Front. Math. China 2015, 10(5): 1169–1178
DOI 10.1007/s11464-015-0483-x

RESEARCH ARTICLE

Lefschetz decomposition for de Rham cohomology on weakly Lefschetz symplectic manifolds

Qiang TAN[1], **Haifeng XU**[2]

1 School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China
2 School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, China

Abstract For a compact symplectic manifold which is s-Lefschetz which is weaker than the hard Lefschetz property, we prove that the Lefschetz decomposition for de Rham cohomology also holds.

Keywords Symplectic manifold, s-Lefschetz, Lefschetz decomposition, s-dd^Λ-lemma
MSC 53D05, 55N20

1 Introduction

Let us recall some definitions and some results of Hodge theory [9]. Let M be an closed oriented Riemannian manifold with metric g. The Hodge star operator

$$*_g\colon \Omega^k(M) \to \Omega^{m-k}(M)$$

is a linear map which satisfies

$$\alpha \wedge *_g\beta = (\alpha,\beta)_g \mathrm{d}\,\mathrm{vol}_g, \quad \forall\, \alpha,\beta \in \Omega^k(M).$$

Here, $\Omega^k(M)$ is the space of the smooth k-forms on M. We denote the adjoint operator of the differential operator d by d^* associated to g. By a direct calculation, we will find that

$$\mathrm{d}^* = (-1)^{mk+m+1} *_g \mathrm{d} *_g$$

on $\Omega^k(M)$. A form α is called harmonic if it is both d-closed and d^*-closed. The Laplacian operator is given by

$$\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}\colon \Omega^k(M) \to \Omega^k(M).$$

Received December 2, 2013; accepted February 10, 2015
Corresponding author: Qiang TAN, E-mail: tanqiang1986@hotmail.com

α is harmonic if and only if $\Delta\alpha = 0$. By the theory of elliptic operator, we conclude that the kernel of Δ is finite dimensional. And the Hodge decomposition tells us every cohomology class has a unique harmonic representative.

Let (M, ω) be a compact symplectic $2n$-manifold, that is, M is a differential manifold with a closed non-degenerate 2-form ω, where ω is called the symplectic structure. Symplectic Hodge theory was introduced by Ehresmann and Libermann [3] and was rediscovered by Brylinski [1]. They defined the symplectic star operator

$$*_\omega \colon \Omega^k(M) \to \Omega^{2n-k}(M)$$

analogously to the Hodge star operator but with respect to the symplectic form ω. As in Riemannian Hodge theory, define

$$\mathrm{d}^\Lambda = (-1)^{k+1} *_\omega \mathrm{d} *_\omega$$

on $\Omega^k(M)$. A form α is called symplectic harmonic if it satisfies

$$\mathrm{d}\alpha = \mathrm{d}^\Lambda\alpha = 0.$$

(Here, we adopt the notion d^Λ as given in [14].) Brylinski conjectured that on a compact symplectic manifold, every de Rham cohomology class contains a symplectic harmonic representative. Some evidence for his conjecture was presented in his paper and he proved the conjecture for compact Kähler manifolds. Several years later, his conjecture for symplectic manifolds was disproved by Mathieu [12]. Brylinski's conjecture is equivalent to the question of the existence of a Hodge decomposition in the symplectic sense. The uniqueness of the decomposition in this case is evidently not true. Mathieu gave two ways to give counter-examples to Brylinski's conjecture. In fact, Mathieu proved that every de Rham cohomology $H^*(M)$ class contains a symplectic harmonic form if and only if the symplectic manifold satisfies the strong (hard) Lefschetz property, that is, the map

$$\begin{aligned} H^k(M) &\to H^{2n-k}(M), \\ A &\mapsto [\omega]^{n-k} \wedge A, \end{aligned}$$

is an isomorphism for all $k \leqslant n$.

Mathieu's theorem is a generalization of the hard Lefschetz theorem for compact Kähler manifolds. His proof involves the representation theory of quivers and Lie superalgebras. Yan [15] provided a simpler, more direct proof of this fact. Yan's proof follows the idea of the standard proof of the hard Lefschetz theorem (see [7]).

