

Alternating link and its generalization

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Abstract The alternating links give a classical class of links. They play an important role in Knot Theory. Ozsváth and Szabó introduced a quasi-alternating link which is a generalization of an alternating link. In this paper we review some results of alternating links and quasi-alternating links on the Jones polynomial and the Khovanov homology. Moreover, we introduce a long pass link. Several problems worthy of further study are provided.

Keywords Alternating links, quasi-alternating links, pass replacement, Jones polynomial, Khovanov homology

MSC2020 57M25, 57M27

1 Introduction

A knot is a piecewise linear simple closed curve without intersection in R^3 (or S^3). A link consists of such finite simple closed curves in R^3 (or S^3). Each curve is called a component of the link. In fact, a knot is a link with one component. If there is an orientation preserving piecewise linear homeomorphism $h: S^3 \rightarrow S^3$ which preserves the orientation such that $h(L_1) = L_2$, then the link L_1 is equivalent to the link L_2 . The unknot is the knot equivalent to a circle. How to distinguish nonequivalent links is an essential problem in Knot Theory. We need to find some invariants of links in order to distinguish the distinct links.

The history of knots can be traced back to the beginning of human being tying knots with ropes. Knot Theory, that is a mathematical study of knots, is probably recognized first by Gauss. He introduced the linking number in the 19th century [19]. Because of the mystery of the problem itself, it is widely concerned by many mathematicians in fields of Topology, Algebra, Geometry, Graph theory Combinatorics, etc. It is used to study related topics by biologists and physicists et al. [71, 55]. In this paper, we review some results on the Jones polynomial and the Khovanov homology of alternating

links and quasi-alternating links. Moreover, a long pass link is introduced and several problems are given. The properties of other polynomial invariants can be referred to the review articles [13, 57].

2 Preliminaries

A link projection is obtained by projecting a link to the plane. If there are a finite number of double crossings in a projection and the over-crossing line and the under-crossing line transverse at each crossing in the plane, then the projection is called a diagram of the link. If the over and under crossing at a crossing are not considered, a 4-regular planar graph is obtained, which is called the *shadow* of the diagram. A *path* is an alternating sequence of vertices and edges. A graph is connected if there is a path for any two distinct vertices. A vertex u of a graph G is called a cut vertex if $E(G)$ is the partition of E_1 and E_2 such that $G[E_1]$ and $G[E_2]$ have just the common vertex v . Here, $G[E_i]$ is the subgraph of G whose vertex set is the set of vertices incident with an edge in E_i and whose edge set is E_i for $i = 1, 2$. A diagram is reduced if its shadow doesn't contain a cut vertex. Reidemeister and Alexander and Briggs independently introduced three Reidemeister moves (Fig. 1) and obtained the following result in 1926 [63, 2]. Then a diagram is *connected* (or non-split) if it isn't equivalent to the disjoint union of two diagrams.

Theorem 1 *If two diagrams differ by a finite sequence of Reidemeister moves, then they are equivalent.*

Jones discovered the famous polynomial invariant—the Jones polynomial in 1984 [28], who won the Fields Prize for this work in 1990. Kauffman introduced the bracket polynomial and then gave a relatively simple expression of the

