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RESEARCH ARTICLE

A fundamental representation of quantum generalized Kac-Moody algebras with one imaginary simple root

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Abstract We consider the Borcherds-Cartan matrix obtained from a symmetrizable generalized Cartan matrix by adding one imaginary simple root. We extend the result of Gebert and Teschner [Lett. Math. Phys., 1994, 31: 327–334] to the quantum case. Moreover, we give a connection between the irreducible dominant representations of quantum Kac-Moody algebras and those of quantum generalized Kac-Moody algebras. As the result, a large class of irreducible dominant representations of quantum generalized Kac-Moody algebras were obtained from representations of quantum Kac-Moody algebras through tensor algebras.

Keywords Quantum generalized Kac-Moody algebra, tensor algebra, fundamental representation
MSC 20G42, 81R10

1 Introduction

Quantum groups were introduced independently by Drinfeld [5] and Jimbo [8] in 1985, which are certain families of Hopf algebras that are deformations of universal enveloping algebras of Kac-Moody algebras. More precisely, let $\mathbf{U}(\mathfrak{g})$ be the universal enveloping algebra of a symmetrizable Kac-Moody algebra $\mathfrak{g}$. Then, to each generic parameter q, we can associate a Hopf algebra $\mathbf{U}_q(\mathfrak{g})$, called the quantum group, whose structure tends to that of $\mathbf{U}(\mathfrak{g})$ as q approaches 1. For the basic structure of $\mathbf{U}_q(\mathfrak{g})$, see [4,14]. Lusztig [13] showed that the integrable highest weight modules over $\mathbf{U}_q(\mathfrak{g})$ can be deformed to those over

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$\mathbf{U}(\mathfrak{g})$ in such a way that the dimensions of weight spaces are invariant under the deformation.

In another direction, Borcherds was led in his study of the Moonshine and the Monster group to consider a new class of infinite-dimensional Lie algebras, now called the *generalized Kac-Moody algebras* [2,3]. The structure and the representation theory of generalized Kac-Moody algebras are similar to those of Kac-Moody algebras, and many basic facts about Kac-Moody algebras can be extended to generalized Kac-Moody algebras. But there are some differences, too. For example, generalized Kac-Moody algebras may have *imaginary simple roots* with norms at most 0 whose multiplicity can be greater than 1. Also, they may have infinitely many simple roots. For more Details, please refer to [9–11].

Kang [12] constructed the quantum group $\mathbf{U}_q(\mathfrak{g})$ associated a generalized Kac-Moody algebra $\mathfrak{g}$. For simplicity, call $\mathbf{U}_q(\mathfrak{g})$ the *quantum generalized Kac-Moody algebra*. Moreover, he showed, for a generic q, Verma modules and irreducible highest weight modules over the universal enveloping algebra $\mathbf{U}(\mathfrak{g})$ of $\mathfrak{g}$ with dominant integral weights can be obtained by deforming from those over $\mathbf{U}_q(\mathfrak{g})$. These results were extended to generalized Kac-Moody superalgebras by Benkart et al. [1].

In this paper, we only consider a special class of Borcherds-Cartan matrices with only one imaginary simple root. As in [6], we prove one of fundamental representations over those quantum generalized Kac-Moody algebras is isomorphic to the tensor algebra of an irreducible representation over the corresponding quantum Kac-Moody algebras. Moreover, by this isomorphism, we further show that a large class of irreducible dominant representations of quantum generalized Kac-Moody algebras can be induced from representations of quantum Kac-Moody algebras through tensor algebras.

2 Preliminaries

Let I be a finite or countably infinite index set. A real square matrix $A = (a_{ij})_{i,j\in I}$ is called a *Borcherds-Cartan* matrix if it satisfies

(1) $a_{ii} = 2$ or $a_{ii} \leqslant 0$ for all $i \in I$,

(2) $a_{ij} \leqslant 0$ if $i \neq j$,

(3) $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$,

(4) $a_{ij} = 0$ if and only if $a_{ji} = 0$.

We say that an index i is real if $a_{ii} = 2$ and imaginary if $a_{ii} \leqslant 0$. We denote by

$$I^{\mathrm{re}} = \{i \in I \mid a_{ii} = 2\}, \quad I^{\mathrm{im}} = \{i \in I \mid a_{ii} \leqslant 0\},$$

the set of real indices and the set of imaginary indices, respectively.

In this paper, we assume that all the entries of A are integers and the diagonal entries are even. Furthermore, we assume that A is *symmetrizable*;

that is, there is a diagonal matrix

$$D = \operatorname{diag}(d_i \in \mathbb{Z}_{>0} \mid i \in I)$$

such that DA is symmetric. In addition, we assume all entries of D are coprime integers.

A *Borcherds-Cartan datum* $(A, P^\vee, P, \Pi^\vee, \Pi)$ consists of

(i) a Borcherds-Cartan matrix A,

(ii) a free abelian group $P^\vee$, called the dual weight lattice,

(iii) $\Pi^\vee = \{\alpha_i^\vee\}_{i\in I}$, the set of simple coroots,

(iv) $\Pi = \{\alpha_i\}_{i\in I} \subset P = \operatorname{Hom}_{\mathbb{Z}}(P^\vee, \mathbb{Z})$, the set of simple roots,

satisfying the following properties:

(a) $\alpha_i(\alpha_j^\vee) = a_{ij}$ for $i, j \in I$,

(b) Π is linearly independent,

(c) for any $i \in I$, there exists $\Lambda_i \in P$ such that $\Lambda_i(\alpha_j^\vee) = \delta_{ij}$ for $i, j \in I$.