On the other hand, for compact symplectic manifolds, Merkulov [13] and Cavalcanti [2] showed that the existence of symplectic harmonic forms in every de Rham cohomology class is equivalent to the symplectic dd^Λ-lemma, that is, to the identities

$$\operatorname{im}\mathrm{d} \cap \ker \mathrm{d}^\Lambda = \operatorname{im}\mathrm{dd}^\Lambda = \operatorname{im}\mathrm{d}^\Lambda \cap \ker \mathrm{d},$$

which mean that if α is a symplectic harmonic k-form, and either is exact or coexact, then $\alpha = \mathrm{dd}^{\Lambda}\beta$ for some k-form β.

Both the existence and uniqueness of symplectic harmonic forms are not expected to hold in de Rham cohomology for symplectic manifolds. So Tseng and Yau thought that the de Rham cohomology maybe not the appropriate cohomology to consider symplectic Hodge theory. They [14] introduced some new cohomologies for compact symplectic manifolds.

The hard Lefschetz property is too strong for symplectic manifolds. Many known non-Kähler symplectic manifolds do not satisfy strong Lefschetz property. Gompf [6] constructed many closed symplectic manifolds which do not satisfy the Lefschetz property. So Fernández et al. [4] introduced s-Lefschetz property which is weaker than hard Lefschetz property. We would observe that every compact symplectic manifold (M, ω) is s-Lefschetz for some integer $s \geqslant 0$. Fernández et al. got many beautiful results on compact symplectic manifold which satisfies the s-Lefschetz property.

It is well known that if a compact symplectic manifold has the hard Lefschetz property, then its de Rham cohomology admits a unique Lefschetz decomposition, i.e.,

$$H^k(M) = \bigoplus_r L^r PH^{k-2r}(M)$$

for any integer k, $0 \leqslant k \leqslant 2n$ (cf. [10,15]). In this paper, we aim to generalize the Lefschetz decomposition for de Rham cohomology on compact symplectic manifolds which are not hard Lefschetz but s-Lefschetz. Here is our main result.

Theorem 1 *Suppose that M is a compact $2n$-dimensional symplectic manifold with the s-Lefschetz property. Then we have the following Lefschetz decomposition:*

1) $H^k(M) = \oplus_r L^r PH^{k-2r}(M)$, $k \leqslant s+1$;
2) $H^{2n-k}(M) = \oplus_r L^r PH^{2n-k-2r}(M)$, $k \leqslant s$.

Remark 1 The similar result was proved by Fernández et al. [5]. Inspired by Tseng and Yau's work, we generalize Fernández et al.'s result by using very different approach.

2 Preliminaries

In this section, we present a brief review of background materials in symplectic Hodge theory. For more details, we refer to [1,11,14,15]. On a compact symplectic manifold (M, ω) of dimension $2n$, the symplectic star operator $*_\omega$ acts on a differential k-form α by

$$\alpha \wedge *_\omega \alpha' = (\omega^{-1})^k(\alpha, \alpha')\mathrm{d}\,\mathrm{vol} = \frac{1}{k!}(\omega^{-1})^{i_1 j_1} \cdots (\omega^{-1})^{i_k j_k} \alpha_{i_1 \cdots i_k} \alpha'_{j_1 \cdots j_k} \frac{\omega^n}{n!}$$

with repeated indices summed over, where $(\omega^{-1})^{ij}$ is the inverse matrix of ω_{ij}.

As in Riemannian Hodge theory, we can define the adjoint operator of d as

$$\mathrm{d}^{\Lambda}\colon \Omega^k(M) \to \Omega^{k-1}(M)$$

via

$$\mathrm{d}^{\Lambda} = (-1)^{k+1} *_{\omega} \mathrm{d} *_{\omega}$$

on $\Omega^k(M)$. We call α coclosed if $\mathrm{d}^{\Lambda}\alpha = 0$, coexact if $\alpha = \mathrm{d}^{\Lambda}\gamma$ for some γ. A form α is called symplectic harmonic if it satisfies

$$\mathrm{d}\alpha = \mathrm{d}^{\Lambda}\alpha = 0.$$

Unlike in the Riemannian case, the symplectic Laplacian

$$\mathrm{d}\mathrm{d}^{\Lambda} + \mathrm{d}^{\Lambda}\mathrm{d} = 0,$$

i.e., d and d^{Λ} are anti-commute.

