Front. Math. China 2015, 10(5): 1179–1202
DOI 10.1007/s11464-015-0468-9

RESEARCH ARTICLE

Color cyclic homology and Steinberg Lie color algebras

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Abstract The present paper contains two interrelated developments. First, the basic properties of the construction theory over the Steinberg Lie color algebras are developed in analogy with Steinberg Lie algebra case. This is done on the example of the central closed of the Steinberg Lie color algebras. The second development is that we define the first ε-cyclic homology group $HC_1(R,\varepsilon)$ of the Γ-graded associative algebra R (which could be seemed as the generalization of cyclic homology group and the $\mathbb{Z}/2\mathbb{Z}$-graded version of cyclic homology that was introduced by Kassel) to calculate the universal central extension of Steinberg Lie color algebras.

Keywords Steinberg Lie color algebra, ε-cyclic homology group, universal central extension
MSC 17B10, 17B35, 17B70

1 Introduction

Lie color algebras, originally introduced by Rittenberg and Wylerin [10], can be seen as a direct generalization of Lie (super) algebras. Indeed, the latter are defined through antisymmetric (commutator) or symmetric (anticommutator) products, although for the former the product is neither symmetric nor anti-symmetric and is defined by means of a commutation factor. In recent years, Lie color algebras have become an interesting subject of mathematics and physics. We assume that k is an algebraically closed field with characteristic zero. Let Γ be an additive group, and let a skew-symmetric bicharacter of Γ be a map $\varepsilon\colon \Gamma\times\Gamma\to k^*$ such that

Received March 7, 2015; accepted March 18, 2015
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$$\varepsilon(\lambda,\mu)=\varepsilon(\mu,\lambda)^{-1},\quad \varepsilon(\lambda,\mu+\nu)=\varepsilon(\lambda,\mu)\varepsilon(\lambda,\nu),\quad \forall\,\lambda,\mu,\nu\in\Gamma.$$

Let $R=\oplus_{\gamma\in\Gamma}R_\gamma$ be a Γ-graded k-vector space. A k-bilinear map

$$[\cdot,\cdot]\colon R\times R\to R$$

is called an ε-Lie color bracket on R if

$$[a,b]=-\varepsilon(\alpha,\beta)[b,a]\quad \text{(Skew symmetry)},$$
$$[a,[b,c]]=[[a,b],c]+\varepsilon(\alpha,\beta)[b,[a,c]]\quad \text{(Jacobi identity)}$$

for all homogeneous elements $a\in R_\alpha$, $b\in R_\beta$, and $c\in R$. The algebra structure $(R,[\cdot,\cdot])$ is called a (Γ,ε)-Lie color algebra or simply a Lie color algebra. If $\Gamma=0$ and $\varepsilon=1$, then a (Γ,ε)-Lie color algebra is a Lie algebra. If $\Gamma=\mathbb{Z}/2\mathbb{Z}$ and

$$\varepsilon(\bar{i},\bar{j})=(-1)^{ij},\quad \forall\,\bar{i},\bar{j}\in\mathbb{Z}/2\mathbb{Z},$$

then a (Γ,ε)-Lie color algebra is a Lie superalgebra.

Let R be a Γ-graded associative algebra, and fix a skew-symmetric bicharacter ε. Then R becomes a (Γ,ε)-Lie color algebra, which we denote by $R^{\varepsilon,-}$. In fact, the bilinear bracket on R is defined by

$$[a,b]=ab-\varepsilon(\alpha,\beta)ba$$

for the homogeneous elements $a\in R_\alpha$ and $b\in R_\beta$. In particular, let $M_n(R)$ be the $n\times n$ matrix with coefficients in R, and let

$$\deg(E_{ij}(a))=\deg(a)$$

for any homogeneous element $a\in R$. Under the Lie color bracket,

$$[E_{ij}(a),E_{kl}(b)]=\delta_{jk}E_{il}(ab)-\delta_{li}\varepsilon(\alpha,\beta)E_{kj}(ba),$$

where E_{ij}, E_{kl} are standard matrix units for $1\leqslant i,j,k,l\leqslant n$ and $a\in R_\alpha$, $b\in R_\beta$, becomes a Lie color algebra, denoted by $\mathfrak{gl}_n(R,\varepsilon)$. The elementary (Γ,ε)-Lie color algebra $\mathfrak{sl}_n(R,\varepsilon)$ is the subalgebra of $\mathfrak{gl}_n(R,\varepsilon)$ generated by the elements $E_{ij}(a)$, $a\in R$, $1\leqslant i\neq j\leqslant n$.

This article is devoted to the study of the basic properties of Steinberg Lie color algebras and the universal central extensions of Lie color algebras. Our present work is motivated by the results and methods in the Lie algebra case [2–4,12], but central extensions of Steinberg Lie algebra, Steinberg Lie superalgebra, and matrix Lie superalgebras over $\mathbb{Z}/2\mathbb{Z}$-graded algebras could be seem as special cases of this article. To study the universal central extension of $\mathfrak{sl}_n(R,\varepsilon)$ in the category of (Γ,ε)-Lie color algebras, we define Steinberg Lie color algebra $\mathfrak{st}_n(R,\varepsilon)$ by generators corresponding to $E_{ij}(a)$ and those same canonical relations in analogy with the Lie algebra case. So one has a Lie color algebra homomorphism ϕ from $\mathfrak{st}_n(R,\varepsilon)$ to $\mathfrak{sl}_n(R,\varepsilon)$. It is known that ϕ yields a central extension. When $n \geqslant 5$, they are the universal central extensions; moreover, the kernel of ϕ can be identified with the first ε-cyclic homology group $HC_1(R,\varepsilon)$ of the Γ-graded associative algebra R. When $n=3$ or $n=4$, $\mathfrak{st}_n(R,\varepsilon)$ is not the universal central extension of $\mathfrak{sl}_n(R,\varepsilon)$. We also construct the universal central extension of the latter two cases.

