Front. Math. China 2015, 10(6): 1483–1496
DOI 10.1007/s11464-015-0480-0

RESEARCH ARTICLE

Flag-transitive 2-(v, k, λ) symmetric designs with $(k, \lambda) = 1$ and alternating socle

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Abstract Consider the flag-transitive 2-(v, k, λ) symmetric designs with $(k, \lambda) = 1$. We prove that if $\mathscr{D}$ is a nontrivial 2-(v, k, λ) symmetric design with $(k, \lambda) = 1$ and $G \leqslant \mathrm{Aut}(\mathscr{D})$ is flag-transitive with $\mathrm{Soc}(G) = A_n$ for $n \geqslant 5$, then $\mathscr{D}$ is the projective space $PG_2(3, 2)$ and $G = A_7$.

Keywords Symmetric design, automorphism group, alternating group, flag-transitive
MSC 05B25, 20B25

1 Introduction

A 2-(v, k, λ) *symmetric design* $\mathscr{D}$ is a pair $(P, \mathscr{B})$, where P is a v-set and $\mathscr{B}$ is a collection of v k-subsets (called blocks) of P such that any 2-subset of P is contained in exactly λ blocks. Given a symmetric design $\mathscr{D}$, the number of blocks through any point in P is a constant, denoted by r, and it is well known that $r = k$. This paper deals only with nontrivial 2-(v, k, λ) symmetric designs, those with $2 < k < v - 1$, and denoted by (v, k, λ) symmetric design for simplicity.

An *automorphism* of $\mathscr{D}$ is a permutation of P which leaves $\mathscr{B}$ invariant. The full automorphism group of $\mathscr{D}$, denoted by $\mathrm{Aut}(\mathscr{D})$, is the group consisting of all automorphisms of $\mathscr{D}$. A *flag* of $\mathscr{D}$ is a point-block pair (x, B) such that $x \in B$. For $G \leqslant \mathrm{Aut}(\mathscr{D})$, G is called *flag-transitive* if G acts transitively on the set of flags, and *point-primitive* if G acts primitively on P.

Flag-transitivity is one of many conditions that can be imposed on the automorphism group G of a design $\mathscr{D}$. Zieschang [20] proved that if G is a flag-transitive automorphism group of a 2-design with $(r, \lambda) = 1$ and T is a minimal normal subgroup of G, then T is abelian, or simple and $C_G(T) = 1$. From

Received March 21, 2014; accepted April 19, 2015
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[3, 2.3.7(a)], we know that if $G \leqslant \mathrm{Aut}(\mathscr{D})$ is flag-transitive with $(r, \lambda) = 1$, then G acts primitively on P. It follows that G is an affine or almost simple group. So it is possible to classify this type of design by using the classification of finite primitive permutation groups, especially for the case where G is almost simple, with alternating socle.

Here, we study flag-transitive symmetric designs. Symmetric designs with $\lambda = 1$ are called projective planes. Kantor almost completely classified flag-transitive projective planes [8, Theorem A]. A lot of work has been done on flag-transitive (v, k, λ) symmetric designs especially when λ is small [12,13,15,16]. Delandtsheer [2], Dong and Zhou [5,6,18], Regueiro [11], and Zhu et al. [19] classified flag-transitive, point-primitive (v, k, λ) symmetric designs with λ up to 100, whose automorphism group is almost simple with an alternating group as socle.

The purpose of this paper is to study the flag-transitive (v, k, λ) symmetric designs with $(k, \lambda) = 1$, whose automorphism group is almost simple with an alternating group as socle. This can be viewed as a first step towards an investigation of this type of design with general λ. It is hoped that the complete characterization of the flag-transitive point-primitive symmetric designs with alternating socle could be solved by using the results of this paper and some new methods.

The main result of this paper is as follows.

Theorem 1 *Let $\mathscr{D}$ be a (v, k, λ) symmetric design with $(k, \lambda) = 1$. If $G \leqslant \mathrm{Aut}(\mathscr{D})$ is flag-transitive with alternating socle, then $\mathscr{D}$ is the projective space $PG_2(3, 2)$ and $G = A_7$.*

The structure of this paper is straightforward. In Section 2, we give some preliminary lemmas on flag-transitive symmetric designs and alternating groups that will apply directly to our situation. In Section 3, we complete the proof of Theorem 1 in three steps.

