} \left\| \partial^{\gamma_1} \varphi_k(\xi) \cdot |\xi|^{-2} \cdot \partial^{\gamma_2} \left(V_{\frac{n-2}{2}}(|\xi|)\right)^{-N} \right\|_{L^2} \left\| \Delta_k \Delta(A_1)^N f_k \right\|_{L^p} \\
&\simeq 2^{(-2 + \frac{n}{2} + \frac{n-1}{2}N)k} \left\| \Delta_k \Delta(A_1)^N f_k \right\|_{L^p},
\end{aligned}$$

which means

$$2^{(2 - \frac{n}{2} - \frac{n-1}{2}N)k} \|f_k\|_{L^p} \preceq \left\| \Delta_k \Delta(A_1)^N f_k \right\|_{L^p}.$$

Lemma 6 is proved.

Now, we continue to prove Theorem 2. Let $f \in S(\mathbb{R}^n)$ satisfy $\widehat{f}(\xi) \in C_0^\infty(B^-)$, where $B^- := \{\xi = (\xi_1, \xi_2, \dots, \xi_n) : |\xi| \leq 1, \xi_1 < 0\}$. Then choose $\xi_k = (i_k\pi + \frac{n\pi}{4} - \frac{3}{4}\pi + \frac{\pi}{2}, 0, \dots, 0)$, where i_k is defined in the proof of Lemma 6. For $0 < \lambda < \frac{1}{2}$, we define smooth function $f_{k,\lambda}$ by

$$\widehat{f}_{k,\lambda}(\xi) = \widehat{f}\left(\frac{\xi - \xi_k}{\lambda}\right).$$

It is obvious that $\text{supp} \widehat{f}_{k,\lambda}(\xi) \subset \{\xi : |\xi| \in [i_k\pi + \frac{n\pi}{4} - \frac{3\pi}{4} + \frac{\pi}{20}, (i_k + 1)\pi + \frac{n\pi}{4} - \frac{3\pi}{4} - \frac{\pi}{20}]\}$. By the definition of $f_{k_i,\lambda}$, we can obtain

$$\Delta_k f_{k,\lambda} = f_{k,\lambda};$$

$$\Delta_j f_{k,\lambda} = 0, \quad j \neq k.$$

By Lemma 6, we have

$$\begin{aligned} \|\Delta(A_1)^N f_{k_i, \lambda}\|_{B_{p_2, q}^{s_2}} &= 2^{ks_2} \|\Delta_k \Delta(A_1)^N f_{k_i, \lambda}\|_{L^{p_2}} \\ &\lesssim 2^{k(s_2 + 2 - \frac{n}{2} - \frac{n-1}{2}N)} \|f_{k_i, \lambda}\|_{L^{p_2}} \\ &\lesssim 2^{k(s_2 + \sigma_2)} \lambda^{n(1 - \frac{1}{p_2})}. \end{aligned}$$

Moreover, by some simple computation, we have

$$\|f_{k_i, \lambda}\|_{B_{p_1, q}^{s_1}} = 2^{ks_1} \|f_{k_i, \lambda}\|_{L^{p_1}} \simeq 2^{ks_1} \lambda^{n(1 - \frac{1}{p_1})}.$$

By the assumption that $\Delta(A_1)^N$ is bounded on Besov space, we can obtain $\|\Delta(A_1)^N f_{k_i, \lambda}\|_{B_{p_2, q}^{s_2}} \lesssim \|f_{k_i, \lambda}\|_{B_{p_1, q}^{s_1}}$. Therefore, when $0 < \lambda < \frac{1}{2}$, we have

$$2^{k(s_2 + \sigma_1)} \lambda^{n(1 - \frac{1}{p_2})} \leq 2^{ks_1} \lambda^{n(1 - \frac{1}{p_1})}.$$

Fixed $k \in \mathbb{Z}^+$, we have

$$\lambda^{n(1 - \frac{1}{p_2})} \lesssim \lambda^{n(1 - \frac{1}{p_1})}.$$

Then let $\lambda \rightarrow 0$, we can obtain

$$p_1 \leq p_2.$$

On the other hand, when fixed $\lambda \in (0, \frac{1}{2})$ and letting $k \rightarrow \infty$, we have

$$2^{k(s_2 + \sigma_1)} \leq 2^{ks_1},$$

which means

$$s_1 \geq s_2 + \sigma_2.$$

Theorem 2 is proved.