2 Basics on $\mathfrak{sl}_n(R,\varepsilon)$ and $\mathfrak{st}_n(R,\varepsilon)$

For convenience, we denote all the homogeneous elements of R by $S(R)$. If $a,b\in S(R)$, we simply write

$$\varepsilon(a,b)=\varepsilon(\alpha,\beta),\quad a\in R_\alpha,\, b\in R_\beta.$$

Let $R=\oplus_{\gamma\in\Gamma}R_\gamma$ be a Γ-graded associative algebra over a commutative ring K with identity $1\in R_0$. We have the (Γ,ε)-Lie color algebra $\mathfrak{gl}_n(R,\varepsilon)$ of $n\times n$ matrices with coefficients in R (see Section 1). For $n\geqslant 3$, the elementary (Γ,ε)-Lie color algebra $\mathfrak{sl}_n(R,\varepsilon)$ is a subalgebra of $\mathfrak{gl}_n(R,\varepsilon)$, which have the following three equivalent definitions.

Lemma 1 *For (Γ,ε)-Lie color algebra $\mathfrak{sl}_n(R,\varepsilon)$, the following statements are equivalent*:

(i) $\mathfrak{sl}_n(R,\varepsilon)$ *is the subalgebra of* $\mathfrak{gl}_n(R,\varepsilon)$ *generated by all the elements* $E_{ij}(a)$, $a\in S(R)$, $1\leqslant i\neq j\leqslant n$;

(ii) $\mathfrak{sl}_n(R,\varepsilon)=[\mathfrak{gl}_n(R,\varepsilon),\mathfrak{gl}_n(R,\varepsilon)]$;

(iii) $\mathfrak{sl}_n(R,\varepsilon)=\{X\in\mathfrak{gl}_n(R,\varepsilon)\mid \mathrm{tr}X\in[R,R]\}$.

Proof (i) $\Rightarrow$ (ii). Since $1\in R_0$ and $n\geqslant 3$, we can take distinct $1\leqslant i,j,k\leqslant n$ such that

$$E_{ij}(a)=[E_{ik}(a),E_{kj}(1)],\quad \forall\, a\in S(R),\, 1\leqslant i\neq j\leqslant n.$$

Obviously, $[\mathfrak{gl}_n(R,\varepsilon),\mathfrak{gl}_n(R,\varepsilon)]$ is a subalgebra of $\mathfrak{gl}_n(R,\varepsilon)$. Thus, the subalgebra of $\mathfrak{gl}_n(R,\varepsilon)$ generated by $E_{ij}(a)$ is contained in $[\mathfrak{gl}_n(R,\varepsilon),\mathfrak{gl}_n(R,\varepsilon)]$.

(ii) $\Rightarrow$ (iii). Recall the bracket product of $\mathfrak{gl}_n(R,\varepsilon)$, we have the trace of any element of $[\mathfrak{gl}_n(R,\varepsilon),\mathfrak{gl}_n(R,\varepsilon)]$ belongs to $[R,R]$.

(iii) $\Rightarrow$ (i). Denote by L' the subalgebra generated by $E_{ij}(a)$, $a\in S(R)$. First, $E_{ij}(R)\subseteq L'$ for $1\leqslant i\neq j\leqslant n$. Since ε is a bicharacter, $\varepsilon(\alpha,0)=1$ for all $\alpha\in\Gamma$. Thus,

$$E_{ii}(a)-E_{jj}(a)=[E_{ij}(a),E_{ji}(1)]\in L',\quad \forall\, a\in R,\, 1\leqslant i\neq j\leqslant n.$$

Finally, if $a,b\in S(R)$, then

$$E_{11}(ab-\varepsilon(a,b)ba)=[E_{1i}(a),E_{i1}(b)]-\varepsilon(a,b)[E_{1i}(ba),E_{i1}(1)],\quad i\neq 1.$$

We have

$$E_{11}([R,R])\subseteq L'.$$

Let

$$X = \sum_{1 \leqslant i,j \leqslant n} E_{ij}(a_{ij}), \quad \sum_{i=1}^{n} a_{ii} \in [R, R].$$

Then

$$\begin{aligned} X &= \sum_{i=1}^{n} E_{ii}(a_{ii}) + \sum_{1 \leqslant i \neq j \leqslant n} E_{ij}(a_{ij}) \\ &= E_{11}\left(\sum_{i=1}^{n} a_{ii}\right) - \sum_{i=2}^{n} (E_{11}(a_{ii}) - E_{ii}(a_{ii})) + \sum_{1 \leqslant i \neq j \leqslant n} E_{ij}(a_{ij}). \end{aligned}$$

Therefore, $X \in L'$. We complete the proof. □

Also, for any $a, b \in R$, we have

$$[E_{ij}(a), E_{jk}(b)] = E_{ik}(ab), \quad i \neq j \neq k \neq i,$$
$$[E_{ij}(a), E_{kl}(b)] = 0, \quad j \neq k,\ i \neq l.$$

Clearly, $\mathfrak{sl}_n(R, \varepsilon)$ has a vector space decomposition

$$\mathfrak{sl}_n(R, \varepsilon) = \xi_0 \bigoplus_{1 \leqslant i \neq j \leqslant n} \xi_{ij}, \tag{1}$$

where

$$\xi_{ij} = \{E_{ij}(a) \mid a \in R\}, \quad 1 \leqslant i \neq j \leqslant n,$$

and ξ_0 is the subalgebra of diagonal matrix whose trace belongs to $[R, R]$.