2 Preliminaries

Lemma 1 [3, 2.3.7(a)] *Let $\mathscr{D}$ be a 2-(v, k, λ) design with flag-transitive automorphism group G. If $(r, \lambda) = 1$, then G is point-primitive.*

Lemma 2 *Let $\mathscr{D}$ be a (v, k, λ) symmetric design. Then*

(i) $k(k-1) = \lambda(v-1)$. *In particular, if $(k, \lambda) = 1$, then $k \mid (v-1)$ and $(k, v) = 1$.*

(ii) $1 + 4\lambda(v-1)$ *is a square.*

Proof Counting in two ways triples (α, β, B), where α and β are distinct points and B is a block incident with both of them, gives (i). Solving the quadratic equation $k(k-1) = \lambda(v-1)$ for k, the discriminant must be a perfect square since k is an integer. □

Lemma 3 *If $\mathscr{D}$ is a (v,k,λ) symmetric design and G is a flag-transitive automorphism group of $\mathscr{D}$, then*

(i) $k^2>\lambda v$, *and* $|G_x|^3>\lambda|G|$, *where* $x\in P$;

(ii) $k\mid \lambda d$, *where d is any subdegree of G. Furthermore, if $(k,\lambda)=1$ then* $k\mid d$.

Proof (i) The equality $k(k-1)=\lambda(v-1)$ implies $k^2=\lambda v-\lambda+k$, and by $k>\lambda$, we have $k^2>\lambda v$. Combining this with $v=|G:G_x|$ and $k\leqslant|G_x|$ gives $|G_x|^3>\lambda|G|$.

(ii) It was proved in [1]. □

Lemma 4 [7, Theorem 6.11] *Let v, k, and λ be integers with $k(k-1)=\lambda(v-1)$ for which there exists a (v,k,λ) symmetric design.*

(i) *If v is even, then $k-\lambda$ is a square.*

(ii) *If v is odd, then the equation*

$$(k-\lambda)x^2+(-1)^{(v-1)/2}\lambda y^2=z^2$$

has a solution in integers x,y,z not all zero.

Lemma 5 [9] *If G is A_n or S_n, where $n\neq 6$, acting on a set Ω of size n, and H is any maximal subgroup of G with $H\neq A_n$, then H satisfies one of the following*:

(i) $H=(S_k\times S_\ell)\cap G$, *with* $n=k+\ell$ *and* $k\neq\ell$ (*intransitive case*);

(ii) $H=(S_k\wr S_\ell)\cap G$, *with* $n=k\ell$, $k>1$, *and* $\ell>1$ (*imprimitive case*);

(iii) $H=AGL_k(p)\cap G$, *with* $n=p^k$ *and p prime* (*affine case*);

(iv) $H=(T^k.(\operatorname{Out}T\times S_k))\cap G$, *with T a nonabelian simple group,* $k\geqslant 2$, *and* $n=|T|^{k-1}$ (*diagonal case*);

(v) $H=(S_k\wr S_\ell)\cap G$, *with* $n=k^\ell$, $k\geqslant 5$ *and* $\ell>1$ (*wreath case*);

(vi) $T\trianglelefteq H\leqslant \operatorname{Aut}(T)$, *with T a nonabelian simple group,* $T\neq A_n$ *and H acting primitively on Ω* (*almost simple case*).

Lemma 6 [10, Theorem (b) (I)] *Let G be a primitive permutation group of odd degree n on a set Ω with simple socle $X:=\operatorname{Soc}(G)$, and let $H=G_x$, $x\in\Omega$. If $X\cong A_c$, an alternating group, then one of the following holds*:

(i) *H is intransitive, and* $H=(S_k\times S_{c-k})\cap G$, *where* $1\leqslant k<c/2$;

(ii) *H is transitive and imprimitive, and* $H=(S_a\wr S_b)\cap G$, *where* $ab=c$, $a>1$, $b>1$;

(iii) *H is primitive,* $n=15$ *and* $G\cong A_7$.

Lemma 7 [17, Theorem 14.2] *Let G be primitive on Ω with $|\Omega|=n$, not containing A_n. Then*

$$|S_n:G|\geqslant\left[\frac{n+1}{2}\right]!.$$

In order to prove Theorem 1, we give the following useful inequalities.

