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RESEARCH ARTICLE

Classification of tilting bundles over a weighted projective line of type $(2, 3, 3)$

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Abstract We give a complete classification of tilting bundles over a weighted projective line of type $(2, 3, 3)$. This yields another realization of the tame concealed algebras of type E_6.

Keywords Tilting bundle, tame concealed algebra, weighted projective line, vector bundle
MSC 14H60, 16G10, 16S50, 18F20

1 Introduction

The notion of weighted projective line was introduced by Geigle and Lenzing [3] for a geometric treatment of the canonical algebras studied by Ringel [9]. The idea arises from the fact that the category of coherent sheaves is derived equivalent to the module category of some canonical algebra (the same type as the weighted projective line). In some sense, we can work with the category of coherent sheaves on a weighted projective line to obtain some property of the canonical algebra of the same type.

As the core concept of the link between geometry and algebra, there is a long story to talk about tilting itself [1,4,6]. Tilting theory arises as a universal method for constructing equivalences between categories. The investigation of tilting object (tilting sheaf in the category of coherent sheaves) is then becoming an attractive work. Geigle and Lenzing [3] showed the existence of a tilting sheaf in the category of coherent sheaves by proving that

$$T_{\mathrm{can}} = \bigoplus_{0 \leqslant \boldsymbol{x} \leqslant \boldsymbol{c}} \mathscr{O}(\boldsymbol{x})$$

is a tilting sheaf. Recently, Chen et al. [2] gave a complete classification of tilting bundles (tilting sheaf which is a vector bundle) on the weighted projective line

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of type $(2,2,n)$ and proved that the endomorphism algebras of tilting bundles gives rise to all the tame concealed algebras of type $\widetilde{\mathbb{D}}_n$.

In this paper, we focus on the category of vector bundles on a weighted projective line of type $(2,3,3)$. A complete classification of tilting bundles is given, see Theorem 4.13 below. Notice that the category of coherent sheaves on a weighted projective line of type $(2,3,3)$ is derived equivalent to the finitely generated module category of the canonical algebra of the same type. Then we actually obtain a classification of the tame concealed algebras of type $\widetilde{\mathbb{E}}_6$, which is in accord with the result given by Happel and Vossieck [5].

This paper is organized as follows. In Section 2, some necessary notations and results about the category of coherent sheaves over a weighted projective line $\mathbb{X}$ of type $(2,3,3)$ are recalled. Morphisms between indecomposable vector bundles are computed in Section 3 in order to check whether an object is rigid or not. Finally, the complete classification of tilting bundles is given in Section 4.

2 Preliminaries

In this section, some basic notations and well-known results on the category of coherent sheaves on a weighted projective line of type (2,3,3) are recalled.

Let k be an algebraically closed field. In [3], the triple $\mathbb{X} = (\mathbb{P}^1, \mathbf{D}, \mathbf{p})$ is called a weighted projective line of type $\mathbf{p}$, where $\mathbb{P}^1$ is the projective line over k, $\mathbf{D} = (\infty, 0, 1)$, and $\mathbf{p} = (2,3,3)$.

Denote by $\mathbb{L}$ the rank one abelian group generated by $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ with respect to relations

$$2\boldsymbol{x}_1 = 3\boldsymbol{x}_2 = 3\boldsymbol{x}_3 =: \boldsymbol{c},$$

where $\boldsymbol{c}$ is called the canonical element of $\mathbb{L}$. Each element $\boldsymbol{x}$ in $\mathbb{L}$ can be uniquely written into a normal form:

$$\boldsymbol{x} = \sum_{i=1}^{3} l_i \boldsymbol{x}_i + l\boldsymbol{c},$$

where $l_1 \in \{0,1\}$, $l_2, l_3 \in \{0,1,2\}$, and $l \in \mathbb{Z}$. By defining

$$\boldsymbol{x} \geqslant 0 \quad \text{if and only if} \quad l \geqslant 0,$$

$\mathbb{L}$ becomes a partially ordered group. In particular, each $\boldsymbol{x}$ satisfies exactly one of the following two situations:

$$\boldsymbol{x} \geqslant 0 \quad \text{or} \quad \boldsymbol{x} \leqslant \boldsymbol{w} + \boldsymbol{c},$$

where

$$\boldsymbol{w} = \boldsymbol{c} - \sum_{i=1}^{3} \boldsymbol{x}_i$$

is called the dualizing element of $\mathbb{L}$.

Let S be the commutative algebra

$$S = k[X_1, X_2, X_3]/\langle X_3^3 - X_2^3 + X_1^2\rangle =: k[x_1, x_2, x_3].$$

Then S has an $\mathbb{L}$-grading by setting

$$\deg(x_i) = \boldsymbol{x}_i, \quad 1 \leqslant i \leqslant 3.$$

By an $\mathbb{L}$-graded version of Serre construction [10], the category of coherent sheaves on a weighted projective line $\mathbb{X}$ is defined as the quotient category:

$$\operatorname{coh}\mathbb{X} = \operatorname{mod}^{\mathbb{L}}(S)/\operatorname{mod}_0^{\mathbb{L}}(S),$$

where $\operatorname{mod}^{\mathbb{L}}(S)$ denotes the category of finite generated $\mathbb{L}$-graded S-modules, while $\operatorname{mod}_0^{\mathbb{L}}(S)$ is the full subcategory of $\operatorname{mod}^{\mathbb{L}}(S)$ consisting of $\mathbb{L}$-graded S-modules of finite length.

The image $\mathscr{O}$ of S in the quotient category serves as the structure sheaf of $\operatorname{coh}\mathbb{X}$, and $\mathbb{L}$ acts on $\operatorname{coh}\mathbb{X}$ by grading shift.

Each line bundle is of the form $\mathscr{O}(\boldsymbol{x})$ for a uniquely determined $\boldsymbol{x}$ in $\mathbb{L}$, and for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{L}$,

$$\operatorname{Hom}(\mathscr{O}(\boldsymbol{x}), \mathscr{O}(\boldsymbol{y})) = S_{\boldsymbol{y}-\boldsymbol{x}},$$

where $S_{\boldsymbol{y}-\boldsymbol{x}}$ denotes the homogeneous subspace of S of degree $\boldsymbol{y} - \boldsymbol{x}$.

Geigle and Lenzing [3] showed that the category $\operatorname{coh}\mathbb{X}$ is a hereditary, abelian, k-linear, Hom-finite, Noetherian category with Serre duality

$$\operatorname{Ext}^1(X, Y) = D\operatorname{Hom}(Y, \tau X),$$

where $D = \operatorname{Hom}(-, k)$ and the k-equivalence $\tau\colon \operatorname{coh}\mathbb{X} \to \operatorname{coh}\mathbb{X}$ is given by the grading shift $X \mapsto X(\boldsymbol{w})$.

Consider the full subcategory $\operatorname{vect}\mathbb{X}$ of $\operatorname{coh}\mathbb{X}$ consisting of coherent sheaves which do not have a simple subsheaf. This subcategory is closed under extension and called the category of vector bundles. Furthermore, the AR-quiver of $\operatorname{vect}\mathbb{X}$ consists of a simple standard component of the form $\mathbb{Z}\widetilde{E_6}$ (see [7, Proposition 5.1]).

Let $K_0(\mathbb{X})$ be the Grothendieck group of $\operatorname{coh}\mathbb{X}$. Then the class $\mathscr{O}(\boldsymbol{x})$, $0 \leqslant \boldsymbol{x} \leqslant \boldsymbol{c}$, forms a $\mathbb{Z}$-basis of $K_0(\operatorname{coh}\mathbb{X})$. We will recall two linear functions on $K_0(\mathbb{X})$, called the rank rk, and the degree deg. The function $\operatorname{rk}\colon K_0(\mathbb{X}) \to \mathbb{Z}$ is determined by putting

$$\operatorname{rk}(\mathscr{O}(\boldsymbol{x})) = 1$$

for any $\boldsymbol{x}$, while the function $\deg\colon K_0(\mathbb{X}) \to \mathbb{Z}$ is characterized by setting

$$\deg(\mathscr{O}(\boldsymbol{x})) = \delta(\boldsymbol{x}),$$

where $\delta\colon \mathbb{L} \to \mathbb{Z}$ is a group homomorphism defined by setting

$$\delta(\boldsymbol{x}_1) = 3, \quad \delta(\boldsymbol{x}_2) = \delta(\boldsymbol{x}_3) = 2.$$

For a coherent sheaf X, its slope is defined to be

$$\mu(X) = \deg(X)/\mathrm{rk}(X),$$

which lies in $\mathbb{Q} \cup \{\infty\}$. Particularly, for any vector bundle X, $\mu(X) \in \mathbb{Q}$.