The $\mathbb{Q}$-vector space $\mathfrak{h} = \mathbb{Q} \otimes_{\mathbb{Z}} P^\vee$ is called the *Cartan subalgebra* and we call Λ_i the *fundamental weights*. The weight lattice P is also defined to be

$$P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbb{Z}\} = \operatorname{Hom}_{\mathbb{Z}}(P^\vee, \mathbb{Z}).$$

Each symmetrizable Borcherds-Cartan matrix gives rise to a Borcherds-Cartan datum as follows. Take

$$P^\vee = \left(\bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee\right) \oplus \left(\bigoplus_{i\in I} \mathbb{Z}s_i\right),$$

and define α_i and Λ_i by

$$\alpha_i(\alpha_j^\vee) = a_{ji}, \quad \alpha_i(s_j) = \delta_{ij},$$

$$\Lambda_i(\alpha_j^\vee) = \delta_{ij}, \quad \Lambda_i(s_j) = 0.$$

We also define ω_i by

$$\omega_i(\alpha_j^\vee) = 0, \quad \omega_i(s_j) = \delta_{ij}.$$

Then

$$P = \left(\bigoplus_{i\in I} \mathbb{Z}\Lambda_i\right) \oplus \left(\bigoplus_{i\in I} \mathbb{Z}\omega_i\right).$$

We denote by P_+ the set $\{\lambda \in P \mid \lambda(\alpha_i^\vee) \geqslant 0 \text{ for every } i \in I\}$ of dominant integral weights. The free abelian group

$$R := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$$

is called the *root lattice*. We set

$$R_+ = \sum_{i\in I} \mathbb{Z}_{\geqslant 0}\alpha_i, \quad R_- = -R_+.$$

For $\alpha \in R_+$, we can write

$$\alpha = \sum_{k=1}^{r} \alpha_{i_k}, \quad i_1, i_2, \ldots, i_r \in I.$$

Let q be an indeterminate and set $q_i = q^{d_i}$ $(i \in I)$. For an integer $n \in \mathbb{Z}$, $i \in I^{\mathrm{re}}$, define

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]! = \prod_{k=1}^{n} [k], \quad \begin{bmatrix} m \\ n \end{bmatrix} = \frac{[m]!}{[n]![m-n]!}.$$

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i, \quad \begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[n]_i![m-n]_i!}.$$

Definition 1 The quantum generalized Kac-Moody algebra $\mathbf{U}_q(\mathfrak{g})$ associated with a Borcherds-Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi)$ is the associative algebra over $\mathbb{Q}(q)$ with unit 1 generated by E_i, F_i $(i \in I)$, and K_h $(h \in P^\vee)$ subject to the following defining relations:

$$K_h K_{h'} = K_{h+h'}, \quad K_h K_{-h} = 1, \quad h, h' \in P^\vee,$$

$$K_h E_i K_{-h} = q^{\alpha_i(h)} E_i, \quad K_h F_i K_{-h} = q^{-\alpha_i(h)} F_i,$$

$$E_i F_j - F_j E_i = \delta_{i,j} \frac{\widetilde{K}_i - \widetilde{K}_i^{-1}}{q_i - q_i^{-1}}, \quad \widetilde{K}_i = K_{d_i \alpha_i^\vee},$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i E_i^{1-a_{ij}-r} E_j E_i^r = 0, \quad a_{ii} = 2,\ i \neq j,$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i F_i^{1-a_{ij}-r} F_j F_i^r = 0, \quad a_{ii} = 2,\ i \neq j,$$

$$E_i E_j - E_j E_i = 0, \quad F_i F_j - F_j F_i = 0, \quad a_{ij} = 0.$$

The quantum generalized Kac-Moody algebra $\mathbf{U}_q(\mathfrak{g})$ has a *Hopf algebra* structure with the comultiplication Δ, the counit ε, and the antipode S defined by

$$\Delta(K_h) = K_h \otimes K_h, \quad \Delta(E_i) = E_i \otimes \widetilde{K}_i^{-1} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + \widetilde{K}_i \otimes F_i,$$

$$\varepsilon(K_h) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0,$$

$$S(K_h) = K_{-h}, \quad S(E_i) = -E_i \widetilde{K}_i, \quad S(F_i) = -\widetilde{K}_i^{-1} F_i,$$

for $h \in P^\vee$ and $i \in I$ ([1,7,12]).

Let $\mathbf{U}_q^+(\mathfrak{g})$ and $\mathbf{U}_q^-(\mathfrak{g})$ be the $\mathbb{Q}(q)$-subalgebras of $\mathbf{U}_q(\mathfrak{g})$ generated by the elements E_i and F_i, respectively, for $i \in I$, and let $\mathbf{U}_q^0(\mathfrak{g})$ be the $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}_q(\mathfrak{g})$ generated by K_h ($h \in P^\vee$), which is isomorphic to $\mathbb{Q}(q)[K_i^{\pm 1}, H_i^{\pm 1}]$. Here, we set

$$K_i^{\pm 1} = K_{\pm \alpha_i^\vee}, \quad H_i^{\pm 1} = K_{\pm s_i}.$$

Then we have the *triangular decomposition* ([1,7,12])

$$\mathbf{U}_q(\mathfrak{g}) \cong \mathbf{U}_q^-(\mathfrak{g}) \otimes \mathbf{U}_q^0(\mathfrak{g}) \otimes \mathbf{U}_q^+(\mathfrak{g}).$$

A $\mathbf{U}_q(\mathfrak{g})$-module V is called a *weight module* if it admits a weight space decomposition

$$V = \bigoplus_{\mu \in P} V_\mu,$$

where

$$V_\mu := \{v \in V \mid K_h v = q^{\mu(h)} v, \forall\, h \in P^\vee\}.$$

We call

$$\mathrm{wt}(V) := \{\mu \in P \mid V_\mu \neq 0\}$$

the set of weights of V.

Let $\mathscr{O}$ be the category consisting of weight modules V with finite-dimensional weight spaces such that

$$\mathrm{wt}(V) \subset \bigcup_{j=1}^{s} (\lambda_j - R_+)$$

for finitely many $\lambda_1, \lambda_2, \ldots, \lambda_s \in P$.

The most interesting examples of $\mathbf{U}_q(\mathfrak{g})$-modules are the highest weight modules in the category $\mathscr{O}$ defined below. A module V over $\mathbf{U}_q(\mathfrak{g})$ is called a *highest weight module with highest weight* $\lambda \in P$ if there exists a non-zero vector $v \in V$ (called a *highest weight vector*) such that

(i) $V = \mathbf{U}_q(\mathfrak{g})v$,

(ii) $v \in V_\lambda$,

(iii) $E_i v = 0$ for every $i \in I$.

Let $J(\lambda)$ denote the left ideal of $\mathbf{U}_q(\mathfrak{g})$ generated by $E_i, K_h - q^{\lambda(h)}$ ($i \in I$, $h \in P^\vee$), and set $M(\lambda) = \mathbf{U}_q(\mathfrak{g})/J(\lambda)$. Then, via left multiplication, $M(\lambda)$ becomes a $\mathbf{U}_q(\mathfrak{g})$-module, which is called the *Verma module*. The basic properties of Verma modules are summarized in the following propositions.