Definition 2 For $n \geqslant 3$, the Steinberg Lie color algebra $\mathfrak{st}_n(R, \varepsilon)$ is defined to be the (Γ, ε)-Lie color algebra over k generated by the homogeneous elements $X_{ij}(a)$ with $\deg(X_{ij}(a)) = \deg a$ for any $a \in S(R)$, $1 \leqslant i \neq j \leqslant n$, subject to the Γ-homogeneous relations:

$$a \mapsto X_{ij}(a) \text{ is a } k\text{-linear map}, \tag{2}$$

$$[X_{ij}(a), X_{jk}(b)] = X_{ik}(ab), \quad i, j, k \text{ distinct}, \tag{3}$$

$$[X_{ij}(a), X_{kl}(b)] = 0, \quad j \neq k,\ i \neq l, \tag{4}$$

where $a, b \in S(R)$, $1 \leqslant i, j, k, l \leqslant n$.

Both the (Γ, ε)-Lie color algebras $\mathfrak{sl}_n(R, \varepsilon)$ and $\mathfrak{st}_n(R, \varepsilon)$ are perfect. There exists a surjective (Γ, ε)-Lie color algebra epimorphism

$$\phi\colon \mathfrak{st}_n(R, \varepsilon) \to \mathfrak{sl}_n(R, \varepsilon)$$

such that

$$\phi(X_{ij}(a)) = E_{ij}(a).$$

Set

$$T_{ij}(a,b) = [X_{ij}(a), X_{ji}(b)], \quad a,b \in S(R),\ 1 \leqslant i \neq j \leqslant n.$$

Then $T_{ij}(a,b)$ is a homogeneous element with degree of $\deg a + \deg b$. Moreover, we have the following lemma.

Lemma 3 *For any $a,b,c \in S(R)$ and distinct i,j,k,l, we have*

$$T_{ij}(a,b) = -\varepsilon(a,b)T_{ji}(b,a), \tag{5}$$

$$[T_{ij}(a,b), X_{kl}(c)] = 0, \tag{6}$$

$$[T_{ij}(a,b), X_{ik}(c)] = X_{ik}(abc), \quad [T_{ij}(a,b), X_{ki}(c)] = -\varepsilon(a+b,c)X_{ki}(cab), \tag{7}$$

$$\begin{aligned} &[T_{ij}(a,b), X_{jk}(c)] = -\varepsilon(a,b)X_{jk}(bac), \\ &[T_{ij}(a,b), X_{kj}(c)] = \varepsilon(a,b)\varepsilon(a+b,c)X_{kj}(cba), \end{aligned} \tag{8}$$

$$[T_{ij}(a,b), X_{ij}(c)] = X_{ij}(abc + \varepsilon(a,b)\varepsilon(a+b,c)cba), \tag{9}$$

$$T_{ij}(ab,c) = T_{ik}(a,bc) - \varepsilon(a+b,c)T_{jk}(ca,b), \tag{10}$$

$$T_{kj}(c,1) = -T_{jk}(c,1) = T_{kj}(1,c). \tag{11}$$

Proof By the ε-skew symmetry, one has

$$T_{ij}(a,b) = -\varepsilon(a,b)[X_{ji}(b), X_{ij}(a)] = -\varepsilon(a,b)T_{ji}(b,a).$$

From the ε-Jacobi identity, we have

$$[x,[y,z]] = [[x,y],z] + \varepsilon(x,y)[y,[x,z]]$$

for the homogeneous elements x,y,z in the (Γ,ε)-Lie color algebra.

Therefore, (6) is obvious, and

$$[T_{ij}(a,b), X_{ik}(c)] = [[X_{ij}(a), X_{ji}(b)], X_{ik}(c)] = X_{ik}(abc),$$

$$\begin{aligned} [T_{ij}(a,b), X_{ki}(c)] &= \varepsilon(b,c)[[X_{ij}(a), X_{ki}(c)], X_{ji}(b)] \\ &= -\varepsilon(b,c)\varepsilon(a,c)[[X_{ki}(c), X_{ij}(a)], X_{ji}(b)] \\ &= -\varepsilon(a+b,c)X_{ki}(cab), \end{aligned}$$

which gives (7).

(8) also comes from the ε-Jacobi identity.