An object X is said to be rigid if $\mathrm{Ext}^1(X, X) = 0$. Geigle and Lenzing [3, Proposition 5.2] showed that for indecomposable vector bundles X, Y, $\mathrm{Hom}(X, Y) \neq 0$ implies $\mu(X) \leqslant \mu(Y)$, and hence, each indecomposable vector bundle is rigid.

Following [7], an indecomposable vector bundle X of rank 2 is said to be an extension bundle if there exists a non-split exact sequence

$$0 \to L(\boldsymbol{w}) \to X \to L(\boldsymbol{x}) \to 0$$

for some line bundle L and

$$0 \leqslant \boldsymbol{x} \leqslant 2\boldsymbol{w} + \boldsymbol{c}.$$

In this case, X is denoted by $E_L\langle \boldsymbol{x} \rangle$. Particularly, if $\boldsymbol{x} = 0$, then $E_L\langle 0 \rangle = E_L$ is called an Auslander bundle, and we will always write $E = E_{\mathscr{O}}$.

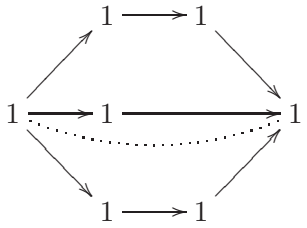
A sheaf T is said to be a tilting sheaf, provided that T is rigid and T generates the category $D^b(\mathrm{coh}\,\mathbb{X})$ as a triangulated category [3]. A tilting sheaf is called a tilting bundle if it is a direct sum of indecomposable vector bundles.

Theorem 2.1 [3] *Every tilting bundle consisting of line bundles is of the form*

$$\bigoplus_{0 \leqslant \boldsymbol{x} \leqslant \boldsymbol{c}} L(\boldsymbol{x})$$

for some line bundle L.

Tilting bundle in Theorem 2.1 is called a canonical tilting bundle, and the quiver of the endomorphism algebra of the canonical tilting bundle is of the form



Notice that the tilting bundle in Theorem has 7 indecomposable direct summands, so to check a bundle T is a tilting bundle, we only need to show that it is rigid and the number of non-isomorphic indecomposable direct summands of T equals to 7 (see [3, Theorem 3.2] and [8, Theorem 9.1.1, Lemma 9.1.3]).

3 Preparations for classification

Thanks to the Serre duality

$$D\mathrm{Ext}^1(X,Y) \simeq \mathrm{Hom}(Y, X(\boldsymbol{w})),$$

in order to check $\mathrm{Ext}^1(X,X)=0$, it is equivalent to proving

$$\mathrm{Hom}(X, X(\boldsymbol{w})) = 0.$$

Hence, in this section, we focus on the computation of morphisms between indecomposable vector bundles. Recall that $E = E_{\mathscr{O}}$ denotes the Auslander bundle determined by $\mathscr{O}$, namely, E is the middle term of the exact sequence

$$0 \to \mathscr{O}(\boldsymbol{w}) \to E \to \mathscr{O} \to 0.$$

Lemma 3.1 *Every vector bundle of rank* 2 *is of the form* $E(\boldsymbol{x})$ *for some* $\boldsymbol{x} \in \mathbb{L}$, *and hence, is an Auslander bundle.*

Proof Notice that each vector bundle of rank 2 is an extension bundle, which is of the form $E_L\langle \boldsymbol{x}\rangle$ for some line bundle L and $0 \leqslant \boldsymbol{x} \leqslant 2\boldsymbol{w}+\boldsymbol{c}$. An easy calculation shows that $\boldsymbol{x} \in \{0, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_2+\boldsymbol{x}_3\}$. For simplicity, we only consider the case when $L = \mathscr{O}$.

Recall that (see [7, Proposition 4.15])

$$E\Big\langle \sum l_i\boldsymbol{x}_i \Big\rangle = E\Big\langle l_i\boldsymbol{x}_i + \sum_{j\neq i}(p_j-2-l_j)\boldsymbol{x}_j \Big\rangle \Big(\sum_{j\neq i}(l_j+1)\boldsymbol{x}_j - \boldsymbol{c}\Big).$$

Then

$$E\langle \boldsymbol{x}_2\rangle = E\langle 0\rangle(\boldsymbol{x}_1+2\boldsymbol{x}_2-\boldsymbol{c}) = E(\boldsymbol{x}_1-\boldsymbol{x}_2),$$

$$E\langle \boldsymbol{x}_3\rangle = E\langle 0\rangle(\boldsymbol{x}_1+2\boldsymbol{x}_3-\boldsymbol{c}) = E(\boldsymbol{x}_1-\boldsymbol{x}_3),$$

$$E\langle \boldsymbol{x}_2+\boldsymbol{x}_3\rangle = E\langle 0\rangle(2\boldsymbol{x}_2+2\boldsymbol{x}_3-\boldsymbol{c}) = E(2\boldsymbol{x}_2-\boldsymbol{x}_3). \qquad \square$$

Lemma 3.2 *Let*

$$\boldsymbol{x} = \sum_i l_i\boldsymbol{x}_i + l\boldsymbol{c}$$

be the normal form of $\boldsymbol{x}$. *Then*

$$\mathrm{Hom}(\mathscr{O}, \mathscr{O}(\boldsymbol{x})) = 0$$

if and only if $l \leqslant -1$.

Proof Note that

$$\mathrm{Hom}(\mathscr{O}, \mathscr{O}(\boldsymbol{x})) = S_{\boldsymbol{x}},$$

and $S_{\boldsymbol{x}} = 0$ if and only if $\boldsymbol{x} \leqslant \boldsymbol{w}+\boldsymbol{c}$.

Let

$$\boldsymbol{x} = \sum_i l_i \boldsymbol{x}_i + l\boldsymbol{c}, \quad 0 \leqslant l_i < p_i,\ l \in \mathbb{Z}.$$

Then

$$\boldsymbol{w} + \boldsymbol{c} - \boldsymbol{x} = (-1 - l)\boldsymbol{c} + (p_i - l_i - 1)\boldsymbol{x}_i$$

is a standard presentation. Hence, $\boldsymbol{w} + \boldsymbol{c} - \boldsymbol{x} \geqslant 0$ if and only if $-1 - l \geqslant 0$, that is, $l \leqslant -1$. □

Remark 3.3 (i) For $\delta(\boldsymbol{x}) \geqslant 0$, if $\mathrm{Hom}(\mathscr{O}, \mathscr{O}(\boldsymbol{x})) = 0$, then $\boldsymbol{x}$ can only appear in the following 9 cases:

δ	$\boldsymbol{x}$
0	$\boldsymbol{x}_2 - \boldsymbol{x}_3,\ \boldsymbol{x}_3 - \boldsymbol{x}_2$
1	$\boldsymbol{x}_1 - \boldsymbol{x}_3,\ \boldsymbol{x}_1 - \boldsymbol{x}_2,\ \boldsymbol{x}_2 + \boldsymbol{x}_3 - \boldsymbol{x}_1$
2	$2\boldsymbol{x}_2 - \boldsymbol{x}_3$
3	$\boldsymbol{x}_1 + \boldsymbol{x}_2 - \boldsymbol{x}_3,\ \boldsymbol{x}_1 + \boldsymbol{x}_3 - \boldsymbol{x}_2$
5	$\boldsymbol{x}_1 + 2\boldsymbol{x}_2 - \boldsymbol{x}_3$

(ii) For $\delta(\boldsymbol{x}) \geqslant 0$, if

$$\mathrm{Hom}(\mathscr{O}, \mathscr{O}(\boldsymbol{x})) = 0, \quad \mathrm{Hom}(\mathscr{O}, \mathscr{O}(\boldsymbol{x} - \boldsymbol{w})) = 0,$$

then

$$\boldsymbol{x} = \boldsymbol{x}_2 - \boldsymbol{x}_3,\ \boldsymbol{x}_3 - \boldsymbol{x}_2,\ \text{or } \boldsymbol{x}_2 + \boldsymbol{x}_3 - \boldsymbol{x}_1.$$

Lemma 3.4 (i) $\mathrm{Hom}(\mathscr{O}, E(\boldsymbol{x})) = 0$ *if and only if* $\delta(\boldsymbol{x}) \leqslant 0$ *or* $\boldsymbol{x} = 2\boldsymbol{x}_2 - \boldsymbol{x}_3$ *or* $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ *or* $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_3$.

(ii) $\mathrm{Hom}(E, \mathscr{O}(\boldsymbol{x})) = 0$ *if and only if* $\delta(\boldsymbol{x}) < 0$ *or* $\boldsymbol{x} = \boldsymbol{x}_3 - \boldsymbol{x}_2$ *or* $\boldsymbol{x} = \boldsymbol{x}_2 - \boldsymbol{x}_3$ *or* $\boldsymbol{x} = -\boldsymbol{w}$.