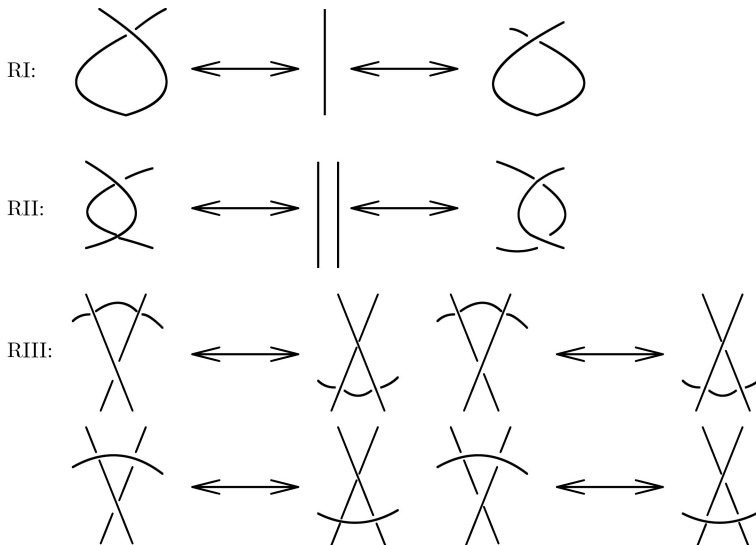


Fig. 1 Reidemeister movers

Jones polynomial in 1987 [31]. Let D be a diagram of a link L . Each crossing can be resolved by using 0-revolution or 1-revolution (Fig. 2). A state of D is obtained by resolving each crossing. Let S_D be the collection of all states of D and let $c(D)$ be the number of crossings in D . Obviously, D has $2^{c(D)}$ states. Let $b(\alpha)$ and $l(\alpha)$ denote the number of 1-revolutions and loops in $\alpha \in S_D$, respectively. Then $a(\alpha)$ is the number of 0-revolutions in α where $a(\alpha) = c(\alpha) - b(\alpha)$. Set $p_i(D) = \sum_{\alpha \in S_D, l(\alpha)=i} A^{a(\alpha)-b(\alpha)}$. The bracket polynomial is as follows:

$$\langle D \rangle = \sum_{i \geq 1} p_i(D) (-A^2 - A^{-2})^{i-1}.$$

Definition 1 [31, 72] Let D be a diagram of an oriented link L . Set $\omega(D)$ to be the writhe of D and set $t = A^{-4}$. Then the Jones polynomial of L is as follows:

$$V_L(t) = (-A)^{-3\omega(D)} \langle D \rangle.$$

The breadth is the difference between the maximal degree of t and the minimal degree t . Suppose that $V_L(t) = t^r \sum_{i=0}^n a_i t^i$ for $n, r \geq 0$. If $a_i \neq 0$ for each $0 \leq i \leq n$, then $V_L(t)$ is gapless.

Khovanov proposed a cohomological invariant of a link – Khovanov homology [33], which is a categorification of the Jones polynomial. The Khovanov homology encodes more information about knots than the Jones polynomial. It detects the unknot, the trefoil knot and the figure-eight knot etc. [4, 5, 6, 9, 23, 35, 41, 42, 46, 81, 82]. Rasmussen defined a knot invariant $s(K)$ on the Khovanov homology by using Lee’s spectral sequence [62]. $s(K)$ provides a lower bound for the smooth slice genus of K . Rasmussen gave a combinatorial proof of the Milnor conjecture which is proved by using Gauge Theory [48, 50]. We recall the Khovanov homology over \mathbb{Q} and the readers are referred to [7, 32, 33, 39, 74, 77, 84].

Let D be a diagram of an oriented link L with crossings $1, 2, \dots, n$. A state $\alpha \in \{0, 1\}^n$ is obtained by resolving each crossing of D where the i th element denotes the revolution of the crossing i in α . Regard every state as a vertex. Two vertices α and α' are adjacent if and only if they differ in exactly one place. Set $\alpha = l_1 \dots l_{k-1} 0 l_{k+1} \dots l_n$, $\alpha' = l_1 \dots l_{k-1} 1 l_{k+1} \dots l_n$. Then

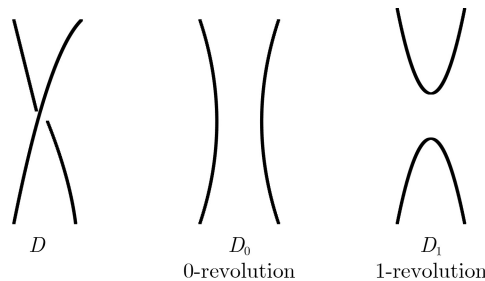


Fig. 2 Two revolutions

$\xi = l_1 \dots l_{k-1} * l_{k+1} \dots l_n : \alpha \rightarrow \alpha'$. We associate each cycle to a graded vector space $V = \langle 1, x \rangle$ where $\deg(1) = 1, \deg(x) = 0$. There is a vector space V_α for each α such that