For (9), we have

$$\begin{aligned} [T_{ij}(a,b), X_{ij}(c)] &= [T_{ij}(a,b), [X_{ik}(c), X_{kj}(1)]] \\ &= [[T_{ij}(a,b), X_{ik}(c)], X_{kj}(1)] - [[T_{ij}(a,b), X_{kj}(1)], X_{ik}(c)] \\ &= [X_{ik}(abc), X_{kj}(1)] - \varepsilon(a,b)[X_{kj}(ba), X_{ik}(c)] \\ &= X_{ij}(abc + \varepsilon(a,b)\varepsilon(a+b,c)cba). \end{aligned}$$

Using ε-Jacobi identity and (3), we have

$$\begin{aligned} T_{ij}(ab,c) &= [[X_{ik}(a), X_{kj}(b)], X_{ji}(c)] \\ &= [X_{ik}(a), [X_{kj}(b), X_{ji}(c)]] + \varepsilon(b,c)[[X_{ik}(a), X_{ji}(c)], X_{kj}(b)] \\ &= [X_{ik}(a), X_{ki}(bc)] - \varepsilon(a+b,c)[X_{jk}(ca), X_{kj}(b)] \\ &= T_{ik}(a,bc) - \varepsilon(a+b,c)T_{jk}(ca,b). \end{aligned}$$

Taking $a = b = 1$ in (10), we get

$$T_{ij}(1,c) = T_{ik}(1,c) - T_{jk}(c,1),$$

and exchanging j and k, we have

$$T_{ik}(1,c) = T_{ij}(1,c) - T_{kj}(c,1).$$

Thus,

$$T_{kj}(c,1) + T_{jk}(c,1) = 0,$$

which gives us

$$T_{kj}(c,1) = -T_{jk}(c,1) = T_{kj}(1,c). \qquad \square$$

By Lemma 3, we have the following result.

Proposition 4 *Let*

$$\mathfrak{T} := \sum_{1\leqslant i<j\leqslant n} [X_{ij}(R), X_{ji}(R)].$$

Then $\mathfrak{T}$ is a subalgebra of $\mathfrak{st}_n(R,\varepsilon)$ with $[\mathfrak{T}, X_{ij}(R)] \subseteq X_{ij}(R)$, and

$$\mathfrak{st}_n(R,\varepsilon) = \mathfrak{T} \bigoplus_{1\leqslant i\neq j\leqslant n} X_{ij}(R). \tag{12}$$

Moreover, $\mathfrak{T}$ contains the center $\mathfrak{Z}$ of $\mathfrak{st}_n(R,\varepsilon)$ and $\ker\phi \subseteq \mathfrak{Z}$. Thus, $(\mathfrak{st}_n(R,\varepsilon), \phi)$ is a central extension of $\mathfrak{sl}_n(R,\varepsilon)$.

Proof By (6)–(9), we have $[\mathfrak{T}, X_{ij}(R)] \subseteq X_{ij}(R)$. Using ε-skew symmetry and ε-Jacobi identity, $\mathfrak{T}$ is closed under the bracket, i.e., it is a subalgebra of $\mathfrak{st}_n(R,\varepsilon)$.

From decomposition (1) of $\mathfrak{sl}_n(R,\varepsilon)$ and the epimorphism ϕ, we have the direct sum decomposition (12). Especially, we see that ϕ is injective restricted on $X_{ij}(R)$ for $1 \leqslant i \neq j \leqslant n$ and $\phi(\mathfrak{T}) = \xi_0$. Since the center of $\mathfrak{sl}_n(R,\varepsilon)$ is contained in ξ_0, we have $\mathfrak{Z} \subseteq \mathfrak{T}$.

Finally, $\ker\phi \subseteq \mathfrak{T}$. For any $1 \leqslant i < j \leqslant n$, $[\ker\phi, X_{ij}(R)] \subseteq X_{ij}(R)$. On the other hand, $\ker\phi$ is an ideal of $\mathfrak{st}_n(R,\varepsilon)$. One has

$$[\ker\phi, X_{ij}(R)] \subseteq \ker\phi \subseteq \mathfrak{T}.$$

Thus,

$$[\ker\phi, X_{ij}(R)] \subseteq \mathfrak{T} \cap X_{ij}(R) = \{0\}.$$

Since $\mathfrak{st}_n(R,\varepsilon)$ is generated by $X_{ij}(a)$, $a \in R$, we have

$$[\ker\phi, \mathfrak{st}_n(R,\varepsilon)] = 0, \quad \ker\phi \subseteq \mathfrak{Z}. \qquad \square$$

Now, setting $b = 1$ in (10), we have

$$T_{ij}(a,c) = T_{ik}(a,c) - \varepsilon(a,c)T_{jk}(ca,1). \tag{13}$$

For $a, b \in S(R)$, we set

$$t(a,b) = T_{1j}(a,b) - \varepsilon(a,b)T_{1j}(ba,1). \tag{14}$$

Then, by (10), (11), and (13), we have

$$\begin{aligned} t(a,b) &= T_{1k}(a,b) - \varepsilon(a,b)T_{jk}(ba,1) - \varepsilon(a,b)T_{1j}(ba,1) \\ &= T_{1k}(a,b) - \varepsilon(a,b)(T_{1j}(ba,1) - T_{kj}(ba,1)) \\ &= T_{1k}(a,b) - \varepsilon(a,b)T_{1k}(ba,1), \end{aligned}$$

which shows that $t(a,b)$ does not depend on the choice of j. Then, both $T_{ij}(a,b)$ and $t(a,b)$ are homogeneous elements with degree $\deg a + \deg b$ for $a, b \in S(R)$.

The following lemma is immediate.