Lemma 8 *Let s and t be two positive integers.*

(i) *If $t > s \geqslant 7$, then*

$$\binom{s+t}{s} > t^4 > s^2t^2.$$

(ii) *If $t \geqslant 4$ and $s \geqslant 3$, then $\binom{s+t}{s} > s^2t^2$ implies*

$$\binom{s+t+1}{s} > s^2(t+1)^2.$$

(iii) *If $s \geqslant 6$ and $t \geqslant 2$, then $2^{(s-1)(t-1)} > s^4\binom{t}{2}^2$ implies*

$$2^{s(t-1)} > (s+1)^4\binom{t}{2}^2.$$

(iv) *If $t \geqslant 6$ and $s \geqslant 2$, then $2^{(s-1)(t-1)} > s^4\binom{t}{2}^2$ implies*

$$2^{(s-1)t} > s^4\binom{t+1}{2}^2.$$

Proof (i) It is necessary to prove that $\binom{s+t}{s} > t^4$ holds. Since $t > s \geqslant 7$, we have

$$\begin{aligned}
\frac{\binom{s+t}{s}}{t^4} &= \frac{t+1}{t^4}\,\frac{t+2}{2}\cdots\frac{t+s}{s} \\
&> \frac{(t+2)(t+3)(t+4)(t+5)(t+6)(t+7)}{7!\,t^3} \\
&= \frac{t^6 + 27t^5 + 295t^4 + 1665t^3 + 5104t^2 + 8028t + 5040}{7!\,t^3} \\
&> \frac{t^3 + 27t^2 + 295t + 1665}{7!} \\
&\geqslant \frac{8^3 + 27 \times 8^2 + 295 \times 8 + 1665}{7!} \\
&= \frac{179}{144} \\
&> 1,
\end{aligned}$$

so that

$$\binom{s+t}{s} > t^4 > s^2t^2.$$

(ii) Suppose that $\binom{s+t}{s} > s^2t^2$. Then

$$\binom{s+t+1}{s} = \binom{s+t}{s}\frac{(s+t+1)}{(t+1)} > s^2t^2\frac{(s+t+1)}{(t+1)} = s^2(t+1)^2\frac{(s+t+1)t^2}{(t+1)^3}.$$

This, together with the inequality

$$(s+t+1)t^2 > (t+1)^3,$$

yields (ii).

(iii) Since $s \geqslant 6$, $t \geqslant 2$, and

$$2^{(s-1)(t-1)} > s^4 \binom{t}{2}^2,$$

we have

$$\begin{aligned} 2^{s(t-1)} &= 2^{(s-1)(t-1)} 2^{t-1} \\ &> s^4 \binom{t}{2}^2 2^{t-1} \\ &= (s+1)^4 \binom{t}{2}^2 \left(1 - \frac{1}{s+1}\right)^4 2^{t-1} \\ &\geqslant (s+1)^4 \binom{t}{2}^2 \left(1 - \frac{1}{7}\right)^4 2^{2-1} \\ &> (s+1)^4 \binom{t}{2}^2. \end{aligned}$$

A similar argument yields (iv). □

3 Proof of Theorem 1

Throughout this paper, we assume that the following hypothesis holds.

Hypothesis Let $\mathscr{D} = (P, \mathscr{B})$ be a (v, k, λ) symmetric design with $(k, \lambda) = 1$, $G \leqslant \mathrm{Aut}(\mathscr{D})$ be a flag-transitive automorphism group G with $\mathrm{Soc}(G) = A_n$. Let x be an element of P and $H = G_x$.

By the fact $k = r$ and Lemma 1, G acts primitively on P. Therefore, H is a maximal subgroup of G by [17, Theorem 8.2] and $v = |G : H|$. Furthermore, by the flag-transitivity of G, we have k divides $|H|$, and $k^2 > v$ by Lemma 3 (i).

An almost simple group with socle A_n must be A_n or S_n, or $n = 6$ and it is one of A_6, S_6, M_{10}, $PGL_2(9)$, or $P\Gamma L_2(9)$. First, we deal with these exceptional cases. Assume that $n = 6$ and $G \cong M_{10}$, $PGL_2(9)$, or $P\Gamma L_2(9)$. Each group has exactly three maximal subgroups with index greater than 2, and their indices are precisely 45, 36, and 10, respectively, by using the computer algebra system GAP [14]. The possible parameters (v, k, λ) such that $2 < k < v - 1$ and $k(k-1) = \lambda(v-1)$ are

$$(36, 15, 6),\ (36, 21, 12),\ (45, 12, 3),\ (45, 33, 24).$$

These can be ruled out since $k \nmid |H|$.

Now, we consider $G = A_n$ or S_n with $n \geqslant 5$. The point stabilizer $H = G_x$ acts both on P and on the set $\Omega_n := \{1, 2, \ldots, n\}$. The action of H on Ω_n is primitive, imprimitive, or intransitive as a subgroup of S_n. Our proof consists of three steps.

Step 1 H acts primitively on Ω_n.