$$V_\alpha = V^{\otimes l_\alpha} \{r_\alpha + n_+ - 2n_-\}.$$

Here, n_+ and n_- are the number of positive crossings and negative crossings in D , respectively. r_α is the number of 1's and l_α is the number of cycles in α . Let

$$C^{i,*} = \bigoplus_{\alpha \in \{0,1\}^n, r_\alpha = i+n_-} V_\alpha,$$

where $i = -n_-, \dots, n_+$. Let $m : V \otimes V \rightarrow V$ where

$$1 \otimes 1 = 1, 1 \otimes x = x \otimes 1 = x, x \otimes x = 0,$$

and let $\Delta : V \rightarrow V \otimes V$ such that

$$\Delta(1) = 1 \otimes x + x \otimes 1, \Delta(x) = x \otimes x.$$

Define the differential

$$d^{i,*} : C^{i,*}(D) \rightarrow C^{i+1,*}(D),$$

where $\forall v \in V_\alpha \subseteq C^{i,*}$,

$$d^{i,*}(v) = \sum_{\text{Tail}(\xi)=\alpha} \text{sign}(\xi) d_\xi(v).$$

Here, $\text{sign}(\xi)$ is determined by the parity of the number of 1's before $*$ in ξ . $d_\xi(v)$ is determined by either m or Δ on the cycles appearing in the changing disk. $d_\xi(v)$ is the identity on cycles not entering the changing disk.

Definition 2 [33] The Khovanov homology (or cohomology) of an orientable link L is the homology of a bi-graded chain complex $(C^{i,*}, d^{i,*})$

$$Kh^{i,j}(L) = \frac{\ker d^{i+1,j}}{\text{im } d^{i,j}},$$

where $C^{i,*}$ and $d^{i,*}$ are given above for $i, j \in \mathbb{Z}$.

Definition 3 [34, 45] If its nontrivial homology groups lie on two adjacent diagonals, then the link L is Khovanov homologically thin (or H-thin). If Khovanov homology is completely determined by its Jones polynomial and its signature σ , then L is Khovanov holomologically σ -thin.

3 Alternating links

The alternating links give a classical class of links. They play an important role in Knot Theory. In this section, we review some classical results on the alternating links. Some related computations are referred to [17, 25, 27, 30, 38, 47, 52, 53, 61, 85].

Definition 4 [70] Let D be a diagram. If one passes the over-crossings and under-crossings alternately while moving along each component of D , then D is an alternating diagram. If a link has an alternating diagram, then it is alternating.

The research on alternating links can be traced back to the work of Tait and others in the end of the 19th century. Tait proposed an important conjecture about alternating knots in 1898.

Conjecture 1 [70] A connected reduced alternating diagram has minimal crossing number for the knot.

Kauffman, Murasugi and Thistlethwaite obtained the following results and then independently proved Tait's conjecture.

Theorem 2 [31, 54] *The number of crossings in a connected reduced alternating diagram of L is a topological invariant for a link L .*

Theorem 3 [54] *Two connected reduced alternating diagrams of an alternating link have the same number of crossings.*

Theorem 4 [72] *If a link L has a connected reduced alternating diagram of n crossings, then:*

- 1) *the breadth of $V_L(t)$ is exactly n ;*
- 2) *$V_L(t)$ is an alternating polynomial;*
- 3) *the coefficients of the terms of $V_L(t)$ of maximal and minimal degree are both ± 1 ;*
- 4) *if L is prime, and is not a $(2, k)$ torus link, then $V_L(t)$ is gapless.*

Wu obtained the following result by analyzing the black and white regions in the shaded shadow of a link diagram.

Theorem 5 [26] *Let D be a connected reduced diagram with n crossings. α is its state. If there exists a region R satisfying the following conditions:*

- 1) *R and each adjacent region have precisely one common edge;*
 - 2) *both the black mark and white mark occur on the boundary of R in the coloring of the shadow,*
- then*

$$l(\alpha) + l(\hat{\alpha}) \leq n,$$

where the revolution of each crossing in $\hat{\alpha}$ is different from that in α .