Now, we prove the boundedness of $\Delta(A_1)^N$ on Triebel space. We firstly assume $s = 0$, $1 \leq p = q \leq \infty$, by the definition of Triebel space and orthogonality of $\{\varphi_k\}_{k=1}^\infty$, we can obtain

$$\begin{aligned} \|\Delta(A_1)^N f\|_{F_{p, p}^0}^p &= \left\| \left(\sum_{k=0}^{+\infty} |\Delta_k \Delta(A_1)^N f|^p \right)^{\frac{1}{p}} \right\|_{L^p(\mathbb{R}^n)}^p \\ &= \left\| \left(\sum_{k=0}^{+\infty} \left| \sum_{j \in \mathbb{Z}^+ \cup \{0\}} \Delta_j \Delta_k \Delta(A_1)^N f \right|^p \right)^{\frac{1}{p}} \right\|_{L^p(\mathbb{R}^n)}^p \\ &= \left\| \left(\sum_{k=0}^{+\infty} \left| \sum_{j=(k-1) \vee 0}^{j=k+1} \Delta_j \Delta_k \Delta(A_1)^N f \right|^p \right)^{\frac{1}{p}} \right\|_{L^p(\mathbb{R}^n)}^p \end{aligned}$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}^n} \sum_{k=0}^{+\infty} \sum_{j=(k-1)\vee 0}^{j=k+1} |\Delta_j \Delta_k \Delta(A_1)^N f|^p d\xi \\
&\lesssim \sum_{k=0}^{+\infty} \sum_{j=(k-1)\vee 0}^{j=k+1} \int_{\mathbb{R}^n} |\Omega_j(x) * \Delta_k f|^p d\xi \\
&= \sum_{k=0}^{+\infty} \sum_{j=(k-1)\vee 0}^{j=k+1} \|\Omega_j(x) * \Delta_k f\|_{L^p(\mathbb{R}^n)}^p,
\end{aligned}$$

where $\Omega_j(x)$ is defined in (9). By (11), when $|j - k| \leq 1$, we have

$$\|\Omega_j\|_{L^1} \lesssim 2^{(2+\frac{n}{2}-\frac{n-1}{2}N)j} \simeq 2^{(2+\frac{n}{2}-\frac{n-1}{2}N)k}.$$

Combining the above estimates, we have

$$\begin{aligned}
\|\Delta(A_1)^N\|_{F_{p,p}^0}^p &\lesssim \sum_{k=0}^{+\infty} \sum_{j=k-1}^{j=k+1} \|\Omega_j(x) * \Delta_k f\|_{L^p(\mathbb{R}^n)}^p \\
&\lesssim \sum_{k=0}^{+\infty} \sum_{j=k-1}^{j=k+1} \|\Omega_j\|_{L^1}^p \|\Delta_k f\|_{L^p(\mathbb{R}^n)}^p \\
&\lesssim \sum_{k=0}^{+\infty} 2^{(2+\frac{n}{2}-\frac{n-1}{2}N)kp} \|\Delta_k f\|_{L^p(\mathbb{R}^n)}^p \\
&= \sum_{k=0}^{+\infty} \int_{\mathbb{R}^n} 2^{(2+\frac{n}{2}-\frac{n-1}{2}N)kp} |\Delta_k f|^p d\xi \\
&\simeq \int_{\mathbb{R}^n} \left(\sum_{k=0}^{+\infty} 2^{(2+\frac{n}{2}-\frac{n-1}{2}N)kp} |\Delta_k f|^p \right) d\xi \\
&= \|f\|_{F_{p,p}^{2+\frac{n}{2}-\frac{n-1}{2}N}}^p.
\end{aligned}$$

Therefore, when $s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2} + \sigma_1$, there exists $\varepsilon > 0$ such that $s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2} + \sigma_1 + 2\varepsilon$. By Proposition 1 and Proposition 2, we can obtain

$$\begin{aligned}
\|\Delta(A_1)^N f\|_{F_{p_2,q_2}^{s_2}} &\lesssim \|\Delta(A_1)^N f\|_{F_{p_2,p_2}^{s_2+\varepsilon}} \\
&\lesssim \|f\|_{F_{p_2,p_2}^{s_2+\sigma_1+\varepsilon}} \\
&\lesssim \|f\|_{F_{p_2,q_1}^{s_2+\sigma_1+2\varepsilon}} \\
&\lesssim \|f\|_{F_{p_1,q_1}^{s_1}}.
\end{aligned}$$

Theorem 3 is proved.

Finally, we prove Corollary 1. The condition $N > \frac{n+4}{n-1}$ yields,

$$\sigma_1 = 2 + \frac{n}{2} - \frac{n-1}{2}N < 0.$$

By choosing $1 \leq p_1 = p_2 = p \leq \infty$, $q_1 = q_2 = 2$, $s_1 = s_2 = 0$ in Theorem 3, we have

$$\|\Delta(A_1)^N(f)\|_{F_{p,2}^0(\mathbb{R}^n)} \leq \|f\|_{F_{p,2}^0(\mathbb{R}^n)}.$$

By the equivalence between Triebel and L^p space (see (2)), we can obtain

$$\|\Delta(A_1)^N(f)\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}.$$

Corollary 1 is proved.

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