Proposition 1 [1,7,12] (a) *The Verma module $M(\lambda)$ is a highest weight $\mathbf{U}_q(\mathfrak{g})$-module with highest weight λ and highest weight vector $v_\lambda := 1 + J(\lambda)$.*

(b) *The Verma module $M(\lambda)$ is a free $\mathbf{U}_q^-(\mathfrak{g})$-module generated by v_λ.*

(c) *Every highest weight $\mathbf{U}_q(\mathfrak{g})$-module with highest weight λ is a quotient of $M(\lambda)$.*

(d) *The Verma module $M(\lambda)$ contains a unique maximal submodule $R(\lambda)$, the corresponding quotient $L(\lambda) = M(\lambda)/R(\lambda)$ is an irreducible highest weight $\mathbf{U}_q(\mathfrak{g})$-module with highest weight λ.*

Proposition 2 [1,7,12] *Let $\lambda \in P_+$ be a dominant integral weight.*

(a) *The highest weight vector v_λ of $L(\lambda)$ satisfies the following relations*:

$$\begin{cases} F_i^{\lambda(h)+1} v_\lambda = 0, & a_{ii} = 2, \\ F_i v_\lambda = 0, & \lambda(\alpha_i^\vee) = 0. \end{cases} \tag{2.1}$$

(b) *Conversely, let V be a highest weight $\mathbf{U}_q(\mathfrak{g})$-module with a highest weight $\lambda \in P_+$ and a highest weight vector v. If v satisfies the relations in* (2.1)*, then V is isomorphic to $L(\lambda)$.*

Proposition 3 [7] *Let μ be a weight of $L(\lambda)$ with $\lambda \in P_+$, and let $i \in I^{\mathrm{im}}$. Then*

(a) $\mu(\alpha_i^\vee) \in \mathbb{Z}_{\geqslant 0}$;

(b) *if* $\mu(\alpha_i^\vee) = 0$, *then* $L(\lambda)_{\mu-\alpha_i} = 0$;

(c) *if* $\mu(\alpha_i^\vee) = 0$, *then* $F_i(L(\lambda)_\mu) = 0$;

(d) *if* $\mu(\alpha_i^\vee) \leqslant -a_{ii}$, *then* $E_i(L(\lambda)_\mu) = 0$.

3 Main result

In this section, we consider the symmetric Borcherds-Cartan matrix of the following shape:

$$\widetilde{A} = (a_{ij}) = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & & & \\ \vdots & & A & \\ a_{n0} & & & \end{pmatrix}_{I \times I}$$

where $a_{00} \leqslant 0$, $I = \{0, 1, 2, \ldots, n\}$, and $A = (a_{ij})_{n \times n}$ is a generalized Cartan matrix.

Denote by $\mathbf{U}_q(A)$ (resp. $\mathbf{U}_q(\widetilde{A})$) the corresponding quantum Kac-Moody algebra (resp. quantum generalized Kac-Moody algebra). Take $P^\vee$ and $\widetilde{P}^\vee$ as the free abelian groups by

$$P^\vee = \Big(\bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee\Big) \oplus \Big(\bigoplus_{i=1}^n \mathbb{Z}s_i\Big) \subset \widetilde{P}^\vee = \Big(\bigoplus_{i=0}^n \mathbb{Z}\alpha_i^\vee\Big) \oplus \Big(\bigoplus_{i=0}^n \mathbb{Z}s_i\Big).$$

And let

$$P = \Big(\bigoplus_{i=1}^n \mathbb{Z}\omega_i\Big) \oplus \Big(\bigoplus_{i=1}^n \mathbb{Z}\Lambda_i\Big), \quad \widetilde{P} = \Big(\bigoplus_{i=0}^n \mathbb{Z}\widetilde{\omega}_i\Big) \oplus \Big(\bigoplus_{i=0}^n \mathbb{Z}\widetilde{\Lambda}_i\Big),$$

where $\omega_i, \Lambda_i, \widetilde{\omega}_i$, and $\widetilde{\Lambda}_i$ are defined as in Section 2. We may consider P as a subset of $\widetilde{P}$ by the map $\iota\colon \omega_i \to \widetilde{\omega}_i$, $\Lambda_i \to \widetilde{\Lambda}_i$. Hereafter, we will not distinguish

ω_i, Λ_i and $\widetilde{\omega}_i, \widetilde{\Lambda}_i$ for $1 \leqslant i \leqslant n$. We also write Λ_0, ω_0 as long as there is no peril of confusion.

For $\lambda \in P_+$, let $L(\lambda)$ be the irreducible highest weight $\mathbf{U}_q(A)$-module with highest weight λ and highest weight vector v_λ. For $\lambda \in \widetilde{P}_+$, let $\widetilde{L(\lambda)}$ be the irreducible highest weight $\mathbf{U}_q(\widetilde{A})$-module with highest weight λ and highest weight vector $\widetilde{v}_\lambda$.

Denote by $\mathrm{T}(L(\lambda))$ the tensor algebra over $L(\lambda)$, that is,

$$\mathrm{T}(L(\lambda)) := \bigoplus_{n=0}^{+\infty} \mathrm{T}^n(L(\lambda)),$$

where

$$\mathrm{T}^0(L(\lambda)) = \mathbb{Q}(q)1, \quad \mathrm{T}^n(L(\lambda)) = \underbrace{L(\lambda) \otimes L(\lambda) \otimes \cdots \otimes L(\lambda)}_{n}, \quad n \geqslant 1.$$

The following theorem is our main result.

Theorem 1 *Keep the notations above. Let $L(\lambda)$ be the irreducible highest weight $\mathbf{U}_q(A)$-module with highest weight λ defined by*

$$\lambda(\alpha_i^\vee) = -a_{0i}, \quad \lambda(s_i) = 0, \quad 1 \leqslant i \leqslant n,$$

and with highest-weight vector v_λ. Then the tensor algebra $\mathrm{T}(L(\lambda))$ over $L(\lambda)$ is a $\mathbf{U}_q(\widetilde{A})$-module isomorphic to the highest-weight module $\widetilde{L(\Lambda_0)}$ of $\mathbf{U}_q(\widetilde{A})$, where $\Lambda_0 \in \widetilde{P}_+$ is defined by $\Lambda_0(\alpha_i^\vee) = \delta_{0i}$, $0 \leqslant i \leqslant n$.