Using the symplectic form

$$\omega = \sum \frac{1}{2}\omega_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j,$$

the Lefschetz operator

$$L\colon \Omega^k(M) \to \Omega^{k+2}(M)$$

and the dual Lefschetz operator

$$\Lambda\colon \Omega^k(M) \to \Omega^{k-2}(M)$$

are defined acting on $\alpha \in \Omega^k(M)$ by

$$L(\alpha) = \omega \wedge \alpha, \quad \Lambda(\alpha) = \frac{1}{2}(\omega^{-1})^{ij} i_{\partial_{x^i}} i_{\partial_{x^j}} \alpha, \tag{1}$$

respectively, where i denotes the interior product. A nonzero differential k-form α with $k \leqslant n$ is called primitive if $L^{n-k+1}(\alpha) = 0$ or equivalently $\Lambda\alpha = 0$. We denote by $P^k(M)$ the set of all primitive k-form. By [15], we can get

$$\Omega^*(M) = \bigoplus_{k\in\mathbb{N}} L^k P^{*-2k}(M).$$

Given any k-form, there is a unique Lefschetz decomposition into primitive forms.

Lemma 1 (Duality on forms [15]) *The map*

$$L^{n-k}\colon \Omega^k(M) \to \Omega^{2n-k}(M)$$

is an isomorphism.

Fernández et al. [4] introduced s-Lefschetz property and $\mathrm{d}\mathrm{d}^{\Lambda}$-lemma up to degree s.

Definition 1 [4] Let (M,ω) be a symplectic manifold of dimension $2n$ and let $0 \leqslant s \leqslant n-1$.

1) M is said to be s-Lefschetz, if the map

$$L^{n-k}\colon H^k(M) \to H^{2n-k}(M)$$

is an epimorphism for all $k \leqslant s$.

2) M is said to satisfy the dd^Λ-lemma up to degree s or the s-dd^Λ-lemma if

$$\begin{aligned} &\operatorname{im} d \cap \ker \mathrm{d}^\Lambda = \operatorname{im} \mathrm{dd}^\Lambda = \operatorname{im} \mathrm{d}^\Lambda \cap \ker \mathrm{d} && \text{on } \Omega^k(M) \text{ for } k \leqslant s,\\ &\operatorname{im} d \cap \ker \mathrm{d}^\Lambda = \operatorname{im} \mathrm{dd}^\Lambda && \text{on } \Omega^{s+1}(M). \end{aligned}$$

Remark 2 Note that (M,ω) is $(n-1)$-Lefschetz if it satisfies the hard Lefschetz property. So s-Lefschetz can be regard as a generalization of hard Lefschetz and dd^Λ-lemma up to degree s can be regard as a generalization of dd^Λ-lemma.

Let

$$\Omega^k_{hr}(M) = \{\alpha \in \Omega^k(M) \mid d\alpha = \mathrm{d}^\Lambda\alpha = 0\}$$

be the space of the symplectic harmonic k-forms. We consider the vector space

$$H^k_{hr}(M) = \frac{\Omega^k_{hr}(M)}{\Omega^k_{hr}(M) \cap \operatorname{im} \mathrm{d}},$$

consisting of the cohomology classes in $H^k(M)$ containing at least one symplectic harmonic form. Fernández et al. got the following theorem.

Theorem 2 [4,5] *Let (M,ω) be a symplectic manifold of dimension $2n$, and let $s \leqslant n-1$. Then the following statements are equivalent*:

1) *(M,ω) is s-Lefschetz*;

2) *(M,ω) satisfies the dd^Λ-lemma up to degree s*;

3)

$$\operatorname{im} \mathrm{d} \cap \ker \mathrm{d}^\Lambda = \operatorname{im} \mathrm{dd}^\Lambda = \operatorname{im} \mathrm{d}^\Lambda \cap \ker \mathrm{d}$$

hold on $\Omega^{\geqslant 2n-s}$, and

$$\operatorname{im} \mathrm{d} \cap \ker \mathrm{d}^\Lambda = \operatorname{im} \mathrm{dd}^\Lambda$$

holds on $\Omega^{\geqslant 2n-s-1}$;

4)

$$H^k_{hr}(M) = H^k(M),\ k \leqslant s+2, \quad H^{2n-k}_{hr}(M) = H^{2n-k}(M),\ k \leqslant s.$$