This deduces the following result.

Corollary 1 *A prime connected reduced alternating diagram of a link L is a diagram with the minimum crossings of L .*

In the topic of the Khovanov homology of an alternating link, Khovanov obtained the Khovanov homology of $T(2, k)$ over \mathbb{Z} as follows [33].

Theorem 6 *The Khovanov homology of torus links $T(2, k)$ for $k \geq 2$ are given by*

$$Kh^{i,j}(T_{2,k}) \cong \begin{cases} \mathbb{Z}, & \text{if } i = 0, j = -k \text{ or } j = 2 - k; \\ \mathbb{Z}, & \text{if } i = -2l - 1, j = -4l - 2 - k, 1 \leq l \leq \frac{k-1}{2}, l \in \mathbb{Z}; \\ \mathbb{Z}_2, & \text{if } i = -2l, j = -4l - k, 1 \leq l \leq \frac{k-1}{2}, l \in \mathbb{Z}; \\ \mathbb{Z}, & \text{if } i = -2l, j = -4l - k + 2, 1 \leq l \leq \frac{k-1}{2}, l \in \mathbb{Z}; \\ \mathbb{Z}, & \text{if } i = -k, j = -3k \text{ or } j = 2 - 3k, k \text{ is even}; \\ 0, & \text{otherwise.} \end{cases}$$

Bar-Natan computed the Khovanov homology over of all prime knots with crossings up to 11 [7, 8]. Bar-Natan, Garoufalidis and Khovanov proposed the following conjectures based on computations.

Conjecture 2 [7] For any prime alternating knot L there exists an even integer σ and a polynomial $Kh'(L)$ in $t^{\pm 1}$ and $q^{\pm 1}$ with only non-negative coefficients such that

$$Kh_{\mathbb{Q}}(L) = q^{\sigma-1}(1 + q^2 + (1 + tq^4)Kh'(L));$$

$$Kh_{\mathbb{Z}_2}(L) = q^{\sigma-1}(1 + q^2)(1 + (1 + tq^2)Kh'(L)).$$

Here, $Kh_{\mathbb{F}}(L)$ is the graded Poincaré polynomial of the complex $\mathcal{C}(L)$ in the variable t over \mathbb{F} , σ is the signature of L and $Kh'(L)$ contains only power of tq^2 .

Conjecture 3 [18] The following result holds for each alternating knot L .

$$Kh_{\mathbb{Q}}(t, 1) = 1 + V_L(it^{\frac{1}{2}})(it^{\frac{1}{2}})^{-\sigma}$$

where $i = \sqrt{-1}$, σ is the signature of L .

Conjecture 4 [7, 18] For each alternating knots L , the support of the Khovanov homology lies in the lines

$$\deg(q) = 2\deg(t) + \sigma \pm 1.$$

Lee obtained that the Khovanov homology of an alternating knot was determined by its Jones polynomial and signature and then proved Conjectures 2–4 over \mathbb{Q} .

Theorem 7 [39] *For any oriented connected alternating link L , the support of $Kh(t, q)$ lies in*

$$\deg(q) = 2\deg(t) + \sigma \pm 1$$

and for some $p \leq m$ with $a_p = b_m = 1$,

$$Kh_{\mathbb{Q}}(t, q) = \sum_{i=p}^m (a_i t^i q^{2i-\sigma(L)-1} + b_i t^i q^{2i-\sigma(L)+1}).$$

Theorem 8 [39] *Let L be an oriented connected alternating link with n components S_1, S_2, \dots, S_n . Then*

$$Kh_{\mathbb{Q}}(t, q) = q^{-\sigma(L)} \left\{ (q^{-1} + q) \left(\sum_{E \subseteq \{2, \dots, n\}} (tq^2)^{j \in E, k \notin E} 2^{l_{jk}} \right) + (q^{-1} + tq^2 \cdot q) Kh'(tq^2) \right\}$$

where $n \geq 2$, l_{jk} is the linking number of S_j and S_k .