Lemma 9 *Let $\mathscr{D}$ and G satisfy Hypothesis. Let the point stabilizers act primitively on Ω_n. Then $\mathscr{D}$ is the projective space $PG_2(3, 2)$.*

Proof Suppose first that k is even. Then v is odd by Lemma 2 (i). From Lemma 6 (iii), $v = 15$ and $G \cong A_7$. But simple calculation implies that there is no parameters set (v, k, λ) satisfying $(k, \lambda) = 1$ and $k > 2$.

Now, assume that k is odd. Let p be any odd prime divisor of k. Then $(p, v) = 1$ according to Lemma 2 (i). Thus, H contains a Sylow p-subgroup R of G. Let $g \in G$ be a p-cycle. Then there is a conjugate of g belonging to H. This implies that H acting on Ω_n contains an even permutation with exactly one cycle of length p and $n-p$ fixed points. Thus, $n - p \leqslant 2$ by [17, Theorem 13.9]. Since $n - 2 \leqslant p \leqslant n$, $p^2 \nmid |G|$, and so $p^2 \nmid k$. Moreover, since k is odd and the prime divisor p of k is one of $n-2$, $n-1$, and n, it follows that k is either a prime, namely, $k = n-2$, $n-1$, or n, or the product of two twin primes, namely, $(n-2)n$.

Moreover, the primitivity of H acting on Ω_n and the fact that $H \ngeqslant A_n$ imply

$$v \geqslant \frac{[(n+1)/2]!}{2}$$

by Lemma 7. Combining this with $k^2 > v$ gives

$$k^2 > \frac{[(n+1)/2]!}{2}.$$

Substituting $k = n-2$, $n-1$, n, or $(n-2)n$ into the inequality, we get all possible pairs (n, k) are

$$(5,3),\ (7,5),\ (6,5),\ (8,7),\ (5,5),\ (7,7),\ (5,15),\ (7,35),\ (13,143).$$

According to Lemmas 2, 3, and the fact $v \geqslant [(n+1)/2]!/2$, we obtain 14 possible parameters (v, k, λ) which are listed as follows:

$$\begin{gathered}(7,3,1),\ (21,5,1),\ (11,5,2),\ (15,7,3),\ (22,7,2),\\ (43,7,1),\ (211,15,1),\ (106,15,2),\ (31,15,7),\ (71,35,17),\\ (596,35,2),\ (1191,35,1),\ (10154,143,2),\ (20307,143,1).\end{gathered}$$

The four parameters with v even can be ruled out by Lemma 4 (i). For the remaining ten cases with v odd, Lemma 6 (iii) implies that only one case can occur, i.e., $(v, k, \lambda) = (15, 7, 3)$. Now, one can check that $G = A_7$, i.e., $\mathscr{D}$ is the projective space $PG_2(3, 2)$. □

Step 2 H acts transitively and imprimitively on Ω_n.

Lemma 10 *Let $\mathscr{D}$ and G satisfy Hypothesis. Then the point stabilizers cannot be transitive and imprimitive on Ω_n.*

Proof Suppose on the contrary that

$$\Sigma := \{\Delta_0, \Delta_1, \dots, \Delta_{t-1}\}$$

is a nontrivial partition of Ω_n preserved by $H = G_x$, where

$$|\Delta_i| = s, \quad 0 \leqslant i \leqslant t-1,\ s,t \geqslant 2,\ st = n.$$

Since H is maximal and $H \leqslant G_\Sigma < G$, it follows that $H = G_\Sigma$. Since $G \cong A_n$ or S_n, we know that H contains all the even permutations of Ω_n preserving Σ. We claim that there is no other nontrivial partition of Ω_n whose full stabilizer in G is H. This has been proved in [18, Case 3, Step 1]. For the case of completeness, we give the proof here.

Suppose that G_x preserves two nontrivial partitions Σ_1 and Σ_2 of Ω_n. Let

$$\Sigma_1 = \{\Delta_0, \Delta_1, \dots, \Delta_{t_1-1}\}$$

with $|\Delta_i| = s_1$, $0 \leqslant i \leqslant t_1 - 1$, $s_1, t_1 \geqslant 2$, and $s_1 t_1 = n$, and let

$$\Sigma_2 = \{\Gamma_0, \Gamma_1, \dots, \Gamma_{t_2-1}\},$$

with $|\Gamma_j| = s_2$, $0 \leqslant j \leqslant t_2 - 1$, $s_2, t_2 \geqslant 2$, and $s_2 t_2 = n$.