Tseng and Yau [14] introduced some new cohomologies and studied their relationships. Having in mind the properties

$$(d)^2 = (\mathrm{d}^\Lambda)^2 = 0$$

and the anti-commutivity

$$\mathrm{dd}^\Lambda = -\mathrm{d}^\Lambda \mathrm{d},$$

they were led to consider the following cohomology groups:

$$H^k_{\mathrm{d}^\Lambda}(M) = \frac{\ker \mathrm{d}^\Lambda \cap \Omega^k(M)}{\mathrm{im}\, \mathrm{d}^\Lambda \cap \Omega^k(M)}, \quad H^k_{\mathrm{d}+\mathrm{d}^\Lambda}(M) = \frac{\ker(\mathrm{d}+\mathrm{d}^\Lambda) \cap \Omega^k(M)}{\mathrm{im}\, \mathrm{dd}^\Lambda \cap \Omega^k(M)},$$

$$H^k_{\mathrm{dd}^\Lambda}(M) = \frac{\ker(\mathrm{dd}^\Lambda) \cap \Omega^k(M)}{(\mathrm{im}\, \mathrm{d} + \mathrm{im}\, \mathrm{d}^\Lambda) \cap \Omega^k(M)}.$$

They [14] obtained the following properties.

Proposition 1 [14, Corollary 3.25] *On a compact symplectic manifold* (M, ω),

$$H^k_{\mathrm{d}+\mathrm{d}^\Lambda}(M) \cong H^{2n-k}_{\mathrm{dd}^\Lambda}(M).$$

Proposition 2 [14, Proposition 3.13] *On a compact symplectic manifold* (M, ω), *the hard Lefschetz property is satisfied if and only if the canonical homomorphism*

$$H^k_{\mathrm{d}+\mathrm{d}^\Lambda}(M) \to H^k(M)$$

is an isomorphism for all k.

Guillemin [8] proved that if (M, ω) is compact, then $H^k(M) \cong H^k_{\mathrm{d}^\Lambda}(M)$ if and only if (M, ω) is hard Lefschetz. So with Propositions 1 and 2, we can get the following result.

Proposition 3 *On a compact symplectic manifold* (M, ω), *the hard Lefschetz property is satisfied if and only if*

$$H^k_{\mathrm{d}^\Lambda}(M) \cong H^k(M) \cong H^k_{\mathrm{d}+\mathrm{d}^\Lambda}(M), \quad \forall\, k.$$

Hard Lefschetz is too strong for general compact symplectic manifold to satisfy. But every compact symplectic manifold (M, ω) is s-Lefschetz for some integer number $s \geqslant 0$. We aim to generalize above results to symplectic manifold which are not hard Lefschetz.

3 Proof of main theorem

In this section, we generalize the Lefschetz decomposition for de Rham cohomology on compact symplectic manifolds which are not hard Lefschetz but s-Lefschetz.

Proposition 4 *Suppose that* (M, ω) *is a compact symplectic manifold of dimension* $2n$. *If* M *is* s*-Lefschetz, then*

1) $H^k_{\mathrm{d}+\mathrm{d}^\Lambda}(M) \to H^k(M)$, $[\alpha]_{\mathrm{d}+\mathrm{d}^\Lambda} \mapsto [\alpha]$ *is an isomorphism for* $0 \leqslant k \leqslant s+1$;

2) $H^{2n-k}_{\mathrm{d}+\mathrm{d}^\Lambda}(M) \to H^{2n-k}(M)$, $[\alpha]_{\mathrm{d}+\mathrm{d}^\Lambda} \mapsto [\alpha]$ *is an isomorphism for* $0 \leqslant k \leqslant s$.

Conversely, if $H^k_{\mathrm{d}+\mathrm{d}^\Lambda}(M) \to H^k(M)$ *is an isomorphism for* $k \leqslant s+2$ *and* $H^{2n-k}_{\mathrm{d}+\mathrm{d}^\Lambda}(M) \to H^{2n-k}(M)$ *is an isomorphism for* $k \leqslant s$, *then* (M,ω) *is* s*-Lefschetz.*

Proof We prove the first statement, the other is similar. Assume first that (M,ω) is s-Lefschetz. Suppose that α is a d-closed and d^Λ-closed k-form such that $[\alpha] = 0$, i.e., $\alpha = \mathrm{d}\beta$ for some $\beta \in \Omega^{k-1}(M)$. Since, by definition, $\alpha \in \ker \mathrm{d} \cap \ker \mathrm{d}^\Lambda$, we have

$$\alpha \in \operatorname{im} \mathrm{d} \cap \ker \mathrm{d}^\Lambda.$$

It follows from Theorem 2 that $\alpha \in \operatorname{im} \mathrm{dd}^\Lambda$. It follows that $[\alpha]_{\mathrm{d}+\mathrm{d}^\Lambda} = 0$. This proves that the homomorphism is injective.