4 Quasi-alternating links

Ozsváth and Szabó introduced a quasi-alternating link [56]. The unknot and an alternating link are quasi-alternating links, which shows that a quasi-alternating link is a generalization of an alternating link. In this section we recall some results related to the Jones polynomial, the Khovanov homology and differentiation of quasi-alternating links.

Definition 5 The set \mathcal{Q} of quasi-alternating links is the smallest set of links which satisfies the following properties:

- the unknot is in \mathcal{Q} ;
- the set \mathcal{Q} is closed under the following operation. Let L be any link which has a diagram with a crossing satisfying the following properties:
 1. both resolutions $L_0, L_1 \in \mathcal{Q}$;
 2. $\det(L) = \det(L_0) + \det(L_1)$.

Then $L \in \mathcal{Q}$.

Manolescu and Ozsváth proved the following result [45].

Theorem 9 *Each quasi-alternating link is Khovanov-thin over \mathbb{Z} .*

Manolescu [44], Baldwin [3], Greene [22], Champanerkar and Kofman [10] obtained quasi-alternating knots up to 10 crossings. Champanerkar and Kofman [10] proved the result as follows.

Theorem 10 *If D_1 and D_2 are any quasi-alternating knot diagrams, then $D_1 \# D_2$ is quasi-alternating.*

Some infinite families of links are considered later. According to Conway representation, each rational tangle $T_{\alpha/\beta}$ associates a continued fraction such that

$$\frac{a}{\beta} = a_m + \frac{1}{a_{m-1} + \frac{1}{\ddots + \frac{1}{a_2 + \frac{1}{a_1}}}}$$

where $m \geq 1$, $0 < |\beta| < \alpha$, $a_i \geq 0$ for $1 \leq i \leq m$. Let u be a crossing of a link diagram D . If we replace u by $T_{\alpha/\beta}$ of $\text{sign}(c)a_i \geq 1$ for $1 \leq i \leq m$, then $T_{\alpha/\beta}$

extends u . If each rational tangle in a product extends u , then the product of rational tangles extends the crossing u . Each Montesinos link $M(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \dots, \frac{\beta_n}{\alpha_n}, e)$ has a diagram which consists of m rational tangle T_{α_i/β_i} and e half-twists for $m \geq 1$ and $1 \leq i \leq m$. Here, $\frac{\alpha_i}{\beta_i}$ associate $(c_{1,i}c_{2,i} \dots c_{m_i,i})$ for $1 \leq i \leq m$ [1, 16]. Then there are the following results.

Theorem 11 [21] *The pretzel link $P(p_1, \dots, p_n, -q_1, \dots, -q_m; -e) = M((p_1, 1), \dots, (p_n, 1), (q_1, -1), \dots, (q_m, -1), -e)$ for $e, n, m \geq 0$, $p_i \geq 2$, $q_j \geq 3$ is quasi-alternating if and only if*

- (1) $e > m - 1$;
- (2) $e = m - 1 > 0$;
- (3) $e = 0, n = 1$, and $p_1 > \min\{q_1, \dots, q_m\}$ or $m \leq 1$;
- (4) $e = 0, m = 1$, and $q_1 > \min\{p_1, \dots, p_n\}$ or $n \leq 1$.

The result holds on permuting on the parameters p_i and q_j .

Theorem 12 [10] *Set D to be a quasi-alternating diagram at some crossing u . Let D' be the diagram obtained from D by a rational tangle extending u . Then D' is quasi-alternating.*

Theorem 13 [10] *The pretzel link $P(p_1, \dots, p_n, -q)$ is quasi-alternating for $n \geq 1$, $1 \leq i \leq n$, each $p_i \geq 1$ and $q > \min(p_1, \dots, p_n)$. The conclusion hold for all permutations of p_i 's and q 's and for reflections of all these pretzel links.*

Theorem 14 [60] *Set D to be a quasi-alternating diagram at some crossing u . Let D' be the diagram obtained from D by a product of rational tangles that extends u . Then D' is quasi-alternating.*

Theorem 15 [60] *The Montesinos link $M(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \dots, \frac{\beta_n}{\alpha_n}, e)$ is quasi-alternating if it satisfies one of the following conditions:*

- (1) $e \leq 0$;
- (2) $e \geq n$;
- (3) $e = 1$ for some $1 \leq i \leq n$, $\frac{\alpha_i}{\alpha_i - \beta_i} > \min\{\frac{\alpha_j}{\beta_j} | j \neq i\}$;
- (4) $e = r - 1$ for some $1 \leq i \leq n$, $\frac{\alpha_i}{\beta_i} > \min\{\frac{\alpha_j}{\alpha_j - \beta_j} | j \neq i\}$.