Lemma 5

$$\ker\phi = \left\{ \sum_i t(a_i,b_i) \,\middle|\, \sum_i [a_i,b_i] = 0,\ a_i, b_i \in S(R) \right\}.$$

Proof First, (10) is equivalent to

$$T_{jk}(ca,b) = \varepsilon(c,a+b)(T_{ik}(a,bc) - T_{ij}(ab,c)).$$

Taking $c = 1$ and $i = 1$, we have

$$T_{jk}(a,b) = T_{1k}(a,b) - T_{1j}(ab,1) = h(a,b) + \varepsilon(a,b)T_{1k}(ba,1) - T_{1j}(ab,1).$$

It follows that every $x \in \mathfrak{T}$ can be written as

$$x = \sum_i t(a_i,b_i) + \sum_{j=2}^{n} T_{1j}(c_j,1), \quad a_i, b_i \in S(R),\ c_j \in R.$$

In Proposition 4, we have seen $\ker\phi \subseteq \mathfrak{T}$. $\phi(x) = 0$ implies

$$\begin{aligned} &\sum_i E_{11}([a_i,b_i]) + \sum_{2\leqslant j\leqslant n} (E_{11}(c_j) - E_{jj}(c_j)) \\ &\quad = E_{11}\left(\sum_i [a_i,b_i]\right) + \sum_{2\leqslant j\leqslant n} (E_{11} - E_{jj})(c_j) \\ &\quad = 0. \end{aligned}$$

It means that
$$\sum_i [a_i, b_i] = 0, \quad c_j = 0, \quad 2 \leqslant j \leqslant n.$$
Therefore,
$$x = \sum_i t(a_i, b_i)$$
with
$$\sum_i [a_i, b_i] = 0, \quad a_i, b_i \in S(R).$$
Notice that
$$[a, b] = ab - \varepsilon(a, b)ba, \quad a, b \in S(R). \qquad \square$$

For more information of the structure of $\ker \phi$, we have the following proposition.

Proposition 6 *For $a, b, c \in S(R)$, the following identities hold*:
$$t(a, 1) = t(1, a) = 0, \tag{15}$$
$$t(a, b) + \varepsilon(a, b)t(b, a) = 0, \tag{16}$$
$$t(ab, c) - t(a, bc) + \varepsilon(a + b, c)t(ca, b) = 0. \tag{17}$$

Proof By (14), we have
$$t(a, 1) = t(1, a) = 0.$$
By the definition of $t(a, b)$, using (13), one has
$$\begin{aligned} t(ab, c) &= T_{1j}(ab, c) - \varepsilon(a + b, c)T_{1j}(1, cab) \\ &= T_{1k}(a, bc) + \varepsilon(a, b + c)T_{kj}(b, ca) - \varepsilon(a + b, c)T_{1j}(1, cab) \\ &= t(a, bc) + \varepsilon(a, b + c)(T_{1k}(1, bca) + T_{kj}(b, ca)) - \varepsilon(a + b, c)T_{1j}(1, cab) \\ &= t(a, bc) + \varepsilon(a, b + c)T_{1j}(b, ca) - \varepsilon(a + b, c)T_{1j}(1, cab) \\ &= t(a, bc) + \varepsilon(a, b + c)(T_{1j}(b, ca) - \varepsilon(b, a + c)T_{1j}(1, cab)) \\ &= t(a, bc) + \varepsilon(a, b + c)t(b, ca). \end{aligned} \tag{18}$$

Since $t(a, 1) = 0$, setting $c = 1$ in (18), we have
$$0 = t(a, b) + \varepsilon(a, b)t(b, a),$$
which gives (16), and it follows that, by rewrite (18),
$$\begin{aligned} t(ab, c) &= t(a, bc) - \varepsilon(a, b + c)\varepsilon(b, a + c)t(ca, b) \\ &= t(a, bc) - \varepsilon(a + b, c)t(ca, b), \end{aligned}$$
which is (17). $\square$

3 ε-Hochschild homology and ε-cyclic homology

The tensor vector space $R^{\otimes(n+1)}$ could be viewed as a Γ-graded vector space by

$$\deg(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} \alpha_i,$$

where $a_i \in S(R)$ and $\deg(a_i) = \alpha_i \in \Gamma$ for $i = 1, 2, \ldots, n$.

One has the ε-Hochschild boundary d^ε with respect to ε,

$$d_n^\varepsilon \colon R^{\otimes(n+1)} \to R^{\otimes n},$$

defined by

$$\begin{aligned} d_n^\varepsilon(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = & \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ & + (-1)^n \varepsilon\Bigg(\sum_{i=0}^{n-1} a_i, a_n\Bigg) a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \end{aligned}$$

for homogeneous elements a_i, $0 \leqslant i \leqslant n$. We always assume $R^{\otimes 0} = k$ and $d_0^\varepsilon = 0$. Note that d_n^ε is a grading-preserving linear map of Γ-graded vector spaces.

Lemma 7

$$d_{n-1}^\varepsilon d_n^\varepsilon = 0, \quad \forall\, n \in \mathbb{N}.$$

Proof Consider $d_{n-1}^\varepsilon d_n^\varepsilon(a_0 \otimes a_1 \otimes \cdots \otimes a_n)$ for homogeneous elements a_i. Comparing with the Hochschild homology, we only need notice the coefficients of $a_n a_0 a_1 \otimes \cdots \otimes a_{n-1}$ and $a_{n-1} a_n a_0 \otimes \cdots \otimes a_{n-2}$ in the result. One is

$$(-1)^n \varepsilon\Bigg(\sum_{i=0}^{n-1} a_i, a_n\Bigg) + (-1)^{n-1} \varepsilon\Bigg(\sum_{i=0}^{n-1} a_i, a_n\Bigg) = 0,$$