If $[\alpha] \in H^k(M)$, then by Theorem 2, $[\alpha]$ is in $H^k_{hr}(M)$. So there is some β, such that $[\alpha] = [\beta]$, where β is both d-closed and d^Λ-closed. It follows that $[\beta]_{\mathrm{d}+\mathrm{d}^\Lambda} \in H^k_{\mathrm{d}+\mathrm{d}^\Lambda}(M)$ whose image under this map is $[\alpha]$. This proves that the homomorphism is also surjective. Therefore, it is an isomorphism.

The converse is an obvious result following Theorem 2. □

Corollary 1 *Suppose that* (M,ω) *is a compact* s*-Lefschetz symplectic manifold. Then*

1) $H^k(M) \cong H^k_{\mathrm{d}+\mathrm{d}^\Lambda}(M) \cong H^{2n-k}_{\mathrm{dd}^\Lambda}(M)$, $k \leqslant s+1$;

2) $H^{2n-k}(M) \cong H^{2n-k}_{\mathrm{d}+\mathrm{d}^\Lambda}(M) \cong H^k_{\mathrm{dd}^\Lambda}(M)$, $k \leqslant s$.

Let us see the following definition which would be analogous on Kähler manifold to the primitive Dolbeault cohomology.

Definition 2 [14] Let (M,ω) be a compact symplectic manifold of dimension $2n$. The associated primitive cohomology of $H^*_{\mathrm{d}+\mathrm{d}^\Lambda}(M)$ is defined by

$$PH^k_{\mathrm{d}+\mathrm{d}^\Lambda}(M) = \frac{\ker \mathrm{d} \cap P^k(M)}{\mathrm{dd}^\Lambda P^k(M)}.$$

Analogously, Lin [10] defined the primitive cohomology of $H^*(M)$ as follows.

Definition 3 Let (M,ω) be a compact symplectic manifold of dimension $2n$. For any $0 \leqslant k \leqslant n$, the k-th primitive cohomology of $H^k(M)$ is defined as follows.

$$PH^k(M) = \ker(L^{n-k+1} \colon H^k(M) \to H^{2n-k+2}(M)).$$

Here is a very useful lemma. For details we refer to [10] or [14].

Lemma 2 [14, Lemma 3.9] *Let* $\alpha \in P^k(M)$ *with* $0 \leqslant k \leqslant n$. *Suppose that there exists a* k*-form* β *such that* $\alpha = \mathrm{dd}^\Lambda\beta$. *Then there exists a primitive* k*-form* γ *such that* $\alpha = \mathrm{dd}^\Lambda\gamma$.

With Lemma 2, we will get the following lemma.

Lemma 3 *Suppose that M is a compact $2n$-dimensional symplectic manifold with the s-Lefschetz property. Then*

1) $PH^k(M) \cong PH^k_{\mathrm{d}+\mathrm{d}^\Lambda}(M)$ *for any* $0 \leqslant k \leqslant s+1$;

2) $PH^{2n-k}(M) \cong PH^{2n-k}_{\mathrm{d}+\mathrm{d}^\Lambda}(M)$ *for any* $0 \leqslant k \leqslant s+1$.