If $s_1 = s_2 = 2$, then $t_1 = t_2 = n/2$ and the conclusion is easily proved. In fact, for any $0 \leqslant i \leqslant t_1 - 1$, there exists j such that $\Delta_i \cap \Gamma_j \neq \emptyset$. Suppose that

$$\Delta_i \cap \Gamma_j = \{a\}, \quad \Delta_i = \{a,b\}, \quad \Gamma_j = \{a,c\}, \quad \Delta_k = \{c,d\}, \quad \Delta_\ell = \{e,f\},$$

where a,b,c,d,e and f are distinct. Let

$$g = (cd)(ef).$$

It is easy to see that g is an even permutation acting on Ω_n and fixes Σ_1. Thus, $g \in H$ and fixes Σ_2. Since g fixes a, we have g fixes $\Gamma_j = \{a,c\}$, and therefore, g fixes c, a contradiction. So $\Delta_i = \Gamma_j$, the conclusion is proved.

If $s_1 \geqslant 3$ or $s_2 \geqslant 3$, we can write

$$|\Delta_i| = s_1 \geqslant 3.$$

Then there exists j such that $\Delta_i = \Gamma_j$. In fact, there exists j such that

$$\Delta_i \cap \Gamma_j \neq \emptyset, \quad \Gamma_j = (\Gamma_j \cap \Delta_i) \cup (\Gamma_j \cap \Delta_i^c).$$

Suppose $H = \mathrm{Alt}(\Delta_i)$, the alternating group on the set Δ_i. If $\Gamma_j \cap \Delta_i^c = \emptyset$, then $\Gamma_j \subseteq \Delta_i$. Since for any $h \in H$, $\Gamma_j^h = \Gamma_j$ or $\Gamma_j^h \cap \Gamma_j = \emptyset$, Γ_j is a nontrivial block of H. Hence, $\Gamma_j = \Delta_i$ by the primitivity of H. When $\Gamma_j \cap \Delta_i^c \neq \emptyset$, let

$y \in \Gamma_j \cap \Delta_i^c$. For any $h \in H$, $y^h = y$, $y \in \Gamma_j \cap \Gamma_j^h$, so $\Gamma_j = \Gamma_j^h$. Let $z \in \Gamma_j \cap \Delta_i$. For any $w \in \Delta_i$, since H is transitive on Δ_i, there exists $h_0 \in H$ such that $w = z^{h_0} \in \Gamma_j^{h_0}$, and thus,

$$\Delta_i \subseteq \Gamma_j, \quad \Delta_i \cap \Gamma_j^c = \emptyset.$$

Let $H' = \mathrm{Alt}(\Gamma_j)$. For any $h' \in H'$, $\Delta_i^{h'} = \Delta_i$ or $\Delta_i^{h'} \cap \Delta_i = \emptyset$, so Δ_i is a nontrivial block of H'. Therefore, $\Delta_i = \Gamma_j$ by the primitivity of H'. This completes the proof of the claim.

Thus, for any point x in P, its stabilizer H preserves exactly one partition of t classes of size s of Ω_n. Since G acts transitively on all the nontrivial partitions of Ω_n into t classes of size s, we may identify the points of $\mathscr{D}$ with the partitions of Ω_n into t classes of size s. Then

$$\begin{aligned} v &= \frac{\binom{ts}{s}\binom{(t-1)s}{s}\cdots\binom{3s}{s}\binom{2s}{s}}{t!} \\ &= \frac{\binom{ts}{s}}{t}\frac{\binom{(t-1)s}{s}}{(t-1)}\cdots\frac{\binom{3s}{s}}{3}\frac{\binom{2s}{s}}{2} \\ &= \binom{ts-1}{s-1}\binom{(t-1)s-1}{s-1}\cdots\binom{3s-1}{s-1}\binom{2s-1}{s-1}. \end{aligned} \tag{1}$$

Now, we introduce a definition (see [2]). Fix a point x of $\mathscr{D}$, that is, a partition X of Ω_n into t classes $C_0, C_1, \dots, C_{t-1}$ each of size s. We say that a partition Y of Ω_n into t classes of size s is *j-cyclic* with respect to X if X and Y have $t-j$ common classes, and if, numbering the other j classes $C_0, C_1, \dots, C_{j-1}$, for each C_i ($i = 0, 1, \dots, j-1$), there is a point c_i of C_i such that the j classes of Y which differ from those of X are $(C_i - \{c_i\}) \cup \{c_{i+1}\}$, with the subscripts computed modulo j. We define the cycle of Y (with respect to X) to be the cycle $(C_0, C_1, \dots, C_{j-1})$. As X is supposed to be fixed, if $s \geqslant 3$, then the cycle of points $(c_0, c_1, \dots, c_{j-1})$ (with $c_i \in C_i$) is uniquely determined by Y, and if $s = 2$, there are two such 'opposite' cycles $(c_0, c_1, \dots, c_{j-1})$ and $(c'_{j-1}, c'_{j-2}, \dots, c'_0)$, where $C_i = \{c_i, c'_i\}$, and this pair of cycles of points determines Y uniquely from X and is uniquely determined by Y.