Proof For convention, in the following, the indices i, j, k run from 1 to n unless stated otherwise. The generators $E_i, F_i, K_i^{\pm 1}, H_i^{\pm 1}$ of $\mathbf{U}_q(A)$ act trivially on the 1-dimensional subspace $\mathbb{Q}(q)1$ and as highest weight representation on $L(\lambda)$. We extend those actions to the tensor algebra $\mathrm{T}(L(\lambda))$ by the comultiplication in the Hopf algebra $\mathbf{U}_q(A)$. First, the generators K_0, H_0 act diagonally as follows:

$$K_0(1) = q, \quad K_0(v_\lambda) = q^{1-a_{00}} v_\lambda, \quad H_0(1) = 1, \quad H_0(v_\lambda) = q^{-1} v_\lambda,$$

$$K_0(F_k \varphi) = q^{-a_{0k}} F_k K_0(\varphi), \quad H_0(F_k \varphi) = F_k H_0(\varphi), \quad \varphi \in L(\lambda),$$

$$\begin{array}{c} K_0(\Phi \otimes \Psi) = q^{-1} K_0(\Phi) \otimes K_0(\Psi), \\ H_0(\Phi \otimes \Psi) = H_0(\Phi) \otimes H_0(\Psi), \end{array} \quad \Phi, \Psi \in \mathrm{T}(L(\lambda)),$$

$$F_0(1) = v_\lambda, \quad F_0(\Psi) = v_\lambda \otimes \Psi, \quad \Psi \in \mathrm{T}(L(\lambda)),$$

$$E_0(1) = 0, \quad E_0(v_\lambda) = 1.$$

Since

$$\mathrm{T}^n(L(\lambda)) = \mathbf{U}_q^{\leqslant 0}(A)(v_\lambda \otimes \mathrm{T}^{n-1}(L(\lambda))), \quad \forall\, n \geqslant 1,$$

we can inductively define the action as follows: for $\Psi \in \mathrm{T}^n(L(\lambda))$, $n \geqslant 0$,

$$E_0(F_i(\Psi)) = F_i(E_0(\Psi)), \quad E_0(v_\lambda \otimes \Psi) = \frac{K_0 - K_0^{-1}}{q - q^{-1}}(\Psi) + v_\lambda \otimes E_0(\Psi).$$

First, we observe that the generators K_0 and H_0 are defined to act diagonally on the tensor algebra. So, for $h \in \widetilde{P}^\vee$, the generators K_h commute with each other. Second, all the relations involving the quantum Kac-Moody generators $E_i, F_i, K_i^{\pm 1}, H_i^{\pm 1}$ are valid by assumption. Hence, we need to check the following relations on $\mathrm{T}(L(\lambda))$:

$$E_0F_i - F_iE_0 = 0, \quad E_iF_0 - F_0E_i = 0, \quad E_0F_0 - F_0E_0 = \frac{K_0 - K_0^{-1}}{q - q^{-1}},$$
$$K_0E_0K_0^{-1} = q^{a_{00}}F_0, \quad K_0F_0K_0^{-1} = q^{-a_{00}}F_0,$$
$$K_iE_0K_i^{-1} = q^{a_{0i}}E_0, \quad K_iF_0K_i^{-1} = q^{-a_{0i}}F_0,$$
$$K_0E_iK_0^{-1} = q^{a_{0i}}E_i, \quad K_0F_iK_0^{-1} = q^{-a_{0i}}F_i,$$
$$H_0E_iH_0^{-1} = E_i, \quad H_0F_iH_0^{-1} = F_i,$$
$$H_0E_0H_0^{-1} = qE_0, \quad H_0F_0H_0^{-1} = q^{-1}F_0.$$

Take $\psi \in L(\lambda)$, $F_k \in \mathbf{U}_q^-(A)$ and $\Psi \in \mathrm{T}^n(L(\lambda))$, $\Phi \in \mathrm{T}^m(L(\lambda))$. Then the following first six relations can be checked immediately on the tensor algebra:

$$(E_0F_0 - F_0E_0)(\Psi) = \frac{K_0 - K_0^{-1}}{q - q^{-1}}(\Psi), \quad (E_0F_i - F_iE_0)(\Psi) = 0,$$
$$(E_iF_0 - F_0E_i)(\Psi) = E_i(v_\lambda \otimes \Psi) - v_\lambda \otimes E_i(\Psi) = 0,$$
$$K_iF_0K_i^{-1}(\Psi) = K_i(v_\lambda \otimes K_i^{-1}(\Psi)) = q^{-a_{0i}}F_0(\Psi),$$
$$K_0F_0K_0^{-1}(\Psi) = K_0(v_\lambda \otimes K_0^{-1}(\Psi)) = q^{-1}K_0v_\lambda \otimes \Psi = q^{-a_{00}}F_0(\Psi),$$
$$H_0F_0H_0^{-1}(\Psi) = H_0(v_\lambda \otimes H_0^{-1}(\Psi)) = H_0v_\lambda \otimes \Psi = q^{-1}F_0(\Psi).$$

Finally, we check the remaining eight relations on $\mathrm{T}(L(\lambda))$:

$$K_0F_iK_0^{-1}(1) = q^{-1}K_0(F_i1) = q^{-1-a_{0i}}F_iK_0(1) = q^{-a_{0i}}F_i(1),$$

$$K_0F_iK_0^{-1}(v_\lambda) = K_0F_i(q^{a_{00}-1}v_\lambda) = q^{a_{00}-a_{0i}-1}F_iK_0v_\lambda = q^{-a_{0i}}(F_iv_\lambda),$$

$$\begin{aligned} K_0F_iK_0^{-1}(F_k\psi) &= K_0F_iq^{a_{0k}}F_kK_0^{-1}\psi \\ &= q^{a_{0k}}K_0(F_iF_kK_0^{-1}\psi) \\ &= q^{a_{0k}-a_{0i}}F_iK_0(F_kK_0^{-1}\psi) \\ &= q^{a_{0k}-a_{0i}}F_i(K_0F_kK_0^{-1}\psi) \\ &= q^{-a_{0i}}F_i(F_k\psi) \quad \text{(by induction)}, \end{aligned}$$

$$\begin{aligned}
&K_0F_iK_0^{-1}(\Psi\otimes\Phi)\\
&\quad= qK_0\Delta(F_i)(K_0^{-1}\Psi\otimes K_0^{-1}\Phi)\\
&\quad= qK_0(F_iK_0^{-1}\Psi\otimes K_0^{-1}\Phi+K_iK_0^{-1}\Psi\otimes F_iK_0^{-1}\Phi)\\
&\quad= K_0F_iK_0^{-1}\Psi\otimes\Psi+K_0K_iK_0^{-1}\Psi\otimes K_0F_iK_0^{-1}\Phi\quad\text{(by induction)}\\
&\quad= q^{-a_{0i}}(F_i\Psi\otimes\Phi+K_i\Psi\otimes F_i\Phi)\\
&\quad= q^{-a_{0i}}F_i(\Psi\otimes\Phi).
\end{aligned}$$