Proof We prove the first statement, the other is similar. It is clear that there is a natural homomorphism

$$PH^k_{\mathrm{d}+\mathrm{d}^\Lambda}(M) \to PH^k(M),$$
$$[\alpha]_{PH_{\mathrm{d}+\mathrm{d}^\Lambda}} \mapsto [\alpha]_{PH}.$$

When $k=0$, we can immediately get that both $PH^k(M)$ and $PH^k_{\mathrm{d}+\mathrm{d}^\Lambda}(M)$ are $\mathbb{R}$ (here, we suppose that M is connected). We may assume that $k>0$. First, we will prove that this homomorphism is injective. Suppose that α is both d-closed and d^Λ-closed. If

$$0 = [\alpha]_{PH} \in PH^k(M),$$

then we can get that α is d-exact. So α is both d-exact and d^Λ-closed. By Theorem 2 and Definition 1, $\alpha = \mathrm{dd}^\Lambda\beta$ for some k-form β. It follows from Lemma 2 that $\alpha = \mathrm{dd}^\Lambda\gamma$ for some primitive k-form γ. Therefore,

$$[\alpha]_{PH_{\mathrm{d}+\mathrm{d}^\Lambda}} = 0.$$

This proves that the homomorphism is injective.

If $[\alpha]_{PH} \in PH^k(M)$, then, by Definition 3, we can get

$$0 = [L^{n-k+1}\alpha] \in H^{2n-k+2}(M).$$

Therefore,

$$L^{n-k+1}\alpha = \mathrm{d}\beta$$

for a $(2n-k+1)$-form β. It follows from Lemma 1 that $\beta = L^{n-k+1}\eta$ for a $(k-1)$-form η. Thus,

$$L^{n-k+1}(\alpha - \mathrm{d}\eta) = 0.$$

We can see that $\alpha - \mathrm{d}\eta$ is both d-closed and primitive. So it must be d^Λ-closed well. It follows that

$$[\alpha - \mathrm{d}\eta]_{PH_{\mathrm{d}+\mathrm{d}^\Lambda}} \in PH^k_{\mathrm{d}+\mathrm{d}^\Lambda}(M),$$

whose image under the homomorphism is $[\alpha]_{PH}$. This proves that the homomorphism is surjective. Therefore, it is an isomorphism. □

With Proposition 4 and Lemma 3, Theorem 1 could be proved easily.

Proof of Theorem 1 Tseng and Yau [14] have proved that there is a Lefschetz decomposition

$$H^k_{\mathrm{d}+\mathrm{d}^\Lambda}(M) = \bigoplus_r L^r PH^{k-2r}_{\mathrm{d}+\mathrm{d}^\Lambda}(M)$$

for any $0 \leqslant k \leqslant 2n$ on any compact symplectic manifold (M, ω) of dimension $2n$. For a detailed proof of the above decomposition, we refer to [14, Theorem 3.11]. When (M, ω) is s-Lefschetz, we have proved that

$$H^k_{\mathrm{d}+\mathrm{d}^\Lambda}(M) \cong H^k(M), \quad PH^k(M) \cong PH^k_{\mathrm{d}+\mathrm{d}^\Lambda}(M)$$

for any $0 \leqslant k \leqslant s+1$. Therefore,

$$H^k(M) = H^k_{\mathrm{d}+\mathrm{d}^\Lambda}(M) = \bigoplus_r L^r PH^{k-2r}_{\mathrm{d}+\mathrm{d}^\Lambda}(M) = \bigoplus_r L^r PH^{k-2r}(M)$$

for $0 \leqslant k \leqslant s+1$. Similarly, we can get

$$H^{2n-k}(M) = H^{2n-k}_{\mathrm{d}+\mathrm{d}^\Lambda}(M) = \bigoplus_r L^r PH^{2n-k-2r}_{\mathrm{d}+\mathrm{d}^\Lambda}(M) = \bigoplus_r L^r PH^{2n-k-2r}(M)$$

for $0 \leqslant k \leqslant s$. This completes the proof of Theorem 1. □

4 Examples

We know that every compact symplectic manifold (M, ω) is s-Lefschetz for some integer number $s \geqslant 0$. For example, compact simply connected symplectic manifolds of dimension 6 are 1-Lefschetz. Actually, on a compact simply connected symplectic manifold of dimension 6, every de Rham cohomology class of degree $k \neq 4$ admits a symplectically harmonic representative ([5, Corollary]). So we have the following decomposition on compact simply connected symplectic manifolds of dimension 6.