and the other is

$$\begin{aligned} & \varepsilon\Bigg(\sum_{i=0}^{n-2} a_i, a_{n-1} + a_n\Bigg) - \varepsilon\Bigg(\sum_{i=0}^{n-1} a_i, a_n\Bigg) \varepsilon\Bigg(\sum_{i=0}^{n-2} a_i + a_n, a_{n-1}\Bigg) \\ & = \varepsilon\Bigg(\sum_{i=0}^{n-2} a_i, a_{n-1} + a_n\Bigg) - \varepsilon\Bigg(\sum_{i=0}^{n-2} a_i, a_n\Bigg) \varepsilon\Bigg(\sum_{i=0}^{n-2} a_i, a_{n-1}\Bigg) \\ & = \varepsilon\Bigg(\sum_{i=0}^{n-2} a_i, a_{n-1} + a_n\Bigg) - \varepsilon\Bigg(\sum_{i=0}^{n-2} a_i, a_{n-1} + a_n\Bigg) \\ & = 0. \end{aligned}$$

$\square$

Definition 8 The ε-Hochschild homology of the Γ-graded associative algebra R is $H_*(R^{\otimes(*+1)}, d^\varepsilon)$.

Denote by

$$[R,R] = \mathrm{span}_k\{ab - \varepsilon(a,b)ba \mid a, b \in S(R)\}$$

the linear span of all the ε-commutator. Then we have

$$H_1(R,\varepsilon) \cong R/[R,R].$$

Let

$$\xi_{n+1} = \langle t_{n+1} \mid t_{n+1}^{n+1} = 1\rangle$$

be the cyclic group and define an action of ξ_{n+1} on $R^{\otimes(n+1)}$ by setting

$$t_{n+1}(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (-1)^n \varepsilon\left(a_0, \sum_{i=1}^n a_i\right) a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes a_0$$

on the generators of $R^{\otimes(n+1)}$. It is then extended to $R^{\otimes(n+1)}$ by linearity. One can see that the action of

$$t_{n+1}^{n+1}(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 \otimes a_1 \otimes \cdots \otimes a_n$$

by using the identity

$$\prod_{i=0}^n \varepsilon\left(a_i, \sum_{0 \leqslant j \leqslant n,\, j \neq i} a_j\right) = 1.$$

Let Δ_{n+1}^ε be the subspace of $R^{\otimes(n+1)}$ spanned by elements

$$a_0 \otimes a_1 \otimes \cdots \otimes a_n + (-1)^{n+1} \varepsilon\left(a_0, \sum_{i=1}^n a_i\right) a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes a_0$$

for all homogeneous elements $a_i \in S(R)$, $0 \leqslant i \leqslant n$.

Lemma 9

$$d_n^\varepsilon(\Delta_{n+1}^\varepsilon) \subseteq \Delta_n^\varepsilon, \quad \forall\, n \in \mathbb{N}.$$

Proof For homogeneous elements $a_i \in S(R)$, by direct computation, we have

$$d_n^\varepsilon\left(a_0 \otimes a_1 \otimes \cdots \otimes a_n + (-1)^{n+1} \varepsilon\left(a_0, \sum_{i=1}^n a_i\right) a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes a_0\right)$$
$$= \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$

$$\begin{aligned}
&+(-1)^{n+1}\varepsilon\Big(a_0,\sum_{i=1}^{n}a_i\Big)\sum_{i=0}^{n-1}(-1)^i a_1\otimes\cdots\otimes a_{i+1}a_{i+2}\otimes\cdots\otimes a_n\otimes a_0\\
&+(-1)^n\varepsilon\Big(\sum_{i=0}^{n-1}a_i,a_n\Big)a_na_0\otimes a_1\otimes\cdots\otimes a_{n-1}\\
&-\varepsilon\Big(a_0,\sum_{i=1}^{n}a_i\Big)\varepsilon\Big(\sum_{i=1}^{n}a_i,a_0\Big)a_0a_1\otimes a_1\otimes\cdots\otimes a_n\\
=&\sum_{i=1}^{n-1}(-1)^i\Big(a_0\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_n\\
&+(-1)^n\varepsilon\Big(a_0,\sum_{i=1}^{n}a_i\Big)a_1\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_n\otimes a_0\Big).
\end{aligned}$$

Therefore,

$$d_n^\varepsilon(\Delta_{n+1}^\varepsilon)\subseteq\Delta_n^\varepsilon. \qquad \square$$

Denote by

$$C_n(R,\varepsilon)=R^{\otimes(n+1)}/\Delta_{n+1}^\varepsilon$$

the quotient k-space. Then d_n^ε induces a linear map

$$d_n^\varepsilon\colon C_n(R,\varepsilon)\to C_{n-1}(R,\varepsilon).$$

The complex $(C_*(R,\varepsilon),d_n^\varepsilon)$ gives that the homology of R is $H_*(C_*(R,\varepsilon),d^\varepsilon)$, which we call it ε-cyclic homology and denote it by $HC_*(R,\varepsilon)$. For our use, we rewrite $HC_1(R,\varepsilon)$ as follows.