Similarly, the equality $H_0F_iH_0^{-1}=F_i$ holds on $\mathrm{T}(L(\lambda))$.

$$K_0E_0K_0^{-1}(1)=0=q^{a_{00}}E_0(1),$$

$$K_0E_0K_0^{-1}(v_\lambda)=K_0E_0(q^{a_{00}-1}v_\lambda)=q^{a_{00}-1}K_0(1)=q^{a_{00}}E_0(v_\lambda),$$

$$\begin{aligned}
K_0E_0K_0^{-1}(F_k\psi)&=K_0E_0(q^{a_{0k}}F_kK_0^{-1}(\psi))\\
&=q^{a_{0k}}K_0(F_kE_0K_0^{-1}(\psi))\\
&=F_kK_0(E_0K_0^{-1}(\psi))\\
&=F_k(K_0E_0K_0^{-1}(\psi))\quad\text{(by induction)}\\
&=F_kq^{a_{00}}E_0(\psi)\\
&=q^{a_{00}}E_0(F_k\psi),
\end{aligned}$$

$$\begin{aligned}
&K_0E_0K_0^{-1}(v_\lambda\otimes\Psi)\\
&\quad= q^{a_{00}}K_0E_0(v_\lambda\otimes K_0^{-1}\Psi)\\
&\quad= q^{a_{00}}K_0\Big(\frac{K_0-K_0^{-1}}{q-q^{-1}}\,K_0^{-1}\Psi+v_\lambda\otimes E_0K_0^{-1}\Psi\Big)\\
&\quad= q^{a_{00}}\Big(\frac{K_0-K_0^{-1}}{q-q^{-1}}\,\Psi+q^{-1}K_0v_\lambda\otimes K_0E_0K_0^{-1}\Psi\Big)\quad\text{(by induction)}\\
&\quad= q^{a_{00}}\Big(\frac{K_0-K_0^{-1}}{q-q^{-1}}\,\Psi+v_\lambda\otimes E_0\Psi\Big)\\
&\quad= q^{a_{00}}E_0(v_\lambda\otimes\Psi).
\end{aligned}$$

From the proof of the equality of $K_0F_iK_0^{-1}=q^{-a_{0i}}F_i$ acting on $\mathrm{T}(L(\lambda))$, we can deduce that for any $\Psi\in\mathrm{T}^n(L(\lambda))$,

$$K_0F_i(\Psi)=q^{-a_{0i}}F_iK_0(\Psi),$$

$$\begin{aligned}
&K_0E_0K_0^{-1}(F_kv_\lambda\otimes\Psi)\\
&\quad= K_0E_0K_0^{-1}(F_k(v_\lambda\otimes\Psi)-K_kv_\lambda\otimes F_k\Psi)\\
&\quad= K_0E_0K_0^{-1}F_k(v_\lambda\otimes\Psi)-K_0E_0K_0^{-1}(K_kv_\lambda\otimes F_k\Psi)\\
&\quad= F_kK_0E_0K_0^{-1}(v_\lambda\otimes\Psi)-K_0E_0K_0^{-1}(K_kv_\lambda\otimes F_k\Psi)\\
&\quad= q^{a_{00}}F_kE_0(v_\lambda\otimes\Psi)-q^{a_{00}}E_0(K_kv_\lambda\otimes F_k\Psi)\quad\text{(by induction)}\\
&\quad= q^{a_{00}}E_0(F_kv_\lambda\otimes\Psi).
\end{aligned}$$

Similarly, the equality $H_0E_0H_0^{-1} = qE_0$ holds on $\mathrm{T}(L(\lambda))$.

$$K_iE_0K_i^{-1}(1) = 0 = q^{a_{0i}}E_0(1),$$

$$K_iE_0K_i^{-1}(v_\lambda) = K_iE_0(q^{a_{0i}}v_\lambda) = q^{a_{0i}}K_i(1) = q^{a_{0i}}E_0(v_\lambda),$$

$$\begin{aligned} K_iE_0K_i^{-1}(F_k\psi) &= K_iE_0(K_i^{-1}F_k\psi) \\ &= K_iE_0q^{a_{ki}}(F_kK_i^{-1}\psi) \\ &= q^{a_{ki}}K_iF_kE_0K_i^{-1}\psi \\ &= F_kK_iE_0K_i^{-1}\psi \\ &= F_kq^{a_{0i}}E_0\psi \quad \text{(by induction)} \\ &= q^{a_{0i}}E_0(F_k\psi), \end{aligned}$$

$$\begin{aligned} K_iE_0K_i^{-1}(v_\lambda\otimes\Psi) &= q^{a_{0i}}K_iE_0(v_\lambda\otimes K_i^{-1}\Psi) \\ &= q^{a_{0i}}K_i\Big(\frac{K_0-K_0^{-1}}{q-q^{-1}}K_i^{-1}\Psi + v_\lambda\otimes E_0K_i^{-1}\Psi\Big) \\ &= q^{a_{0i}}\Big(\frac{K_i-K_i^{-1}}{q-q^{-1}}\Psi + K_iv_\lambda\otimes K_iE_0K_i^{-1}\Psi\Big) \\ &= q^{a_{0i}}E_0(v_\lambda\otimes\Psi) \quad \text{(by induction)}, \end{aligned}$$

$$\begin{aligned} &K_iE_0K_i^{-1}(F_kv_\lambda\otimes\Psi) \\ &= K_iE_0K_i^{-1}(F_k(v_\lambda\otimes\Psi) - K_kv_\lambda\otimes F_k\Psi) \\ &= F_kK_iE_0K_i^{-1}(v_\lambda\otimes\Psi) - K_iE_0(K_i^{-1}K_kv_\lambda\otimes K_i^{-1}F_k\Psi) \\ &= q^{a_{0i}}E_0F_k(v_\lambda\otimes\Psi) - q^{a_{0i}-a_{0k}}K_iE_0(v_\lambda\otimes K_i^{-1}F_k\Psi) \\ &= q^{a_{0i}}E_0F_k(v_\lambda\otimes\Psi) - q^{a_{0i}-a_{0k}}K_i\Big(\frac{K_0-K_0^{-1}}{q-q^{-1}}K_i^{-1}F_k\Psi + v_\lambda\otimes E_0K_i^{-1}F_k\Psi\Big) \\ &= q^{a_{0i}}E_0F_k(v_\lambda\otimes\Psi) - q^{a_{0i}-a_{0k}}\Big(\frac{K_0-K_0^{-1}}{q-q^{-1}}F_k\Psi + K_iv_\lambda\otimes K_iE_0K_i^{-1}F_k\Psi\Big) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by induction)} \\ &= q^{a_{0i}}E_0F_k(v_\lambda\otimes\Psi) - q^{a_{0i}-a_{0k}}E_0(v_\lambda\otimes F_k\Psi) \\ &= q^{a_{0i}}E_0(F_kv_\lambda\otimes\Psi), \end{aligned}$$