Example 1 Let (M, ω) be a compact simply connected symplectic manifold of dimension 6, then we have

$$\begin{aligned}
H^0(M) &= L^0PH^0(M) = PH^0(M),\\
H^1(M) &= L^0PH^1(M) = PH^1(M),\\
H^2(M) &= L^0PH^2(M) \oplus L^1PH^0(M),\\
H^5(M) &= L^0PH^5(M) \oplus L^1PH^3(M) \oplus L^2PH^1(M),\\
H^6(M) &= L^0PH^6(M) \oplus L^1PH^4(M) \oplus L^2PH^2(M) \oplus L^3PH^0(M).
\end{aligned}$$

The above example, compact simply connected symplectic 6-manifold, may satisfy the hard Lefschetz property in some cases and may not satisfy the hard Lefschetz property in the other. It is hard to distinguish. But the following example is very clear that is not hard Lefschetz but only weakly Lefschetz.

Example 2 Any closed symplectic 4-manifold (M, ω) with b^1 odd is just 0-Lefschetz but not 1-Lefschetz, i.e., hard Lefschetz. We know that, on any symplectic manifold, any cohomology class in $H^k(M)$ contains a harmonic

representative when $k = 0, 1, 2$ ([15, Corollary 3.1]). So on a closed symplectic 4-manifold (M, ω), every de Rham cohomology class of degree $k \neq 3$ admits a symplectically harmonic representative. Yan has proved that $\dim H^3_{hr}(M)$ is even number on any closed symplectic 4-manifold ([15, Corollary 4.5]). When b^1 is odd, it is easy to see that b^3 is odd number by Poincaré duality. So (M, ω) cannot be hard Lefschetz but just 0-Lefschetz. A practical example is the Kodaira-Thurston nilmanifold whose first Betti number is 3. It is well known that the Kodaira-Thurston manifold does not satisfy the hard Lefschetz property. Hence, we have the following weakly Lefschetz decomposition:

$$\begin{aligned} H^0(M) &= L^0PH^0(M) = PH^0(M), \\ H^1(M) &= L^0PH^1(M) = PH^1(M), \\ H^4(M) &= L^0PH^4(M) \oplus L^1PH^2(M) \oplus L^2PH^0(M). \end{aligned}$$

Acknowledgements The authors were very grateful to their advisor Prof. Hongyu Wang for discussion and suggestions. This work was supported in part by the National Natural Science Foundation of China (Grant Nos. 11371309, 11471145, 11401514), the University Science Research Project of Jiangsu Province (14KJB110027), and the Foundation of Yangzhou University (2014CXJ004).

References

1. Brylinski J -L. A differential complex for Poisson manifolds. J Differential Geom, 1988, 28: 93–114
2. Cavalcanti G R. New aspects of the dd^c-lemma. Ph D Thesis, Oxford Univ, 2005, arXiv: math/0501406v1
3. Ehresmann C, Libermann P. Sur le problème déquivalence des formes différentielles extérieures quadratiques. C R Acad Sci Paris, 1949, 229: 697–698
4. Fernández M, Muñoz V, Ugarte L. The $d\delta$-lemma for weakly Lefschetz symplectic manifolds. In: Differential Geometry and Its Applications. Prague: Matfyzpress, 2005, 229–246
5. Fernández M, Muñoz V, Ugarte L. Weakly Lefschetz symplectic manifolds. Trans Amer Math Soc, 2007, 359: 1851–1873
6. Gompf R. A new construction of symplectic manifolds. Ann Math, 1995, 142: 527–597
7. Griffiths P, Harris J. Principles of Algebraic Geometry. New York: John Wiley and Sons, 1978
8. Guillemin V. Symplectic Hodge theory and the $d\delta$-lemma. Preprint, 2001
9. Guillemin V. Hodge theory. http://math.mit.edu/∼vwg/shlomo-notes.pdf, 1997
10. Lin Y. Currents, primitive cohomology classes and symplectic Hodge theory. 2011, arXiv: math/1112.2442v1
11. Lin Y, Sjamaar R. Equivariant symplectic Hodge theory and the $d_G\delta$-lemma. J Symplectic Geom, 2004, 2: 267–278
12. Mathieu O. Harmonic cohomology classes of symplectic manifolds. Comment Math Helv, 1995, 70: 1–9
13. Merkulov S A. Formality of canonical symplectic complexes and Frobenius manifolds. Int Math Res Not, 1998, 14: 727–733
14. Tseng L S, Yau S T. Cohomology and Hodge theory on symplectic manifolds: I. J Differential Geom, 2012, 91: 383–416
15. Yan D. Hodge structure on symplectic manifolds. Adv Math, 1996, 120: 143–154