Proof We only prove (i) while (ii) could be proved similarly. Since E is an Auslander bundle, there is an AR-sequence η:

$$0 \to \mathscr{O}(\boldsymbol{w} + \boldsymbol{x}) \to E(\boldsymbol{x}) \to \mathscr{O}(\boldsymbol{x}) \to 0.$$

Now, suppose $\delta(\boldsymbol{x}) \leqslant 0$. Then

$$\deg(E(\boldsymbol{x})) = \deg \mathscr{O}(\boldsymbol{w} + \boldsymbol{x}) + \deg \mathscr{O}(\boldsymbol{x}) = -1 + 2\delta(\boldsymbol{x}) < 0.$$

Thus, $\mu(E(\boldsymbol{x})) < \mu(\mathscr{O})$, and hence, $\mathrm{Hom}(\mathscr{O}, E(\boldsymbol{x})) = 0$.

Otherwise, by applying $\mathrm{Hom}(\mathscr{O}, -)$ to η, we obtain a long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(\mathscr{O}, \mathscr{O}(\boldsymbol{w} + \boldsymbol{x})) &\to \mathrm{Hom}(\mathscr{O}, E(\boldsymbol{x})) \\ &\to \mathrm{Hom}(\mathscr{O}, \mathscr{O}(\boldsymbol{x})) \to \mathrm{Ext}^1(\mathscr{O}, \mathscr{O}(\boldsymbol{w} + \boldsymbol{x})). \end{aligned}$$

Since $\delta(x) > 0$, $\mathrm{Ext}^1(\mathscr{O}, \mathscr{O}(\boldsymbol{w} + \boldsymbol{x})) = 0$, which yields a short exact sequence

$$0 \to \mathrm{Hom}(\mathscr{O}, \mathscr{O}(\boldsymbol{w} + \boldsymbol{x})) \to \mathrm{Hom}(\mathscr{O}, E(\boldsymbol{x})) \to \mathrm{Hom}(\mathscr{O}, \mathscr{O}(\boldsymbol{x})) \to 0.$$

Thus, $\mathrm{Hom}(\mathscr{O}, E(\boldsymbol{x})) = 0$ if and only if

$$\mathrm{Hom}(\mathscr{O}, \mathscr{O}(\boldsymbol{w}+\boldsymbol{x})) = 0 = \mathrm{Hom}(\mathscr{O}, \mathscr{O}(\boldsymbol{x})).$$

Due to Remark 3.3, one can immediately get the required result. □

Lemma 3.5 $\mathrm{Hom}(E, E(\boldsymbol{x})) \neq 0$ *if and only if* $\delta(\boldsymbol{x}) > 0$ *or* $\boldsymbol{x} = 0$. *Particularly, if* X *is an Auslander bundle and* $\mathrm{Hom}(E, X) \neq 0$, *then*

$$\mathrm{Hom}(E, \tau^{-m}X) \neq 0, \quad \forall\, m \geqslant 0.$$

Proof Consider the following two AR-sequences:

$$\psi\colon 0 \to \mathscr{O}(\boldsymbol{w}) \to E \to \mathscr{O} \to 0,$$
$$\eta\colon 0 \to \mathscr{O}(\boldsymbol{w}+\boldsymbol{x}) \to E(\boldsymbol{x}) \to \mathscr{O}(\boldsymbol{x}) \to 0.$$

If $\delta(\boldsymbol{x}) < 0$, then

$$\deg E(\boldsymbol{x}) = \deg \mathscr{O}(\boldsymbol{x}) + \deg \mathscr{O}(\boldsymbol{w}+\boldsymbol{x}) = 2\delta(\boldsymbol{x}) - 1 < -1$$

and

$$\deg E = \deg \mathscr{O} + \deg \mathscr{O}(\boldsymbol{w}) = -1.$$

Thus, $\deg E(\boldsymbol{x}) < \deg E$ and hence $\mu E(\boldsymbol{x}) < \mu E$. As a result,

$$\mathrm{Hom}(E, E(\boldsymbol{x})) = 0.$$

Now, assume $\delta(\boldsymbol{x}) \geqslant 0$.

By applying $\mathrm{Hom}(\mathscr{O}, -)$ and $\mathrm{Hom}(\mathscr{O}(\boldsymbol{w}), -)$ to η, we obtain the following two long exact sequences:

$$\begin{aligned} 0 \to \mathrm{Hom}(\mathscr{O}, \mathscr{O}(\boldsymbol{w}+\boldsymbol{x})) &\to \mathrm{Hom}(\mathscr{O}, E(\boldsymbol{x})) \to \mathrm{Hom}(\mathscr{O}, \mathscr{O}(\boldsymbol{x})) \\ &\to \mathrm{Ext}^1(\mathscr{O}, \mathscr{O}(\boldsymbol{w}+\boldsymbol{x})) \to \mathrm{Ext}^1(\mathscr{O}, E(\boldsymbol{x})) \to \mathrm{Ext}^1(\mathscr{O}, \mathscr{O}(\boldsymbol{x})) \to 0, \end{aligned} \tag{1}$$

$$\begin{aligned} 0 \to \mathrm{Hom}(\mathscr{O}(\boldsymbol{w}), \mathscr{O}(\boldsymbol{w}+\boldsymbol{x})) &\to \mathrm{Hom}(\mathscr{O}(\boldsymbol{w}), E(\boldsymbol{x})) \\ &\to \mathrm{Hom}(\mathscr{O}(\boldsymbol{w}), \mathscr{O}(\boldsymbol{x})) \to \mathrm{Ext}^1(\mathscr{O}(\boldsymbol{w}), \mathscr{O}(\boldsymbol{w}+\boldsymbol{x})) \\ &\to \mathrm{Ext}^1(\mathscr{O}(\boldsymbol{w}), E(\boldsymbol{x})) \to \mathrm{Ext}^1(\mathscr{O}(\boldsymbol{w}), \mathscr{O}(\boldsymbol{x})) \to 0. \end{aligned} \tag{2}$$

Since $\delta(\boldsymbol{x}) \geqslant 0$, we have

$$\mathrm{Ext}^1(\mathscr{O}, \mathscr{O}(\boldsymbol{x})) = \mathrm{Ext}^1(\mathscr{O}(\boldsymbol{w}), \mathscr{O}(\boldsymbol{w}+\boldsymbol{x})) = \mathrm{Ext}^1(\mathscr{O}(\boldsymbol{w}), \mathscr{O}(\boldsymbol{x})) = 0.$$

Applying these to (2), we deduce that

$$\mathrm{Ext}^1(\mathscr{O}(\boldsymbol{w}), E(\boldsymbol{x})) = 0.$$

Furthermore, by applying $\mathrm{Hom}(-, E(\boldsymbol{x}))$ to ψ, one gets the following exact sequence:

$$\begin{aligned} 0 \to \mathrm{Hom}(\mathscr{O}, E(\boldsymbol{x})) &\to \mathrm{Hom}(E, E(\boldsymbol{x})) \to \mathrm{Hom}(\mathscr{O}(\boldsymbol{w}), E(\boldsymbol{x})) \\ &\to \mathrm{Ext}^1(\mathscr{O}, E(\boldsymbol{x})) \to \mathrm{Ext}^1(E, E(\boldsymbol{x})) \to 0. \end{aligned} \tag{3}$$

If $\delta(\boldsymbol{x}) > 0$, then

$$\mathrm{Ext}^1(\mathscr{O}, \mathscr{O}(\boldsymbol{w}+\boldsymbol{x}) = 0.$$

According to the exact sequence (1), we deduce that

$$\mathrm{Ext}^1(\mathscr{O}, E(\boldsymbol{x})) = 0,$$

and (1) and (3) yield the following two short exact sequences:

$$0 \to \mathrm{Hom}(\mathscr{O}, \mathscr{O}(\boldsymbol{w}+\boldsymbol{x})) \to \mathrm{Hom}(\mathscr{O}, E(\boldsymbol{x})) \to \mathrm{Hom}(\mathscr{O}, \mathscr{O}(\boldsymbol{x})) \to 0, \quad (4)$$

$$0 \to \mathrm{Hom}(\mathscr{O}, E(\boldsymbol{x})) \to \mathrm{Hom}(E, E(\boldsymbol{x})) \to \mathrm{Hom}(\mathscr{O}(\boldsymbol{w}), E(\boldsymbol{x})) \to 0. \quad (5)$$

If $\mathrm{Hom}(\mathscr{O}, E(\boldsymbol{x})) \neq 0$, then $\mathrm{Hom}(E, E(\boldsymbol{x})) \neq 0$. Otherwise, $\mathrm{Hom}(\mathscr{O}, E(\boldsymbol{x})) = 0$. By applying this to (4), we obtain

$$\mathrm{Hom}(\mathscr{O}, \mathscr{O}(\boldsymbol{w}+\boldsymbol{x})) = 0 = \mathrm{Hom}(\mathscr{O}, \mathscr{O}(\boldsymbol{x})).$$

By Remark 3.3 (ii), we have

$$\boldsymbol{x}+\boldsymbol{w} \in \{\boldsymbol{x}_2-\boldsymbol{x}_3, \boldsymbol{x}_3-\boldsymbol{x}_2, \boldsymbol{x}_2+\boldsymbol{x}_3-\boldsymbol{x}_1\}.$$

Thus, $\boldsymbol{x}-\boldsymbol{w} \in \{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$, which shows $\mathrm{Hom}(\mathscr{O}(\boldsymbol{w}), \mathscr{O}(\boldsymbol{x})) \neq 0$. Furthermore, the long exact sequence (2) gives the exact sequence

$$0 \to \mathrm{Hom}(\mathscr{O}(\boldsymbol{w}), E(\boldsymbol{x})) \to \mathrm{Hom}(\mathscr{O}(\boldsymbol{w}), \mathscr{O}(\boldsymbol{x})) \to 0.$$

As a result, $\mathrm{Hom}(\mathscr{O}(\boldsymbol{w}), E(\boldsymbol{x})) \neq 0$. Then by (5), we get $\mathrm{Hom}(E, E(\boldsymbol{x})) \neq 0$.