Proposition 10 *Let*

$$\langle R,R\rangle^\varepsilon=(R\otimes_k R)/I^\varepsilon$$

be the quotient space, where I^ε is the subspace of $R\otimes_k R$ spanned by elements

$$a\otimes b+\varepsilon(a,b)b\otimes a,\quad ab\otimes c-a\otimes bc+\varepsilon(a+b,c)ca\otimes b,\quad \forall\, a,b,c\in S(R).$$

Set

$$\langle a,b\rangle^\varepsilon=a\otimes b+I^\varepsilon,\quad \forall\, a,b\in S(R).$$

Then

$$HC_1(R,\varepsilon)=\Big\{\sum_i\langle a_i,b_i\rangle^\varepsilon\,\Big|\,\sum_i(a_ib_i-\varepsilon(a_i,b_i)b_ia_i)=0,\ a_i,b_i\in S(R)\Big\}$$

is a subspace of $\langle R,R\rangle^\varepsilon$. Moreover, we have the following exact sequence:

$$0\longrightarrow HC_1(R,\varepsilon)\longrightarrow\langle R,R\rangle^\varepsilon=\frac{C_1(R)}{\mathrm{Im}d_2^\varepsilon}\xrightarrow{d_1^\varepsilon}[R,R]\longrightarrow 0.$$

Remark 11 We would like to emphasize the Γ-grading of $\langle R,R\rangle_c^\varepsilon$ and

$HC_1(R,\varepsilon)$ here. First, $R^{\otimes(n+1)}$ is naturally graded by Γ induced by R. Since Δ_n^{ε} is spanned by homogeneous elements of $R^{\otimes(n+1)}$, the quotient k-space $C_n(R,\varepsilon)$ is also Γ-graded. We have mentioned that d_n^{ε} is a grading-preserving linear map, which shows that the kernel and image of d_n^{ε} are graded subspaces. Thus, their quotient spaces are also Γ-graded. More directly, the Γ-grading of $\langle R,R\rangle^{\varepsilon}$ is given by

$$(\langle R,R\rangle^{\varepsilon})_c = \mathrm{span}_k\{\langle a,b\rangle^{\varepsilon} \in \langle R,R\rangle^{\varepsilon} \mid a,b\in S(R),\, \deg a+\deg b = c\}, \quad \forall\, c\in\Gamma.$$

Remark 12 If we take $\Gamma=\{0\}$ and $\varepsilon=1$, then the ε-Hochschild (cyclic) homology is nothing but the common Hochschild (cyclic) homology of the associative algebra R (see [7] or [8]). For the super case, we need take $\Gamma=\mathbb{Z}_2$ and

$$\varepsilon(\overline{\alpha},\overline{\beta}) = (-1)^{\alpha\beta}, \quad \overline{\alpha},\overline{\beta}\in\mathbb{Z}_2.$$

The Hochschild (cyclic) homology of superalgebras was first introduced by Kassel (see [6] or [1]).

Now, we state one of the main theorem of this section. In order to prove this theorem, we need the following definitions.

Definition 13 Let R be a Lie color algebra. An R-module M is a Γ-graded vector space

$$M=\bigoplus_{c\in\Gamma} M_c$$

such that

$$R_aM_b\subseteq M_{a+b}, \quad \forall\, a,b\in C,$$

and

$$[x,y]\cdot m = x(y\cdot m) - \varepsilon(x,y)y\cdot(x\cdot m), \quad \forall\, m\in M.$$

Definition 14 Let R be a Lie color algebra, and let $\mathscr{C}$ be a Γ-graded free module over K. A bilinear form ψ on $R\times R\to\mathscr{C}$ is called a color 2-cocycle on R if it satisfies

$$\psi(R_a,R_b)\subseteq \mathscr{C}_{a+b},$$
$$\psi(x,y) = -\varepsilon(x,y)\psi(y,x),$$
$$\varepsilon(z,x)\psi(x,[y,z]) + \varepsilon(x,y)\psi(y,[z,x]) + \varepsilon(y,z)\psi(z,[x,y]) = 0$$

for all $x,y,z\in S(R)$.

Theorem 15 $\ker\phi \simeq HC_1(R,\varepsilon)$.

Proof Consider the following two exact sequences:

$$\begin{array}{ccccccccc}
0 \longrightarrow & HC_1(R,\varepsilon) & \longrightarrow & \langle R,R\rangle^{\varepsilon} & \xrightarrow{d_1^{\varepsilon}} & [R,R] & \longrightarrow 0 \\
 & & & \downarrow \eta & & \downarrow E_{11} & \\
0 \longrightarrow & \ker\phi & \longrightarrow & \mathfrak{st}_n(R,\varepsilon) & \xrightarrow{\phi} & \mathfrak{sl}_n(R,\varepsilon) & \longrightarrow 0
\end{array}$$

Define

$$\eta\colon \langle R,R\rangle^{\varepsilon} \to \mathfrak{st}_n(R,\varepsilon)$$

by

$$\eta(\langle a,b\rangle^{\varepsilon}) = t(a,b), \quad a,b \in S(R).$$

From Proposition 10, (16), and (17), we know that η is well defined. Moreover,

$$\phi\circ\eta(\langle a,b\rangle^{\varepsilon}) = \phi(t(a,b)) = E_{11}(ab-\varepsilon(a,b)ba) = E_{11}(d_1^{\varepsilon}(\langle a,b\rangle^{\varepsilon})),$$

which shows that the above diagram commutes and defines, by restriction, a surjective homomorphism of $HC_1(R,\varepsilon)$ onto $\ker\phi$.