These conclusions contain the results related to quasi-alternating Montesinos links [80]. In addition, Chbili and Qazaqzeh considered the Jones polynomial of a quasi-alternating link and then obtained the following result [15].

Theorem 16 *Let L be a quasi-alternating link whose diagram D is quasi-alternating at u . Let D^* be a diagram obtained from D by replacing u by a product of rational tangles extending u . If $V_L(t)$ is gapless, then $V_{L^*}(t)$ is gapless where D^* is a diagram of L^* .*

Chbili and Kaur considered the diagram from a quasi-alternating diagram by replacing the crossing u by a reduced alternating tangle that extends u . They obtained the conclusion as follows.

Theorem 17 [14] *Set D to be a quasi-alternating link diagram at a crossing u . Let D^* be the diagram obtained from D by replacing u by a reduced alternating tangle T extending u . Then D^* is quasi-alternating at every crossing of T .*

5 Long pass links

A long pass link is introduced in this section. It is different from the notation of the quasi-alternating link. Related problems are provided.

Definition 6 [78] Let D be a link diagram. Assign a letter to each crossing and denote two adjacent crossings a and b by an unordered pair (a^r, b^s) . Then an embedding presentation of D is obtained. Here, $r, s \in \{+, -\}$. If (a^r, b^s) is an overcrossing at a , then $a^r = a$; otherwise $r = -$. Each crossing has an anti-clockwise rotation.

Definition 7 [78] Let $P = a_0^{r_0} a_1^{r_1} \dots a_k^{r_k} a_{k+1}^{r_{k+1}}$ be a path in a link diagram D . If $k \geq 0$, $a_0 = a_{k+1}$, $r_j = r_{j+1} \in \{+, -\}$ for any $0 \leq j \leq k$ and $r_0 = -r_{k+1}$, then P is removable. If $a_0 \neq a_{k+1}$, $k \geq 1$, $r_j = r_{j+1}$ for each $1 \leq j \leq k-1$, then $P = a_0^{r_0} a_1^{r_1} \dots a_k^{r_k} a_{k+1}^{r_{k+1}}$ is a pass of length k . If there doesn't exist a pass P_1 of l ($l > k$) such that $a_1^{r_1} \dots a_k^{r_k} \in P_1$, then P is maximal.

If a diagram D has a removable pass, then it is equivalent to a diagram without any removable pass. Now consider the related operations.

Definition 8 [78] Let D be a link diagram and let $e_i = (a_i^{r_i}, b_i^{s_i})$ be edges for $i = 1, 2$. If $b_1^{s_1} = a_2^{r_2}$, then $e = (a_1^{r_1}, b_2^{s_2})$ is the union of e_1 and e_2 . Accordingly, e_1 and e_2 are obtained from e by adding a vertex b_1 on e . e_1 and e_2 is a subdivision of e .

The union implies to remove a crossing b_1 . The subdivision implies to add a crossing b_1 . Next consider the operations on passes.

Definition 9 [78] Set D_1 to be a link diagram. Suppose that $P_1 = a_0^r a_1^\epsilon a_2^\epsilon \dots a_k^\epsilon a_{k+1}^s$ is a pass of length k ($k \geq 1$), and $a_j^{-\epsilon}$ is adjacent to $x_j^{r_j}, y_j^{s_j}$, $(a_j^\epsilon, a_{j+1}^\epsilon)$ is on the boundary of the face f_j for each $1 \leq j \leq k$ where $r, \epsilon, s, r_j, s_j \in \{+, -\}$. Remove the pass P_1 , that is to remove a_j and to add $(x_j^{r_j}, y_j^{s_j})$ for each $1 \leq j \leq k$. Then a figure denoted by $D_1 \setminus P_1$ is obtained. Denote the face containing f_j of D_1 by f_j in $D_1 \setminus P_1$. Add a pass P , that is to add a crossing b_i on e_i ($l \geq 1$, $1 \leq i \leq l$) in $D_1 \setminus P_1$ such that $P = a_0^{r_0} b_1^\epsilon \dots b_l^\epsilon a_{l+1}^{r_{l+1}}$ is a pass of length l ($l \geq 1$). Then a link diagram D is obtained.