$$K_0E_iK_0^{-1}(1) = 0 = q^{a_{0i}}E_i(1),$$

$$K_0E_iK_0^{-1}(v_\lambda) = K_0(E_iq^{a_{00}-1}v_\lambda) = 0 = q^{a_{0i}}E_iv_\lambda,$$

$$K_0E_iK_0^{-1}(F_k\psi) = K_0E_iq^{a_{0k}}F_kK_0^{-1}\psi = q^{a_{0k}}K_0E_iF_kK_0^{-1}(\psi).$$

If $k \neq i$, then

$$\begin{aligned} q^{a_{0k}}K_0E_iF_kK_0^{-1}(\psi) &= q^{a_{0k}}K_0F_kE_iK_0^{-1}\psi \\ &= q^{a_{0k}-a_{0k}}F_kK_0E_iK_0^{-1}\psi \\ &= q^{a_{0i}}E_i(F_k\psi). \end{aligned}$$

If $k = i$, then

$$\begin{aligned} q^{a_{0k}} K_0 E_i F_k K_0^{-1}(\psi) &= q^{a_{0k}} K_0 E_k F_k K_0^{-1}\psi \\ &= q^{a_{0k}} K_0 \Big(F_k E_k - \frac{K_k - K_k^{-1}}{q - q^{-1}} \Big) K_0^{-1}\psi \\ &= F_k K_0 E_k K_0^{-1}\psi - q^{a_{0k}} K_0 \frac{K_k - K_k^{-1}}{q - q^{-1}} K_0^{-1}\psi \\ &= q^{a_{0k}} \Big(F_k E_k - \frac{K_k - K_k^{-1}}{q - q^{-1}} \Big)(\psi) \\ &= q^{a_{0k}} E_k (F_k \psi), \end{aligned}$$

$$\begin{aligned} K_0 E_i K_0^{-1}(\Psi \otimes \Phi) &= q K_0 \Delta(E_i)(K_0^{-1}\Psi \otimes K_0^{-1}\Phi) \\ &= q K_0 (E_i K_0^{-1}\Psi \otimes K_i^{-1} K_0^{-1}\Phi + K_0^{-1}\Psi \otimes E_i K_0^{-1}\Phi) \\ &= K_0 E_i K_0^{-1}\Psi \otimes K_0 K_i^{-1} K_0^{-1}\Psi + K_0 K_0^{-1}\Psi \otimes K_0 E_i K_0^{-1}\Phi \\ &= q^{a_{0i}} (E_i \Psi \otimes K_i^{-1}\Phi + \Psi \otimes E_i \Phi) \\ &= q^{a_{0i}} E_i (\Psi \otimes \Phi) \quad \text{(by induction)}. \end{aligned}$$

Similarly, the equality $H_0 E_i H_0^{-1} = E_i$ also holds on $\mathrm{T}(L(\lambda))$.

Having proved that $T(L(\lambda))$ as a $\mathbf{U}_q(\widetilde{A})$-module, now we need to prove that $\mathrm{T}(L(\lambda))$ is indeed isomorphic to the irreducible hightest weight module $\widetilde{L(\Lambda_0)}$.

Suppose that $\widetilde{\mathfrak{g}}$ is the generalized Kac-Moody algebra associated with the Borcherds-Cartan matrix $\widetilde{A}$. Then we have the triangular decomposition

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}}_+.$$

Here, $\widetilde{\mathfrak{n}}_-$ is the subalgebra obtained from the quotient of the free algebra $\hat{\mathfrak{n}}_-$ generated by $f_0, f_1, \ldots, f_n$ by the ideal generated by $(adf_i)^{1-a_{ij}} f_j$ if $a_{ii} = 2$, $j \neq i$ and $f_i f_j - f_j f_i$ if $a_{ij} = 0$.

Denote the highest weight vector of $\widetilde{L(\Lambda_0)}$ by v_0. Define a linear map

$$\rho'' \colon \mathbf{U}_q(\hat{\mathfrak{n}}_-) v_0 \to \mathrm{T}(L(\lambda))$$

by

$$\rho''(F_{i_1} F_{i_2} \cdots F_{i_n} v_0) = F_{i_1} F_{i_2} \cdots F_{i_n}(1),$$

where $i_1, i_2, \ldots, i_n \in \{0, 1, \ldots, n\}$. To prove that ρ'' induces a well-defined $\mathbf{U}_q(\widetilde{A})$-module homomorphism

$$\rho' \colon \mathbf{U}_q(\widetilde{\mathfrak{n}}_-) v_0 \to \mathrm{T}(L(\lambda)),$$

we need to check the Serre relations are valid. That is, for $1 \leqslant i \leqslant n$, $0 \leqslant j \leqslant n$, $j \neq i$, we need to prove

$$\rho'' \Big(\sum_{n=0}^{b=1-a_{ij}} (-1)^n \begin{bmatrix} b \\ n \end{bmatrix} F_i^n F_j F_i^{b-n} v_0 \Big) = 0.$$

We only consider $j=0$. By definition, it suffices to show

$$\rho''\Big(\sum_{n=0}^{b}(-1)^n\begin{bmatrix} b\\ n\end{bmatrix}F_i^nF_0F_i^{b-n}v_0\Big)=0.$$

Since F_i $(1\leqslant i\leqslant n)$ acts trivially on 1, we have

$$\rho''\Big(\sum_{n=0}^{b}(-1)^n\begin{bmatrix} b\\ n\end{bmatrix}F_i^nF_0F_i^{b-n}v_0\Big)=(-1)^{1-a_{i0}}F_i^{1-a_{i0}}F_0(1).$$