Let

$$\Sigma_j = \{Y \mid Y \text{ is a } j\text{-cyclic partition with respect to } X\}.$$

Suppose that Y is any j-cyclic partition with cycle $(C_0, C_1, \dots, C_{j-1})$ with respect to X, i.e.,

$$\begin{aligned} Y\colon\ & (C_0 - \{c_0\}) \cup \{c_1\},\ (C_1 - \{c_1\}) \cup \{c_2\},\ \dots,\ (C_{j-1} - \{c_{j-1}\}) \cup \{c_0\}, \\ & C_j,\ C_{j+1},\ \dots,\ C_{t-1}, \end{aligned}$$

where $c_i \in C_i$, $i = 0, 1, \dots, j-1$. Let H be the stabilizer of X. For any $h \in H$, we have

$$\begin{aligned} Y^h\colon\ & (C_0^h - \{c_0^h\}) \cup \{c_1^h\}, (C_1^h - \{c_1^h\}) \cup \{c_2^h\}, \dots, (C_{j-1}^h - \{c_{j-1}^h\}) \cup \{c_0^h\}, \\ & C_j^h,\ C_{j+1}^h,\ \dots,\ C_{t-1}^h. \end{aligned}$$

Since $c_k^h \in C_k^h$ for $k=0,1,\ldots,j-1$, Y^h is also a j-cycle partition with cycle $(C_0^h, C_1^h, \ldots, C_{j-1}^h)$ with respect to X, and $Y^h \in \Sigma_j$.

Therefore, the set Σ_j of j-cyclic partitions with respect to X is a union of orbits of H on P for $j=2,3,\ldots,t$. It follows that k divides the length of Σ_j by Lemma 3 (ii).

(a) Suppose first that $s=2$. Then $t \geqslant 3$,

$$v=(2t-1)(2t-3)\cdots 5\cdot 3,$$

and

$$d_j=|\Sigma_j|=\frac{1}{2}\binom{t}{j}\binom{s}{1}^j=2^{j-1}\binom{t}{j}.$$

We claim that $t<7$. If $t \geqslant 7$, then

$$\begin{aligned}v&=(2t-1)(2t-3)\cdots 5\cdot 3\\&\geqslant (2t-1)(2t-3)(2t-5)(2t-7)\cdot 5\cdot 3\\&=t^2(t-1)^2+(239t^4-1918t^3+5159t^2-5280t+1575)\\&\geqslant t^2(t-1)^2+133371\\&>t^2(t-1)^2,\end{aligned}$$

and as k divides $d_2=t(t-1)$, it follows that $t(t-1)\geqslant k$. Hence,

$$v>t^2(t-1)^2\geqslant k^2,$$

which is a contradiction. Thus, $t<7$. For $t=3,4,5,$ or 6, we calculate $d=\gcd(d_2,d_3)$ and list the results in Table 1.

Table 1 Possible d when $s=2$

t	n	v	d_2	d_3	d
3	6	15	6	4	2
4	8	105	12	16	4
5	10	945	20	40	20
6	12	10395	30	80	10

In each case, $k \leqslant d$, which contradicts the fact that $k^2>v$.

(b) Suppose second that $s \geqslant 3$. Then

$$d_j=|\Sigma_j|=\binom{t}{j}\binom{s}{1}^j=s^j\binom{t}{j}.$$

In particular,

$$d_2=\binom{t}{2}\binom{s}{1}^2=s^2\binom{t}{2}$$

and $k \mid d_2$. Moreover, from

$$\binom{is-1}{s-1} = \frac{is-1}{s-1} \cdot \frac{is-2}{s-2} \cdots \frac{is-(s-1)}{1} > i^{s-1}, \quad i = 2, 3, \ldots, t,$$

we have

$$v > 2^{(s-1)(t-1)}.$$

Then

$$2^{(s-1)(t-1)} < v < k^2 \leqslant s^4 \binom{t}{2}^2,$$

and so

$$2^{(s-1)(t-1)} < s^4 \binom{t}{2}^2. \tag{2}$$

We will calculate all pairs (s, t) satisfying inequality (2). First, we claim that $s < 6$ or $t < 6$.