Definition 10 [78] A pass replacement is an operation that one removes a pass D_1 , adds a pass P and then obtains a link diagram D based on a link diagram D_1 where D_1, P_1, P, D are given above. If $k = l$ and $b_i \in f_i$ for each $1 \leq i \leq k$, then P isn't different from P_1 ; otherwise P is different

from P_1 . If $l < k$, then the replacement is a short pass replacement. If P and P_1 are different and $l = k$, then the replacement is an equal pass replacement. If $l > k$, then the replacement is a long pass replacement.

The pass replacement is always a replacement between different passes later in the paper.

Definition 11 Let D be a reduced link diagram. If there isn't a short pass replacement in D , then D is a diagram without a short pass. If there is neither a short pass replacement nor an equal pass replacement, then D is a long pass diagram. If a link (or knot) has a long pass diagram, then it is a long pass link (or knot).

An alternating diagram is obvious to be a diagram without a short pass. The non-alternating knot 10_{152} in the Rolfsen knot table is a long pass knot. Since it is Khovanov homologically thick [64], it isn't quasi-alternating by Theorem 9. So a diagram without an equal pass is a generalization of an alternating diagram and it is different from the notation of the quasi-alternating diagram. Obviously, the Thistlethwaite's unknot [36] is a diagram without a short pass (Fig. 3), which shows that there exists a diagram without a short pass is equivalent to the unknot. Is a long pass link equivalent to the unlink? Several problems are given as follows.

Conjecture 5 Let D be a long pass diagram of a link L with n ($n \geq 2$) crossings. Then D is its diagram with the minimum crossings.

If D is alternating, this conjecture holds. 10_{152} is a non-alternating link with the minimum crossings satisfying the conjecture.

Problem 1 Characterize the properties of a quasi-alternating and a long pass link, especially diagrams of the unknot.

Problem 2 Study the Jones polynomials of quasi-alternating knots and

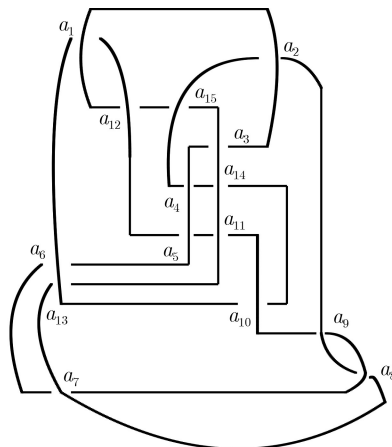


Fig. 3 Thistlethwaite K_T

long pass knots and then study whether the Jones polynomial detects the unknot.

Jones proposed an important conjecture that the Jones polynomial detects the unknot [29]. The conjecture holds for an alternating knot and certain knots and a knot with crossings less than 25. Some results and a detailed introduction are given in [31, 54, 72, 11, 37, 24, 67, 75, 76, 83].

Problem 3 Find an effective method to determine the Khovanov homology of some infinite families of links and study the ability of the Khovanov homology to detect links.

It is hard to determine the Khovanov homology of a link. It can be determined for a link with crossings less than 50 and certain links [65, 12, 20, 40, 43, 51, 49, 58, 59, 66, 68, 73, 69]. It is worth finding methods to determine the Khovanov homology of a general link. The Khovanov homology isn't a complete invariant of a link [79]. Mathematicians have been paying attention to the following important problem:

Find an effective complete invariant of a link, especially a knot.

Acknowledgements The author is grateful to Professor Zhongtao Wu for his helpful comments.

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