By the action of F_0, we have

$$F_i^{1-a_{i0}}F_0(1)=F_i^{1-a_{i0}}(v_\lambda)=F_i^{1-a_{i0}}v_\lambda=F_i^{1+\lambda(\alpha_i^\vee)}v_\lambda=0.$$

So we obtain

$$\rho'\colon \mathbf{U}_q(\widetilde{\mathfrak{n}}_-)v_0\to \mathrm{T}(L(\lambda)).$$

Since $\widetilde{L(\Lambda_0)}$ is obtained from the Verma module $M(\Lambda_0)$ modulo the subspace generated by the primitive vectors $F_i^{\Lambda_0(\alpha_i)+1}v_0$ $(1\leqslant i\leqslant n)$, we have

$$\rho'(F_i^{\Lambda_0(\alpha_i^\vee)+1}v_0)=F_i^{\Lambda_0(\alpha_i^\vee)+1}(1)=0.$$

Finally, we obtain

$$\rho\colon \widetilde{L(\Lambda_0)}\to \mathrm{T}(L(\lambda)),$$

which is injective since $\widetilde{L(\Lambda_0)}$ is a simple $\mathbf{U}_q(\widetilde{A})$-module. Note that any element of $\mathrm{T}^n(L(\lambda))$ is of the form

$$u_1v_\lambda\otimes u_2v_\lambda\otimes\cdots\otimes u_nv_\lambda,$$

where

$$u_i=F_{i,1}F_{i,2}\cdots F_{i,t_i},\quad F_{i,1},F_{i,2},\dots,F_{i,t_i}\in\{F_0,F_1,\dots,F_n\}.$$

First, we take

$$\rho(F_{n,1}F_{n,2}\cdots F_{n,t_n}F_0v_0)=F_{n,1}F_{n,2}\cdots F_{n,t_n}v_\lambda=u_nv_\lambda.$$

To obtain

$$u_{n-1}v_\lambda\otimes u_nv_\lambda=F_{n-1,1}F_{n-1,2}\cdots F_{n-1,t_{n-1}}v_\lambda\otimes u_nv_\lambda,$$

we take

$$\begin{aligned}&\rho\big(F_{n-1,t_{n-1}}F_0F_{n,1}F_{n,2}\cdots F_{n,t_n}F_0-q^{\lambda(\alpha_{n-1,t_{n-1}}^\vee)}F_0F_{n,1}F_{n,2}\cdots F_{n,t_n}F_0\big)(v_0)\\ &\quad=F_{n-1,t_{n-1}}v_\lambda\otimes u_nv_\lambda.\end{aligned}$$

Inductively, we can find an element $u \in \widetilde{L(\Lambda_0)}$ such that

$$\rho(u) = u_1 v_\lambda \otimes \cdots \otimes u_n v_\lambda,$$

and ρ is surjective. This completes the proof. $\square$

Remark 1 Theorem 1 can be generalized to the symmetrizable Borcherds-Cartan matrices.

Taking

$$\widetilde{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix}_{I \times I},$$

we will show that certain irreducible representations of $\mathbf{U}_q(\widetilde{A})$ can be deduced from the irreducible representations of $\mathbf{U}_q(A)$.

Since

$$\mathbf{U}_q(\widetilde{A}) \cong \mathbf{U}_q(A) \otimes_{\mathbb{Q}(q)} [E_0, F_0],$$

we can view $\mathbf{U}_q(A)$ as a subalgebra of $\mathbf{U}_q(\widetilde{A})$. Given a $\mathbf{U}_q(A)$-module M, we write $\mathrm{Ind}_A^{\widetilde{A}} M$ (or $\mathrm{Ind}M$) for the induced module $\mathbf{U}_q(\widetilde{A}) \otimes_{\mathbf{U}_q(A)} M$. Thus, we may consider Ind as a functor from the category of $\mathbf{U}_q(A)$-modules to the category of $\mathbf{U}_q(\widetilde{A})$-modules. It is known that this functor is left adjoint to the restriction functor $\mathrm{Res}_A^{\widetilde{A}}$ (or Res) going in the other direction.

Corollary 1 *Keep the notations above. Then we have the following statements.*

(1) *For any $\mu \in P_+$, define an action of $\mathbf{U}_q(\widetilde{A})$ on $L(\mu)$ as follows*:

$$E_0(F_{i_1} F_{i_2} \cdots F_{i_r} v_\mu) = F_0(F_{i_1} F_{i_2} \cdots F_{i_r} v_\mu) = 0,$$
$$K_0(F_{i_1} F_{i_2} \cdots F_{i_r} v_\mu) = H_0(F_{i_1} F_{i_2} \cdots F_{i_r} v_\mu) = F_{i_1} F_{i_2} \cdots F_{i_r} v_\mu,$$

for any $F_{i_1}, F_{i_2}, \ldots, F_{i_r} \in \mathbf{U}_q^-(\widetilde{A})$. Then $L(\mu)$, as a $\mathbf{U}_q(\widetilde{A})$-module, is isomorphic to $\widetilde{L(\mu)}$.

(2) *For any $\mu \in P_+$, there is a unique $\mathbf{U}_q(\widetilde{A})$-module monomorphism*

$$\phi\colon \widetilde{L(\Lambda_0 + \mu)} \to \widetilde{L(\Lambda_0)} \otimes \widetilde{L(\mu)} \xrightarrow{\sim} \mathrm{T}(L(\lambda_0)) \otimes L(\mu),$$

where $\lambda_0 \in P_+$ satisfies $\lambda_0(\alpha_i^\vee) = 0$ $(1 \leqslant i \leqslant n)$ and $\mathrm{T}(L(\lambda_0))$ is the corresponding tensor algebra. Moreover, for any

$$u = F_{i_1} F_{i_2} \cdots F_{i_r} \widetilde{v}_{\Lambda_0 + \mu} \in \widetilde{L(\Lambda_0 + \mu)},$$

we have

$$\phi(u) = \sum_{(\delta_1, \delta_2, \ldots, \delta_r) \in \{0,1\}^r} K_{i_1}^{\delta_1} F_{i_1}^{1-\delta_1} K_{i_2}^{\delta_2} F_{i_2}^{1-\delta_2} \cdots K_{i_r}^{\delta_r} F_{i_r}^{1-\delta_r}(1) \otimes F_{i_1}^{\delta_1} F_{i_2}^{\delta_2} \cdots F_{i_r}^{\delta_r} v_\mu.$$

(3) *For $\lambda \in P_+$, $a_0 \in \mathbb{Z}_{\geqslant 0}$, $b_0 \in \mathbb{Z}$, we have*

$$\operatorname{Res} L(\lambda + \widetilde{a_0\Lambda_0} + b_0\omega_0) \cong \begin{cases} L(\lambda), & a_0 = 0, \\ \bigoplus_{0}^{+\infty} L(\lambda), & a_0 \in \mathbb{Z}_{\geqslant 1}. \end{cases}$$

Proof (1) Since all the other relations can be checked trivially on $L(\lambda)$, we only need to check the following two relations on $L(\mu)$:

$$K_0 E_i K_0^{-1} = E_i, \quad K_0 F_i K_0^{-1} = F_i.$$

It is easy to check that this indeed defines a $\mathbf{U}_q(\widetilde{A})$-module structure on $L(\mu)$.