Let $s = 6$ and $t = 6$. Then

$$2^{(s-1)(t-1)} = 2^{25} > s^4 \binom{t}{2}^2 = 2^4 \cdot 3^6 \cdot 5^2,$$

i.e., the pair $(s, t) = (6, 6)$ does not satisfy inequality (2) but satisfies the conditions of Lemma 8 (iii) and (iv). Thus, $s < 6$ or $t < 6$. We proceed our proof in two cases as follows.

Case (i) $3 \leqslant s < 6$ and $t \geqslant 6$.

If $s = 3$, then by (2),

$$2^{2(t-1)} < 3^4 \binom{t}{2}^2,$$

and so $2^t < 9t(t-1)$, which gives $6 \leqslant t \leqslant 9$.

Similarly, if $s = 4$, then

$$2^{3(t-1)} < 4^4 \binom{t}{2}^2,$$

and $t = 6$.

If $s = 5$, then

$$2^{4(t-1)} < 5^4 \binom{t}{2}^2,$$

and so

$$2^{2t-1} < 25t(t-1),$$

which is impossible.

Case (ii) $2 \leqslant t < 6$.

If $t = 2$, then by (2), $2^{s-1} < s^4$, and it follows that $3 \leqslant s \leqslant 17$. Similarly, if $t = 3$, then

$$2^{2(s-1)} < 9s^4,$$

and hence, $3 \leqslant s \leqslant 8$.

If $t = 4$, then

$$2^{3(s-1)} < 36s^4,$$

which gives $3 \leqslant s \leqslant 6$.

If $t = 5$, then from

$$2^{4(s-1)} < 100s^4,$$

we get $s = 3$ or 4.

To sum up, we get 32 pairs (s, t) satisfying inequality (2) as follows:

$$\begin{gathered}(3,2),\ (3,3),\ (3,4),\ (3,5),\ (3,6),\ (3,7),\ (3,8),\ (3,9),\\ (4,2),\ (4,3),\ (4,4),\ (4,5),\ (4,6),\ (5,2),\ (5,3),\ (5,4),\\ (6,2),\ (6,3),\ (6,4),\ (7,2),\ (7,3),\ (8,2),\ (8,3),\ (9,2),\\ (10,2),\ (11,2),\ (12,2),\ (13,2),\ (14,2),\ (15,2),\ (16,2),\ (17,2).\end{gathered}$$

For each (s, t), we calculate the parameters (v, k, λ) satisfying equation (1), Lemmas 2, 3, and $k \mid d_2$. Then we obtain only one triple

$$(v, k, \lambda) = (10, 9, 8)$$

corresponding to $(s, t) = (3, 2)$. But it contradicts the assumption that $k < v-1$. □

Step 3 H acts intransitively on Ω_n.

Lemma 11 *Let $\mathscr{D}$ and G satisfy Hypothesis. Then the point stabilizers cannot be intransitive on Ω_n.*

Proof Suppose on the contrary that H acts intransitively on Ω_n. Since H is a maximal subgroup of G, we have

$$H = (\mathrm{Sym}(S) \times \mathrm{Sym}(\Omega_n \setminus S)) \cap G,$$

where $\emptyset \neq S \subset \Omega_n$ and $|S| = s < n/2$ by Lemma 5 (i). By the flag-transitivity of G, H is transitive on the blocks through x, and so H fixes exactly one point in P. Since H stabilizes only one s-subset of Ω_n, we can identify the point x with S. As the orbit of S under G consists of all the s-subsets of Ω_n, we can identify P with the set of s-subsets of Ω_n. Thus, we may assume that $x = \{1, 2, \ldots, s\}$, and then the orbits of G_x are as follows:

$$\begin{gathered}\Delta_0 = \{x\}, \quad \Delta_1 = \{\{i_1, i_2, \ldots, i_s\} \mid s < i_1 < i_2 < \cdots < i_s\},\\ \Delta_2 = \{\{i_1, i_2, \ldots, i_s\} \mid i_1 \leqslant s < i_2 < \cdots < i_s\},\ \ldots,\\ \Delta_s = \{\{i_1, i_2, \ldots, i_{s-1}, i_s\} \mid i_1 < i_2 < \cdots < i_{s-1} \leqslant s < i_s\}.\end{gathered}$$

It follows that the subdegrees of G are

$$n_0 = 1, \quad n_{i+1} = \binom{s}{i}\binom{n-s}{s-i}, \quad i = 0, 1, \dots, s-1. \tag{3}$$

First, we claim that $s \leqslant 6$. Since $k \mid n_i$ for any subdegree n_i of G by Lemma 3 (ii) and $n_s = s(n-s)$ is a subdegree of G by (3), we have $k \mid s(n-s)$. Combining this with $k^2 > v$, we get

$$s^2(n-s)^2 > \binom{n}{s}.$$

Since the condition $s < n/2$ is equivalent to $s < t := n - s$, we have

$$s^2t^2 > \binom{s+t}{s}. \tag{4}$$

Combining it with Lemma 8 (i), we get $s \leqslant 6$.