If $\delta(\boldsymbol{x}) = 0$, then $\boldsymbol{x} \in \{0, \boldsymbol{x}_2-\boldsymbol{x}_3, \boldsymbol{x}_3-\boldsymbol{x}_2\}$. If $\boldsymbol{x} = 0$, then the result obviously holds. Otherwise, by Remark 3.3 (i), we have

$$\begin{aligned}\mathrm{Hom}(\mathscr{O}, \mathscr{O}(\boldsymbol{x})) &= \mathrm{Hom}(\mathscr{O}(\boldsymbol{w}), \mathscr{O}(\boldsymbol{w}+\boldsymbol{x})) \\ &= \mathrm{Hom}(\mathscr{O}(\boldsymbol{w}), \mathscr{O}(\boldsymbol{x})) \\ &= \mathrm{Hom}(\mathscr{O}, \mathscr{O}(\boldsymbol{w}+\boldsymbol{x})) \\ &= 0.\end{aligned}$$

Then by (1), $\mathrm{Hom}(\mathscr{O}, E(\boldsymbol{x})) = 0$, and by (2),$\mathrm{Hom}(\mathscr{O}(\boldsymbol{w}), E(\boldsymbol{x})) = 0$. Thus, $\mathrm{Hom}(E, E(\boldsymbol{x})) = 0$ according to (3). This finishes the proof. □

Immediately, we have the following consequence.

Corollary 3.6 *Let T be an arbitrary tilting bundle, and let X be an arbitrary Auslander bundle. Then there is at most one indecomposable vector bundle in the τ-orbit of X which occurs as a direct summand of T.*

Proof Only need to notice that $\mathrm{Hom}(X, \tau^{-m}X) \neq 0$ for any Auslander bundle X and any $m \geqslant 0$. □

For later usage, we describe all the AR-sequences in $\mathrm{vect}(\mathbb{X})$. Notice that all the indecomposable vector bundles of rank 3 are in one τ-orbit.

Proposition 3.7 *If $0 \neq f\colon \mathscr{O} \to X$ is an irreducible morphism, then $X = E(-\boldsymbol{w})$. Dually, if $0 \neq g\colon Y \to \mathscr{O}$ is an irreducible morphism, then $Y = E$.*

Proof Consider the following two AR-sequences:

$$0 \to \mathscr{O} \to X \oplus X' \to \tau^{-1}\mathscr{O} \to 0,$$
$$0 \to \mathscr{O} \to E(-\boldsymbol{w}) \to \tau^{-1}\mathscr{O} \to 0.$$

Since

$$\mathrm{Ext}^1(\mathscr{O}(-\boldsymbol{w}), \mathscr{O}) = D\mathrm{Hom}(\mathscr{O}, \mathscr{O}) = k,$$

we have

$$E(-\boldsymbol{w}) = X \oplus X'.$$

Because $E(-\boldsymbol{w})$ is indecomposable, $X' = 0$, and thus, $X = E(-\boldsymbol{w})$. □

Proposition 3.8 *If $0 \neq f\colon E \to X$ is an irreducible morphism, then $X = \mathscr{O}$ or X is an indecomposable vector bundle of rank 3.*

Proof Consider the AR-sequence

$$0 \to E \to \mathscr{O} \oplus F \to \tau^{-1}E \to 0.$$

Notice that $\mathrm{rk}F = 3$. If $F = E(\boldsymbol{x}) \oplus F'$, then $\mathrm{rk}(F') = 1$ and it is thus a line bundle. Since there is an irreducible morphism $E \to F'$, we have $F' = \mathscr{O}$, and thus, the dimension of the space consisting of irreducible morphisms from E to $\mathscr{O}$ is at least 2, which is impossible for $\dim \mathrm{Hom}(E, \mathscr{O}) = 1$. Particularly, F does not have a line bundle as its direct summand. Hence, F is indecomposable. □

Hence, there is always an AR-sequence

$$0 \to E \to \mathscr{O} \oplus F \to \tau^{-1}E \to 0,$$

and we will fix F as expressed above. Furthermore, for any vector bundle X, we have $1 \leqslant \mathrm{rk}X \leqslant 3$, and hence, vector bundles of rank 3 all lie in one τ-orbit. The following result can be found in [7, Proposition 5.1], and we give a direct proof here.

Proposition 3.9 *There is an AR-sequence*

$$0 \to \tau F \to \bigoplus_{\delta(\boldsymbol{x})=0} E(\boldsymbol{x}) \to F \to 0.$$

Proof First, consider the AR-sequence

$$0 \to \mathscr{O}(\boldsymbol{w} + \boldsymbol{x}) \to E(\boldsymbol{x}) \to \mathscr{O}(\boldsymbol{x}) \to 0.$$

Then

$$\deg E(\boldsymbol{x}) = \deg \mathscr{O}(\boldsymbol{w} + \boldsymbol{x}) + \deg \mathscr{O}(\boldsymbol{x}) = \delta(\boldsymbol{x} + \boldsymbol{w}) + \delta(\boldsymbol{x}) = 2\delta(\boldsymbol{x}) - 1.$$

Notice the AR-sequence

$$0 \to E(\boldsymbol{x}) \to \mathscr{O}(\boldsymbol{x}) \oplus F(\boldsymbol{x}) \to E(\boldsymbol{x}-\boldsymbol{w}) \to 0. \tag{6}$$

Then

$$\deg F(\boldsymbol{x}) = \deg E(\boldsymbol{x}) + \deg E(\boldsymbol{x}-\boldsymbol{w}) - \deg \mathscr{O}(\boldsymbol{x}) = 3\delta(\boldsymbol{x}),$$

and hence, $\mu F(\boldsymbol{x}) = \delta(\boldsymbol{x})$.

Thus, if $\delta(\boldsymbol{x}) = 0$, then

$$\mu(F) = \mu(F(\boldsymbol{x})) = 0.$$

Recall F and $F(\boldsymbol{x})$ are in the same τ-orbit, then $F = F(\boldsymbol{x})$.

Consider the AR-sequence (6). There exists irreducible morphism $E(\boldsymbol{x}) \to F$ for $\delta(\boldsymbol{x}) = 0$. Then we have the following AR-sequence:

$$0 \to \tau F \to \bigoplus_{\delta(\boldsymbol{x})=0} E(\boldsymbol{x}) \oplus X \to F \to 0$$

for some vector bundle X. If $\delta(\boldsymbol{x}) = 0$, then $\boldsymbol{x} \in \{0, \boldsymbol{x}_2 - \boldsymbol{x}_3, \boldsymbol{x}_3 - \boldsymbol{x}_2\}$. Thus,

$$\mathrm{rk} \bigoplus_{\delta(\boldsymbol{x})=0} E(\boldsymbol{x}) = 6 = \mathrm{rk}\tau F + \mathrm{rk}F,$$

and therefore, $X = 0$. □

Lemma 3.10 *Let X be an arbitrary indecomposable vector bundle.*

(i) *If* $\mathrm{Hom}(F, X) \neq 0$, *then* $\mathrm{Hom}(F, \tau^{-m}X) \neq 0$ *for any* $m \geqslant 0$.

(ii) *If* $\mathrm{Hom}(X, F) \neq 0$, *then* $\mathrm{Hom}(X, \tau^{-m}F) \neq 0$ *for any* $m \geqslant 0$.

Proof We only prove (ii) while (i) can be proved similarly.