Since

$$K_0 v_\mu = v_\mu = q^{\mu(\alpha_0^\vee)} v_\mu, \quad E_i v_\mu = 0, \ i \in I,$$

there is a unique $\mathbf{U}_q(\widetilde{A})$-module homomorphism

$$\theta\colon \widetilde{L(\mu)} \to L(\mu), \qquad \widetilde{v}_\mu \mapsto v_\mu.$$

Obviously, it is surjective. Since $\widetilde{L(\mu)}$ is irreducible, it is also injective. Thus, θ is an isomorphism.

(2) For any $h \in \widetilde{P}^\vee$ and $i \in I$, we have

$$K_h(\widetilde{v}_{\Lambda_0} \otimes \widetilde{v}_\mu) = q^{(\Lambda_0+\mu)(h)} \widetilde{v}_{\Lambda_0} \otimes \widetilde{v}_\mu, \quad E_i(\widetilde{v}_{\Lambda_0} \otimes \widetilde{v}_\mu) = 0.$$

There is a unique $\mathbf{U}_q(\widetilde{A})$-module homomorphism

$$\pi\colon L(\widetilde{\Lambda_0 + \mu}) \to \widetilde{L(\Lambda_0)} \otimes \widetilde{L(\mu)}, \qquad \widetilde{v}_{\Lambda_0+\mu} \mapsto \widetilde{v}_{\Lambda_0} \otimes \widetilde{v}_\mu.$$

Take

$$\phi = \rho \circ \pi\colon L(\widetilde{\Lambda_0 + \mu}) \to \mathrm{T}(L(\lambda_0)) \otimes L(\mu).$$

Then the result follows.

(3) Denote by $v_{\widetilde{\lambda}}$ the highest weight vector of $L(\lambda + \widetilde{a_0\Lambda_0} + b_0\omega_0)$. Then

$$L(\lambda + \widetilde{a_0\Lambda_0} + b_0\omega_0) = \mathbf{U}_q(\widetilde{A}) v_{\widetilde{\lambda}}.$$

Note that all the elements of $L(\lambda + \widetilde{a_0\Lambda_0} + b_0\omega_0)$ are of the form $F_{i_1}F_{i_2}\cdots F_{i_r}v_{\widetilde{\lambda}}$, where $i_1, i_2, \ldots, i_r \in \{0, 1, 2, \ldots, n\}$. Since $a_{0i} = 0$, we have

$$F_{i_1}F_{i_2}\cdots F_{i_r}v_{\widetilde{\lambda}} = F_{j_1}F_{j_2}\cdots F_{j_t}F_0^m v_{\widetilde{\lambda}}$$

for some $m \in \mathbb{Z}_{\geqslant 0}$, $j_1, j_2, \dots, j_t \in \{1, 2, \dots, n\}$. If

$$a_0 = 0, \quad (\lambda + a_0\Lambda_0 + b_0\omega_0)(\alpha_0^\vee) = 0,$$

then we have

$$F_0 v_{\widetilde{\lambda}} = 0.$$

In this case, we have $v_{\widetilde{\lambda}} \in (\mathrm{Res}L(\lambda + \widetilde{a_0\Lambda_0} + b_0\omega_0))_\lambda$ and all the non-zero elements of $L(\lambda + \widetilde{a_0\Lambda_0} + b_0\omega_0)$ must be of the form $F_{i_1}F_{i_2}\cdots F_{i_r}v_{\widetilde{\lambda}}$, where $i_1, i_2, \dots, i_r \in \{1, 2, \dots, n\}$. So we have a non-zero map

$$\begin{aligned} \theta\colon M(\lambda) &\to \mathrm{Res}L(\lambda + \widetilde{a_0\Lambda_0} + b_0\omega_0), \\ v_\lambda &\mapsto v_{\widetilde{\lambda}}. \end{aligned}$$

It is easy to show that θ is surjective as $\mathbf{U}_q(A)$-modules and

$$L(\lambda) \cong M(\lambda)/\ker\theta \cong \mathrm{Res}L(\lambda + \widetilde{a_0\Lambda_0} + b_0\omega_0).$$

If

$$a_0 \neq 0, \quad (\lambda + a_0\Lambda_0 + b_0\omega_0)(\alpha_0^\vee) = a_0,$$

then the non-empty set $\{v_{\widetilde{\lambda}}, F_0 v_{\widetilde{\lambda}}, F_0^2 v_{\widetilde{\lambda}}, \dots\}$ is infinite and

$$(\mathrm{Res}L(\lambda + \widetilde{a_0\Lambda_0} + b_0\omega_0))_\lambda = \{v_{\widetilde{\lambda}}, F_0 v_{\widetilde{\lambda}}, F_0^2 v_{\widetilde{\lambda}}, \dots\}.$$

There is a non-zero map ψ defined by

$$\begin{aligned} \psi\colon \ \oplus_0^{+\infty} M(\lambda) &\to \mathrm{Res}L(\lambda + \widetilde{a_0\Lambda_0} + b_0\omega_0), \\ (v_\lambda, v_\lambda, \dots) &\mapsto v_{\widetilde{\lambda}} + F_0 v_{\widetilde{\lambda}} + F_0^2 v_{\widetilde{\lambda}} + \cdots . \end{aligned}$$

It is easy to check that ψ is surjective as $\mathbf{U}_q(A)$-modules and

$$\oplus_0^{+\infty} M(\lambda)/\ker\psi \cong \oplus_0^{+\infty} L(\lambda) \cong \mathrm{Res}L(\lambda + \widetilde{a_0\Lambda_0} + b_0\omega_0). \qquad \square$$

Remark 2 Corollary 1 (1) can be generalized to those $\widetilde{A}$ obtained from A by adding several imaginary simple roots with all the added entries being zero.

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