If $s = 1$, then $v = n \geqslant 5$. The group G is $(v-2)$-transitive on P. Since $k \leqslant v-2$, G acts k-transitively on P. Then

$$v = |\mathscr{B}| = |B^G| = \binom{n}{k}$$

for every block $B \in \mathscr{B}$. Therefore, $\binom{n}{k} = n$, and so $k = 1$ or $n-1$. Thus, $\mathscr{D}$ is trivial, a contradiction.

If $s = 2$, then the subdegrees are 1, $\binom{n-2}{2}$, $2(n-2)$, and G is a primitive rank 3 group acting on the points (and blocks) of $\mathscr{D}$. By the classification of primitive rank 3 symmetric designs [4], we know that $(v, k, \lambda) = (15, 7, 3)$ with $\mathrm{Soc}(G) = A_6$, or $(35, 17, 8)$ with $\mathrm{Soc}(G) = A_8$. However, in each case, we have $k \nmid |H|$, so G is not flag-transitive.

Suppose that $3 \leqslant s \leqslant 6$. Now, for each value of s, using inequality (4) and Lemma 8 (ii), we know that t (and hence n) is bounded. For example, let $s = 3$. Since

$$\binom{3+48}{3} > 3^2 48^2,$$

we must have $4 \leqslant t \leqslant 47$ by Lemma 8 (ii), and so $7 \leqslant n \leqslant 50$. The bounds that we obtain for n are listed in Table 2.

Table 2 Bound of n when $3 \leqslant s \leqslant 6$

s	t	n
3	$4 \leqslant t \leqslant 47$	$7 \leqslant n \leqslant 50$
4	$5 \leqslant t \leqslant 14$	$9 \leqslant n \leqslant 18$
5	$6, 7, 8, 9$	$11, 12, 13, 14$
6	7	13

Note that $v=\binom{n}{s}$, and $n_1=\binom{n-s}{s}$, $n_s=s(n-s)$ are two subdegrees of G acting on P. Therefore, the 4-tuple (v,k,λ,n) satisfies the following conditions:

(i) $k \mid d$, where $d=\gcd(n_1,n_s)$ (Lemma 3 (ii));

(ii) $k^2>v=\binom{n}{s}$ (Lemma 3 (i));

(iii) $k(k-1)=\lambda(v-1)$, where λ is an integer (Lemma 2 (i));

(iv) $(k,\lambda)=1$.

If $s=3$, by using GAP for (i) and (ii), we obtain the following possibilities for parameters (v,k,n):

$$(286,30,13),\ (364,33,14),\ (1540,57,22),\ (1771,60,23),$$
$$(4495,84,31),\ (4960,87,32),\ (9880,111,40),$$
$$(10660,114,41),\ (18424,138,49),\ (19600,141,50).$$

But each case does not satisfy condition (iii) or (iv). Take $(v,k,n)=(286,30,13)$, for example. By (iii),

$$\lambda=\frac{k(k-1)}{v-1}=\frac{58}{19}$$

is not an integer. The other cases except $(1771,60,23)$ can be ruled out similarly. For $(v,k,n)=(1771,60,23)$, we get $\lambda=2$, which contradicts (iv).

If $s=4$, 5, or 6, by using GAP, there are no parameters (v,k,n) satisfying conditions (i) and (ii). For example, if $s=6$, then

$$n=13,\quad v=1716,\quad d=7.$$

It follows that $k\leqslant 7$ by (i), and then $k^2<v$, which is impossible. If $s=5$, then $n=11$, 12, 13, or 14, $v=462$, 792, 1287, or 2002 and $d=6$, 7, 8, or 9, respectively. It is easy to check that $k^2<v$ for every case. This is the final contradiction. □

Lemmas 9–11 finish the proof of Theorem 1.

Acknowledgements The authors would like to thank the anonymous referees for their valuable suggestions and comments which helped to improve this paper. This work was supported by the National Natural Science Foundation of China (Grant No. 11471123) and the Natural Science Foundation of Guangdong Province (Grant No. S2013010011928).

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