To prove (ii), it suffices to deal with the case $m = 1$. Consider the exact sequence

$$0 \to F \to \bigoplus_i E_i \to F(-\boldsymbol{w}) \to 0.$$

Since $\mathrm{Hom}(X, F) \neq 0$, there exists i such that $\mathrm{Hom}(X, E_i) \neq 0$. Notice that

$$\mathrm{rk}E_i = 2 < \mathrm{rk}F = \mathrm{rk}F(-\boldsymbol{w}).$$

Then the irreducible morphism from E_i to $F(-\boldsymbol{w})$ is a monomorphism. Thus, $\mathrm{Hom}(X, F(-\boldsymbol{w})) \neq 0$. □

4 Classification of tilting bundles

Now, we are able to classify all the tilting bundles over weighted projective line of type $(2, 3, 3)$. When considering the relationship between the category of

coherent sheaves on a weighted projective line of type $(2,3,3)$ and the category of finitely generated modules over a canonical algebra of the same type, we actually obtain the classification of tame concealed algebras of type $\widetilde{\mathbb{E}}_6$, which is first given by Happel and Vossieck [5], see also [9, p. 365].

4.1 Domain

For an indecomposable vector bundle X, denote $S(X \to)$ to be the slice which starts from X. In our situation, $S(X \to)$ is the set of indecomposable vector bundles Y that satisfy $\mathrm{Hom}(X,Y) \neq 0$ and $\mathrm{Hom}(X, \tau^m Y) = 0$ for any $m \geqslant 1$. Dually, one can define $S(\to X)$. For example, $S(\mathscr{O} \to)$ is of the form (as a subquiver of $\Gamma(\mathrm{vect}\,\mathbb{X})$)

$$\begin{array}{ccccccc} \mathscr{O} & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & X_4 \longrightarrow \mathscr{O}(\boldsymbol{x}) \\ & & & & & \searrow & \\ & & & & & & X_4' \longrightarrow \mathscr{O}(\boldsymbol{x}) \end{array}$$

Let $\mathscr{X}_i$ to be the τ-orbit of X_i, and let $\mathscr{X}_1$, $\mathscr{X}_4'$, $\mathscr{X}_5$, and $\mathscr{X}_5'$ to be the τ-orbits of $\mathscr{O}$, X_4', $\mathscr{O}(\boldsymbol{x})$, and $\mathscr{O}(\boldsymbol{x})$, respectively. For an indecomposable vector bundle X, if $X \in \mathscr{X}_i$ or $X \in \mathscr{X}_i'$, define the level of X to be i.

Suppose that X is an indecomposable vector bundle, its domain in $\Gamma(\mathrm{vect}\,\mathbb{X})$ is the set of indecomposable vector bundle Y, which satisfies that there exists $m, n \geqslant 0$ such that

$$\tau^m Y \in S(\to X), \quad \tau^{-n} Y \in S(X \to).$$

Denote by $\mathrm{Dom}(X)$ the domain of X. Particularly, let $\mathrm{Dom}^+(X)$ be the subset of $\mathrm{Dom}(X)$ consisting of indecomposable vector bundles whose levels are less than that of X. And $\mathrm{Dom}^-(X)$ is defined dually.

The following two results are due to [2].

Proposition 4.1 *Let X and Y be two indecomposable vector bundles. Then $Y \in \mathrm{Dom}(X)$ if and only if $X \in \mathrm{Dom}(Y)$.*

Proposition 4.2 *Let X and Y be two indecomposable vector bundles. Then the following statements are equivalent*:

(i) $X \in \mathrm{Dom}^+(Y)$;

(ii) $\mathrm{Dom}^+(X) \subseteq \mathrm{Dom}^+(Y)$;

(iii) $Y \in \mathrm{Dom}^-(X)$;

(iv) $\mathrm{Dom}^-(Y) \subseteq \mathrm{Dom}^-(X)$.

Lemma 4.3 *Let X be an indecomposable vector bundle.*

(i) *$X \in \mathrm{Dom}(F)$ if and only if $F \oplus X$ is rigid.*

(ii) *$E \oplus X$ is rigid if and only if $X \in \mathrm{Dom}(E)$ or $X = \mathscr{O}(3\boldsymbol{w})$ or $X = \mathscr{O}(-2\boldsymbol{w})$.*

(iii) *Let $X \in \mathrm{Dom}^+(E)$ and $Y \in \mathrm{Dom}^-(E)$. Then $X \oplus Y$ is rigid.*

Proof Thanks to the Serre duality

$$D\mathrm{Hom}(X,Y)=\mathrm{Ext}^1(Y,X(\boldsymbol{w})),$$

we obtain these results by direct checking. □

4.2 Tilting bundle containing an indecomposable vector bundle of rank 3 as a direct summand

Since every indecomposable vector bundle of rank 3 is of the form $F(k\boldsymbol{w})$ for some unique $k\in\mathbb{Z}$, without loss of generality, we may assume in this case that the tilting bundle T contains F as a direct summand.

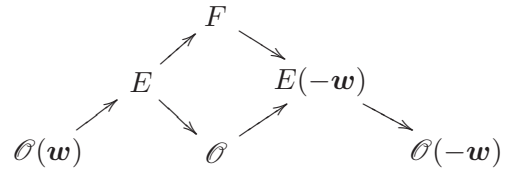
Set

$$\mathrm{Dom}^{x_1}(F)=\mathrm{Dom}^{+}(F),$$

$$\mathrm{Dom}^{x_2}(F)=(\mathscr{X}_3\cup\mathscr{X}_4\cup\mathscr{X}_5)\cap\mathrm{Dom}^{-}(F),$$

$$\mathrm{Dom}^{x_3}(F)=(\mathscr{X}_3\cup\mathscr{X}_4'\cup\mathscr{X}_5')\cap\mathrm{Dom}^{-}(F).$$

The shape of $\mathrm{Dom}^{x_1}(F)$ (as a subquiver of the AR-quiver of $\mathrm{vect}\,\mathbb{X}$) is of the form



And notice further that the shapes of the above three sets are the same.

Lemma 4.4 *Let $X_i\in\mathrm{Dom}^{x_i}(F)$ and $X_j\in\mathrm{Dom}^{x_j}(F)$. If $i\neq j$, then $X_i\oplus X_j$ is rigid.*

Proof The fact that $\mathrm{Dom}^{x_j}(F)\subseteq\mathrm{Dom}(X_i)$ implies the result. □

Lemma 4.5 *Let M be a direct sum of some indecomposable vector bundles in $\mathrm{Dom}^{x_i}(F)\backslash\{F\}$. If M is rigid, then $|M|\leqslant 2$.*

Proof Actually, each $\mathrm{Dom}^{x_i}(F)$ is of the form

$$\{L(\boldsymbol{w}),L,L(-\boldsymbol{w}),E_L,E_L(-\boldsymbol{w}),F\}$$

for

$$L(\boldsymbol{w})=\mathscr{L}\cap\mathrm{Dom}^{x_i}(F)\cap S(\to F),$$

where $\mathscr{L}$ is the category of line bundles. Then by Lemmas 3.2 and 4.3, the result holds. Particularly, elements in the set

$$\{L(\boldsymbol{w})\oplus L(-\boldsymbol{w}),E_L\oplus L(\boldsymbol{w}),E_L\oplus L,E_L(-\boldsymbol{w})\oplus L(-\boldsymbol{w}),E_L(-\boldsymbol{w})\oplus L\}$$

are just those M with $|M|=2$. □

Put

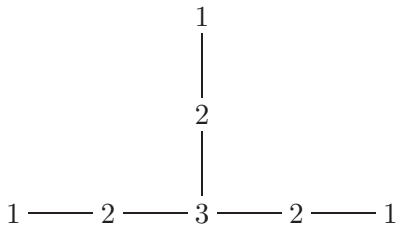
$$\begin{aligned}\mathrm{Br}^i(F) = \{&L(\boldsymbol{w}) \oplus L(-\boldsymbol{w}), E_L \oplus L(\boldsymbol{w}), E_L \oplus L,\\ &E_L(-\boldsymbol{w}) \oplus L(-\boldsymbol{w}), E_L(-\boldsymbol{w}) \oplus L\},\end{aligned} \tag{7}$$

where

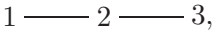
$$L(\boldsymbol{w}) = \mathscr{L} \cap \mathrm{Dom}^{x_i}(F) \cap S(\to F).$$

Proposition 4.6 *Every tilting bundle T containing F as a direct summand is of the form $F \oplus (\oplus_i T^{x_i})$, where $T^{x_i} \in \mathrm{Br}^i(F)$, and every object of this form is a tilting bundle.*

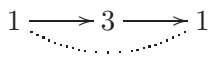
The quiver of the endomorphism algebra of tilting bundle of this form is like



where the orientation can be chosen freely, and any one of the arms, which is of the form



could be replaced by a branch as



with zero relations.

Proof (a) Let $T = F \oplus T'$ be a tilting bundle. Then T is rigid. By Lemma 4.3 (i), every indecomposable direct summand of T' belongs to $\mathrm{Dom}(F)$. Thus, T is of the form $F \oplus (\oplus_i T^{x_i})$, where $T^{x_i} \in \mathrm{Dom}^{x_i}(F) \cup \{0\}$. Particularly, when there does not exist a non-zero summand belonging to $\mathrm{Dom}^{x_i}(F)$, put $T^{x_i} = 0$. Since T is rigid, T^{x_i} is rigid too. By Lemma 4.5, $|T^{x_i}| \leqslant 2$. Thus,

$$|T| \leqslant 1 + 2 + 2 + 2 = 7.$$

For T is a tilting bundle, $|T| = 7$. Hence, $|T^{x_i}| = 2$ for every i, which means $T^{x_i} \in \mathrm{Br}^i(F)$ by Lemma 4.5.

(b) To show the object in the above form is a tilting bundle, we only need to prove it is rigid, since it has exactly 7 indecomposable direct summands. And the rigidness follows from Lemmas 4.4 and 4.5. □

In the following, tilting bundle which does not have an indecomposable vector bundle of rank 3 as its direct summand is considered. These tilting bundles are then direct sums of some line bundles and some Auslander bundles. Tilting bundle consisting of line bundles is described in Theorem 2.1.

By Corollary 3.6, we know that the number of Auslander bundles appearing in the tilting bundle is not greater than three.

4.3 Tilting bundle consisting of one Auslander bundle and 6 line bundles

Without loss of generality, assume that the unique Auslander bundle in the tilting bundle T is E. According to Lemma 4.3, the other six line bundles belong to

$$\mathrm{Dom}(E) \cup \{\mathscr{O}(3\boldsymbol{w})\} \cup \{\mathscr{O}(-2\boldsymbol{w})\}.$$

Note that $\mathscr{O}(3\boldsymbol{w}) \oplus \mathscr{O}(-2\boldsymbol{w})$ is not rigid.

The line bundles in $\mathrm{Dom}^-(E)$ are of the form $L_i(m\boldsymbol{w})$, $0 \leqslant m \leqslant 3$, where

$$L_i = \mathscr{O}(\boldsymbol{x}_1 - \boldsymbol{x}_i), \quad i = 2, 3.$$

Actually,

$$L_2 = S(E \to) \cap \mathscr{L} \cap \mathscr{X}_5', \quad L_3 = S(E \to) \cap \mathscr{L} \cap \mathscr{X}_5.$$

We claim that if Y is a direct sum of some line bundles in $\{L_i(m\boldsymbol{w}) \mid 0 \leqslant m \leqslant 3\}$ and is rigid, then $|Y| \leqslant 2$. In fact, if $|Y| \geqslant 3$, then there exists $1 \leqslant k \leqslant 2$, such that $L_i(k\boldsymbol{w}) \oplus L_i((k+1)\boldsymbol{w})$ is a direct summand of Y. However, $L_i(k\boldsymbol{w}) \oplus L_i((k+1)\boldsymbol{w})$ is not rigid, which is a contradiction, and hence, the claim holds. Furthermore, according to Remark 3.3, if $|Y| = 2$, then

$$Y \in \{L_i \oplus L_i(2\boldsymbol{w}), L_i \oplus L_i(3\boldsymbol{w}), L_i(\boldsymbol{w}) \oplus L_i(3\boldsymbol{w})\}.$$

Thus, if X is a direct sum of some line bundles in $\mathrm{Dom}^-(E)$ and is rigid, then $|X| \leqslant 4$.

Furthermore, we show the equality can be reached. Notice that

$$L_2(m\boldsymbol{w}) \in \mathrm{Dom}^{x_3}(F), \quad L_3(m\boldsymbol{w}) \in \mathrm{Dom}^{x_2}(F), \quad 0 \leqslant m \leqslant 2,$$
$$L_2(m\boldsymbol{w}) \in \mathrm{Dom}^{x_3}(F(\boldsymbol{w})), \quad L_3(m\boldsymbol{w}) \in \mathrm{Dom}^{x_2}(F(\boldsymbol{w})), \quad 1 \leqslant m \leqslant 3.$$

Then

$$\bigoplus_{i=2}^{3}(L_i \oplus L_i(2\boldsymbol{w})), \quad \bigoplus_{i=2}^{3}(L_i(\boldsymbol{w}) \oplus L_i(3\boldsymbol{w}))$$

are both rigid.

Note that $L_2 \oplus L_3(3\boldsymbol{w})$ is not rigid. Thus, if X is a direct sum of some line bundles in $\mathrm{Dom}^-(E)$ and is rigid with $|X| = 4$, then

$$X \in \left\{ \bigoplus_{i=2}^{3}(L_i \oplus L_i(2\boldsymbol{w})), \bigoplus_{i=2}^{3}(L_i(\boldsymbol{w}) \oplus L_i(3\boldsymbol{w})) \right\}.$$

Put

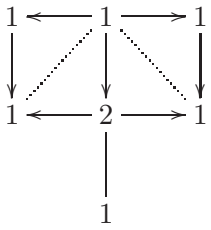
$$C_1 = \left\{ \bigoplus_{i=2}^{3}(L_i \oplus L_i(2\boldsymbol{w})) \oplus \mathscr{O}(3\boldsymbol{w}), \bigoplus_{i=2}^{3}(L_i(\boldsymbol{w}) \oplus L_i(3\boldsymbol{w})) \oplus \mathscr{O}(-2\boldsymbol{w}) \right\}, \quad (8)$$

where $L_i = \mathscr{O}(\boldsymbol{x}_1 - \boldsymbol{x}_i)$, $i = 2, 3$.

From the argument above, we can deduce the main result of this subsection.

Proposition 4.7 *Tilting bundle T consisting of E and 6 line bundles is of the form $E \oplus \mathscr{O}(k\boldsymbol{w}) \oplus T^*$, where $k \in \{0, 1\}$, $T^* \in C_1$. Conversely, each object of this form is a tilting bundle.*

The quiver of the endomorphism algebra of tilting bundles in this form is like



where we may choose any orientation to the unoriented edge.

Proof Let $T = E \oplus T'$ be the tilting bundle. Then by Lemma 4.3, we have

$$T' \in \operatorname{add}(\operatorname{Dom}(E) \cup \{\mathscr{O}(-2\boldsymbol{w})\} \cup \{\mathscr{O}(3\boldsymbol{w})\}).$$

Let $T' = T^+ \oplus T^-$, where $T^- \in \operatorname{add}(\operatorname{Dom}^-(E))$ and T^+ does not have a direct summand belonging to $\operatorname{Dom}^-(E)$. Then $|T^-| \leqslant 4$. Furthermore, T^+ is a direct sum of some of the line bundles

$$\mathscr{O}(3\boldsymbol{w}),\ \mathscr{O}(\boldsymbol{w}),\ \mathscr{O},\ \mathscr{O}(-2\boldsymbol{w}).$$

Then $|T^+| \leqslant 2$ since T^+ is rigid. Notice $|T| = 7$. Then $|T^+| = 2$ and $|T^-| = 4$. Hence,

$$T^- \in \left\{ \bigoplus_{i=2}^{3} (L_i \oplus L_i(2\boldsymbol{w})),\ \bigoplus_{i=2}^{3} (L_i(\boldsymbol{w}) \oplus L_i(3\boldsymbol{w})) \right\}$$

and

$$T^+ \in \{\mathscr{O}(k\boldsymbol{w}) \oplus \mathscr{O}(3\boldsymbol{w}), \mathscr{O}(k\boldsymbol{w}) \oplus \mathscr{O}(-2\boldsymbol{w}) \mid k = 0, 1\}.$$

Finally, note that $\mathscr{O}(-2\boldsymbol{w}) \oplus L_2(2\boldsymbol{w})$ and $\mathscr{O}(3\boldsymbol{w}) \oplus L_2(\boldsymbol{w})$ are not rigid, and the elements in C_1 are all rigid. Thus, if

$$T^+ = \mathscr{O}(k\boldsymbol{w}) \oplus \mathscr{O}(3\boldsymbol{w}),$$

then

$$T^- = \bigoplus_{i=2}^{3} (L_i \oplus L_i(2\boldsymbol{w})),$$

and if

$$T^+ = \mathscr{O}(k\boldsymbol{w}) \oplus \mathscr{O}(-2\boldsymbol{w}),$$

then

$$T^- = \bigoplus_{i=2}^{3}(L_i(\boldsymbol{w}) \oplus L_i(3\boldsymbol{w})),$$

where $k \in \{0, 1\}$. Therefore, T is of the desired form.

In order to prove that object of the given form is a tilting object, it suffices to show it is rigid. $\mathscr{O}(k\boldsymbol{w}) \in \mathrm{Dom}^+E$ shows that $E \oplus \mathscr{O}(k\boldsymbol{w})$ is rigid. Each indecomposable direct summand of T^* is contained in Dom^-E and this implies that $E \oplus T^*$ is rigid. And the rigidness of $\mathscr{O}(k\boldsymbol{w}) \oplus T^*$ is due to Lemma 4.3 (iii). □

4.4 Tilting bundle consisting of 2 Auslander bundles and 5 line bundles

Without loss of generality, we may assume that the 2 Auslander bundles are E and E^*. According to Lemma 4.3 (ii), the Auslander bundle E^* belongs to $\mathrm{Dom}(E)$, and hence, is of the form $E(k\boldsymbol{w} + \boldsymbol{x}_i)$ for $i \in \{2, 3\}$ and $k \in \{1, 2, 3\}$.

Lemma 4.8 *Let X be an indecomposable vector bundle, and let $X \oplus E$ be rigid. Take $i \in \{2, 3\}$. Then*

(i) *$X \oplus E(2\boldsymbol{w} + \boldsymbol{x}_i)$ is rigid if and only if*

$$X \in \mathrm{Dom}(E) \cap \mathrm{Dom}(E(2\boldsymbol{w} + \boldsymbol{x}_i));$$

(ii) *$X \oplus E(\boldsymbol{w} + \boldsymbol{x}_i)$ is rigid if and only if*

$$X \in (\mathrm{Dom}(E) \cap \mathrm{Dom}(E(\boldsymbol{w} + \boldsymbol{x}_i))) \cup \{\mathscr{O}(-2\boldsymbol{w})\} \cup \{\mathscr{O}(4\boldsymbol{w} + \boldsymbol{x}_i)\};$$

(iii) *$X \oplus E(3\boldsymbol{w} + \boldsymbol{x}_i)$ is rigid if and only if*

$$X \in (\mathrm{Dom}(E) \cap \mathrm{Dom}(E(3\boldsymbol{w} + \boldsymbol{x}_i))) \cup \{\mathscr{O}(3\boldsymbol{w})\} \cup \{\mathscr{O}(\boldsymbol{w} + \boldsymbol{x}_i)\}.$$

Proof We only prove (i), while the proofs of (ii) and (iii) are similar. Since the grading shift is an equivalence, we have

$$\mathrm{Dom}(E(\boldsymbol{x})) = \mathrm{Dom}(E)(\boldsymbol{x}).$$

The rigidness of $X \oplus E$ implies

$$X \in \mathrm{Dom}(E) \cup \{\mathscr{O}(3\boldsymbol{w})\} \cup \{\mathscr{O}(-2\boldsymbol{w})\},$$

and if $X \oplus E(2\boldsymbol{w} + \boldsymbol{x}_i)$ is rigid, then

$$X \in \mathrm{Dom}(E(2\boldsymbol{w} + \boldsymbol{x}_i)) \cup \{\mathscr{O}(5\boldsymbol{w} + \boldsymbol{x}_i)\} \cup \{\mathscr{O}(\boldsymbol{x}_i)\}.$$

The fact that

$$\mathscr{O}(3\boldsymbol{w}), \mathscr{O}(-2\boldsymbol{w}) \notin \mathrm{Dom}(E(2\boldsymbol{w} + \boldsymbol{x}_i)), \quad \mathscr{O}(5\boldsymbol{w} + \boldsymbol{x}_i), \mathscr{O}(\boldsymbol{x}_i) \notin \mathrm{Dom}(E),$$

shows the result. □

Now, we aim to determine a vector bundle X consisting of line bundles such that $X \oplus E \oplus E^*$ is a tilting bundle. Take $i \in \{2,3\}$.

By the definition of domain, line bundles in $\mathrm{Dom}(E) \cap \mathrm{Dom}(E(2\boldsymbol{w}+\boldsymbol{x}_i))$ are of the forms

$$\mathscr{O},\ \mathscr{O}(\boldsymbol{w}),\ \mathscr{O}(2\boldsymbol{w}+\boldsymbol{x}_i),\ \mathscr{O}(3\boldsymbol{w}+\boldsymbol{x}_i),\ \mathscr{O}(k\boldsymbol{w}+\boldsymbol{x}_j),$$

where $i \neq j \in \{2,3\}$ and $k \in \{1,2,3,4\}$. Hence, if X is a direct sum of some of these line bundles and is rigid, then $|X| \leqslant 4$. Therefore, E^* is not isomorphic to $E(2\boldsymbol{w}+\boldsymbol{x}_i)$ since $X \oplus E \oplus E^*$ is a tilting bundle.

Line bundles in $\mathrm{Dom}(E) \cap \mathrm{Dom}(E(\boldsymbol{w}+\boldsymbol{x}_i))$ are of the forms

$$\mathscr{O},\ \mathscr{O}(\boldsymbol{w}),\ \mathscr{O}(\boldsymbol{w}+\boldsymbol{x}_i),\ \mathscr{O}(2\boldsymbol{w}+\boldsymbol{x}_i),\ \mathscr{O}(k\boldsymbol{w}+\boldsymbol{x}_j),$$

where $i \neq j \in \{2,3\}$ and $k \in \{1,2,3\}$. Notice that if L is one of these line bundles except $\mathscr{O}(\boldsymbol{w}+\boldsymbol{x}_j)$ and $\mathscr{O}(3\boldsymbol{w}+\boldsymbol{x}_j)$, then $L \oplus \mathscr{O}(-2\boldsymbol{w}) \oplus \mathscr{O}(4\boldsymbol{w}+\boldsymbol{x}_i)$ is rigid. Furthermore,

$$\mathscr{O}, \mathscr{O}(\boldsymbol{w}) \in \mathrm{Dom}^{x_1}(F),$$

$$\mathscr{O}(\boldsymbol{w}+\boldsymbol{x}_i), \mathscr{O}(2\boldsymbol{w}+\boldsymbol{x}_i) \in \mathrm{Dom}^{x_i}(F)$$

and

$$\mathscr{O}(k\boldsymbol{w}+\boldsymbol{x}_j) \in \mathrm{Dom}^{x_j}(F), \quad k \in \{1,2,3\}.$$

By Lemma 4.4, if X is a direct sum of line bundles and $X \oplus E \oplus E(\boldsymbol{w}+\boldsymbol{x}_i)$ is rigid, then $|X| \leqslant 5$.

Denote by $C_2(1,i)$ the set

$$\{\mathscr{O}(m\boldsymbol{w}) \oplus \mathscr{O}(n\boldsymbol{w}+\boldsymbol{x}_i) \oplus \mathscr{O}(2\boldsymbol{w}+\boldsymbol{x}_j) \oplus \mathscr{O}(-2\boldsymbol{w}) \oplus \mathscr{O}(4\boldsymbol{w}+\boldsymbol{x}_i)\}, \quad (9)$$

where $0 \leqslant m \leqslant 1$, $1 \leqslant n \leqslant 2$, $i \neq j \in \{2,3\}$. It is then the set consisting of X such that $X \oplus E \oplus E(\boldsymbol{w}+\boldsymbol{x}_i)$ is rigid and $|X| = 5$.

Similarly, when $E^* = E(3\boldsymbol{w}+\boldsymbol{x}_i)$, we also obtain a set denoted by $C_2(3,i)$:

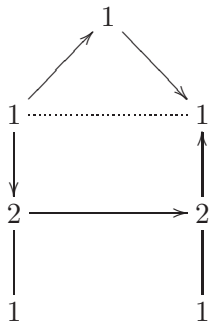
$$\{\mathscr{O}(m\boldsymbol{w}) \oplus \mathscr{O}(n\boldsymbol{w}+\boldsymbol{x}_i) \oplus \mathscr{O}(3\boldsymbol{w}+\boldsymbol{x}_j) \oplus \mathscr{O}(3\boldsymbol{w}) \oplus \mathscr{O}(\boldsymbol{w}+\boldsymbol{x}_i)\}, \quad (9')$$

where $0 \leqslant m \leqslant 1$, $3 \leqslant n \leqslant 4$, $i \neq j \in \{2,3\}$, such that if $X \in B_2(3,i)$, then $X \oplus E \oplus E(3\boldsymbol{w}+\boldsymbol{x}_i)$ is rigid and $|X| = 5$; the converse is also true.

By the discussion above, we immediately have the following result.

Proposition 4.9 *Tilting bundle which contains E as a direct summand and consists of 2 Auslander bundles and 5 line bundles is of the form $E \oplus E(k\boldsymbol{w}+\boldsymbol{x}_i) \oplus X$, where $X \in C_2(k,i)$ for $k \in \{1,3\}$ and $i \in \{2,3\}$. Conversely, vector bundle of this form is a tilting bundle.*

The quiver of the endomorphism algebra of tilting bundle in this case has the following form:



where the orientation of the two unoriented edges can be chosen freely.

4.5 Tilting bundle consisting of 3 Auslander bundles and 4 line bundles

We will first show the forms of the three Auslander bundles.

Lemma 4.10 *Let E_1, E_2, and E_3 be three Auslander bundles. If $E_1 \oplus E_2 \oplus E_3$ is rigid, then there exists $i \neq j$ such that $\mu(E_i) = \mu(E_j)$.*

Proof Since $E_1 \oplus E_2 \oplus E_3$ is rigid, $E_2, E_3 \in \mathrm{Dom}(E_1)$. Notice that the Auslander bundles in $\mathrm{Dom}(E_1)$ are of the forms

$$E_1(k\boldsymbol{w} + \boldsymbol{x}_i), \quad 1 \leqslant k \leqslant 3,\ i = 2, 3$$

If $\mu(E_2) = \mu(E_1)$, then we are done. Otherwise,

$$E_2 = E_1(k\boldsymbol{w} + \boldsymbol{x}_i), \quad k = 1, 3,\ i = 2, 3.$$

Assume $E_2 = E_1(\boldsymbol{w} + \boldsymbol{x}_2)$. If $\mu(E_2) = \mu(E_3)$, then we are done. Otherwise, $E_3 = E_1(3\boldsymbol{w} + \boldsymbol{x}_3)$. However, $E_3 \notin \mathrm{Dom}(E_2)$. Thus, $E_2 \oplus E_3$ is not rigid, which is a contradiction. □

Let

$$T = E \oplus E(\boldsymbol{x}) \oplus E(\boldsymbol{y}) \oplus M$$

be a tilting bundle. Without loss of generality, we may assume $\mu(E) = \mu(E(\boldsymbol{x}))$. For $\mu(E(\boldsymbol{x})) = \mu(E(\boldsymbol{y}))$, we will come to the first case by shifting under $-\boldsymbol{y}$. Thus,

$$T = E \oplus E(2\boldsymbol{w} + \boldsymbol{x}_i) \oplus E(k\boldsymbol{w} + \boldsymbol{x}_j) \oplus M, \quad 1 \leqslant k \leqslant 3,\ 2 \leqslant i \neq j \leqslant 3.$$

Lemma 4.11 *Let*

$$E(k) = E \oplus E(2\boldsymbol{w} + \boldsymbol{x}_i) \oplus E(k\boldsymbol{w} + \boldsymbol{x}_j),$$

and let L be a line bundle, where $2 \leqslant i \neq j \leqslant 3$. Then

(i) *$L \oplus E(1)$ is rigid if and only if*

$$L \in \mathrm{Dom}(E) \cap \mathrm{Dom}(E(2\boldsymbol{w} + \boldsymbol{x}_i)) \cap \mathrm{Dom}(E(\boldsymbol{w} + \boldsymbol{x}_j))$$

or $L = \mathscr{O}(4\boldsymbol{w} + \boldsymbol{x}_j)$;

(ii) $L \oplus E(2)$ *is rigid if and only if*

$$L \in \mathrm{Dom}(E) \cap \mathrm{Dom}(E(2\boldsymbol{w} + \boldsymbol{x}_2)) \cap \mathrm{Dom}(E(2\boldsymbol{w} + \boldsymbol{x}_3));$$

(iii) $L \oplus E(3)$ *is rigid if and only if*

$$L \in \mathrm{Dom}(E) \cap \mathrm{Dom}(E(2\boldsymbol{w} + \boldsymbol{x}_i)) \cap \mathrm{Dom}(E(3\boldsymbol{w} + \boldsymbol{x}_j))$$

or $L = \mathscr{O}(\boldsymbol{w} + \boldsymbol{x}_j)$.

Proof Almost the same as Lemma 4.8 by repeatedly using Lemma 4.3. □

Now, we are going to determine M consisting of some line bundles such that $M \oplus E(k)$ is rigid.

Line bundles in

$$\mathrm{Dom}(E) \cap \mathrm{Dom}(E(2\boldsymbol{w} + \boldsymbol{x}_i)) \cap \mathrm{Dom}(E(\boldsymbol{w} + \boldsymbol{x}_j))$$

are of the forms

$$\mathscr{O},\ \mathscr{O}(\boldsymbol{w}),\ \mathscr{O}(2\boldsymbol{w} + \boldsymbol{x}_i),\ \mathscr{O}(3\boldsymbol{w} + \boldsymbol{x}_i),\ \mathscr{O}(\boldsymbol{w} + \boldsymbol{x}_j),\ \mathscr{O}(2\boldsymbol{w} + \boldsymbol{x}_j).$$

Let L be one of these line bundles. Then $L \oplus \mathscr{O}(4\boldsymbol{w} + \boldsymbol{x}_j)$ is rigid. And notice further that

$$\begin{gathered}\mathscr{O}, \mathscr{O}(\boldsymbol{w}) \in \mathrm{Dom}^{x_1}(F),\\ \mathscr{O}(2\boldsymbol{w} + \boldsymbol{x}_i), \mathscr{O}(3\boldsymbol{w} + \boldsymbol{x}_i) \in \mathrm{Dom}^{x_i}(F),\\ \mathscr{O}(\boldsymbol{w} + \boldsymbol{x}_j), \mathscr{O}(2\boldsymbol{w} + \boldsymbol{x}_j) \in \mathrm{Dom}^{x_j}(F).\end{gathered}$$

Hence, if M is a direct sum of some line bundles and $M \oplus E(1)$ is rigid, then $|M| \leqslant 4$. Denote by $C_3(1, i)$ the set

$$\{\mathscr{O}(m\boldsymbol{w}) \oplus \mathscr{O}(n\boldsymbol{w} + \boldsymbol{x}_i) \oplus \mathscr{O}(l\boldsymbol{w} + \boldsymbol{x}_j) \oplus \mathscr{O}(4\boldsymbol{w} + \boldsymbol{x}_j) \mid m = 0, 1,\ n = 2, 3,\ l = 1, 2\}, \tag{10}$$

whose elements are just those M consisting of some line bundles such that $M \oplus E(1)$ is rigid and $|M| = 4$.

By a similar argument as above, if M is a direct sum of some line bundles such that $M \oplus E(2)$ is rigid, then $|M| \leqslant 3$. And hence, $M \oplus E(2)$ cannot be completed into a tilting bundle of the desired form of this subsection.

Finally, if M is a direct sum of some line bundles such that $M \oplus E(3)$ is rigid, then $|M| \leqslant 4$. Similar as above, we obtain a set consisting of such M with $|M| = 4$, which is denoted by $C_3(3, i)$:

$$\{\mathscr{O}(m\boldsymbol{w}) \oplus \mathscr{O}(n\boldsymbol{w} + \boldsymbol{x}_i) \oplus \mathscr{O}(l\boldsymbol{w} + \boldsymbol{x}_j) \oplus \mathscr{O}(\boldsymbol{w} + \boldsymbol{x}_j) \mid m = 0, 1,\ n = 2, 3,\ l = 3, 4\}. \tag{10$'$}$$

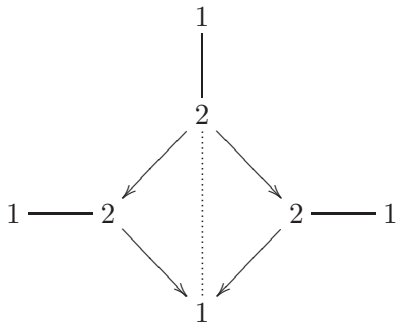
By the discussion and similar arguments as above, the following result is obvious.

Proposition 4.12 *A tilting bundle, which is direct sum of* 4 *line bundles and* 3 *Auslander bundles and contains* E *as a direct summand, is of the form*

$$E \oplus E(2\boldsymbol{w} + \boldsymbol{x}_i) \oplus E(k\boldsymbol{w} + \boldsymbol{x}_j) \oplus M,$$

where $M \in C_3(k, i)$, $k \in \{1, 3\}$ *and* $2 \leqslant i \neq j \leqslant 3$. *Conversely, object of this form is a tilting bundle.*

The quiver of the endomorphism algebra of tilting bundle in this form is like



where the orientation of the three unoriented edges could be chosen freely.

Now, we are able to summerize the above results into the main theorem of this paper.

Theorem 4.13 *A vector bundle* T *is a tilting bundle if and only if* T *is one of the following forms*:

(i) (*canonical*) $\oplus_{0 \leqslant \boldsymbol{x} \leqslant \boldsymbol{c}} L(\boldsymbol{x})$ *for some line bundle* L;

(ii) $F \oplus (\oplus_i T^{x_i})$, *where* $T^{x_i} \in \mathrm{Br}^i(F)$;

(iii) $E \oplus \mathscr{O}(k\boldsymbol{w}) \oplus T^*$, *where* $k \in \{0, 1\}$, $T^* \in C_1$;

(iv) $E \oplus E(k\boldsymbol{w} + \boldsymbol{x}_i) \oplus X$, *where* $X \in C_2(k, i)$ *for* $k \in \{1, 3\}$ *and* $i \in \{2, 3\}$;

(v) $E \oplus E(2\boldsymbol{w} + \boldsymbol{x}_i) \oplus E(k\boldsymbol{w} + \boldsymbol{x}_j) \oplus M$, *where* $M \in C_3(k, i)$, $k \in \{1, 3\}$ *and* $2 \leqslant i \neq j \leqslant 3$.

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