To get the injectivity of η, first we begin by finding a color 2-cocycle

$$f\colon \mathfrak{gl}_n(R,\varepsilon)\times\mathfrak{gl}_n(R,\varepsilon)\to\langle R,R\rangle^{\varepsilon}.$$

More precisely, if $x=E_{ij}(a)$, $y=E_{kl}(b)\in\mathfrak{gl}_n(R,\varepsilon)$, $a,b\in S(R)$, $1\leqslant i,j,k,l\leqslant n$, we define

$$f(x,y)=\delta_{jk}\delta_{li}\langle a,b\rangle^{\varepsilon}.$$

Extending f on the whole $\mathfrak{gl}_n(R,\varepsilon)$ by k-bilinearity, one can check that

$$f(x,y)+\varepsilon(x,y)f(y,x)=0,$$
$$\varepsilon(z,x)f(x,[y,z])+\varepsilon(x,y)f(y,[z,x])+\varepsilon(y,z)f(z,[x,y])=0$$

for the homogeneous elements $x,y,z\in\mathfrak{gl}_n(R,\varepsilon)$.

Therefore, f defines a color 2-cocycle on $\mathfrak{gl}_n(R,\varepsilon)$ with values in $\langle R,R\rangle^{\varepsilon}$.

We can restrict f on $\mathfrak{sl}_n(R,\varepsilon)$, which is also a color 2-cocycle. Now, we can form a (Γ,ε)-Lie color algebra

$$L=\mathfrak{sl}_n(R,\varepsilon)\oplus\langle R,R\rangle^{\varepsilon}$$

with Lie color bracket

$$[(x,c),(y,c')]=([x,y],f(x,y)), \quad \forall\, x,y\in\mathfrak{sl}_n(R,\varepsilon),\ \forall\, c,c'\in\langle R,R\rangle^{\varepsilon}.$$

L is a (Γ,ε)-Lie color algebra as f is a color 2-cocycle. Clearly,

$$f(E_{ij}(a),E_{kl}(b))\neq 0$$

only if $j=k$ and $l=i$.

Thus, the elements $(E_{ij}(a),0)$ in L satisfy relations (2)–(4). By the universal property of $\mathfrak{st}_n(R,\varepsilon)$, there exists a (unique) Lie algebra homomorphism

$$\rho\colon \mathfrak{st}_n(R,\varepsilon)\to L$$

such that

$$\rho(X_{ij}(a))=E_{ij}(a).$$

Hence,

$$\begin{aligned}\rho(T_{ij}(a,b)) &= [(E_{ij}(a),0),(E_{ji}(b),0)]\\ &= (E_{ii}(ab)-\varepsilon(a,b)E_{jj}(ba), f(E_{ij}(a),E_{ji}(b)))\\ &= (E_{ii}(ab)-\varepsilon(a,b)E_{jj}(ba), \langle a,b\rangle^{\varepsilon}).\end{aligned}$$

Therefore,

$$\begin{aligned}\rho(t(a,b)) &= \rho(T_{1j}(a,b)-\varepsilon(a,b)T_{1j}(1,ba))\\ &= (E_{11}(ab-\varepsilon(a,b)ba), \langle a,b\rangle^{\varepsilon}-\varepsilon(a,b)\langle 1,ba\rangle^{\varepsilon})\\ &= (E_{11}(d_1^{\varepsilon}(\langle a,b\rangle^{\varepsilon})), \langle a,b\rangle^{\varepsilon})\end{aligned}$$

as $\langle 1,ba\rangle^{\varepsilon}=0$ in $\langle R,R\rangle^{\varepsilon}$.

It follows that

$$\rho(\eta(\langle a,b\rangle^{\varepsilon})) = (E_{11}(d_1^{\varepsilon}(\langle a,b\rangle^{\varepsilon})), \langle a,b\rangle^{\varepsilon}),$$

and so

$$\rho(\eta(c)) = (E_{11}(d_1^{\varepsilon}(c)), c), \quad \forall\, c\in\langle R,R\rangle^{\varepsilon}.$$

Thus, η is injective. The proof is complete. □

The cohomology theory of the (Γ,ε)-Lie color algebra R is studied systematically by Scheunert and Zhang [11]. In this paper, we only use the homology $H_*(R,k)$ of the (Γ,ε)-Lie color algebra R with the trivial coefficient k. A (Γ,ε)-Lie color algebra R over k is called perfect if $[R,R]=R$. It is well known that R is perfect means that the first homology $H_1(R,k)$ is trivial. The pair $(\widehat{R},\pi)$, where $\widehat{R}$ is a (Γ,ε)-Lie color algebra and $\pi\colon \widehat{R}\to R$ an epimorphism, is called a central extension of R if $[\ker(\pi),\widehat{R}]=0$. A central extension $(\widetilde{R},\varphi)$ of R is called universal if for any central extension $(\widehat{R},\pi)$ of R, there exists a unique homomorphism $\rho\colon \widetilde{R}\to\widehat{R}$ such that $\pi\circ\rho=\varphi$. A (Γ,ε)-Lie color algebra R is called centrally closed if any central extension of R splits. Similarly, with the Lie algebras, we have the following result.

Theorem 16 *If R is a perfect (Γ,ε)-Lie color algebra, then there exists a unique universal central extension $(\widetilde{R},\varphi)$ of R and $\widetilde{R}$ is centrally closed. If $\widetilde{R}$ is centrally closed and $\widetilde{R}$ is a central extension of R, then $\widetilde{R}$ is the universal central extension of R and the kernel is isomorphic to $H_2(R,k)$.*

Remark 17 The details for the theory of the universal central extension of Lie algebras can be found in [5], while Neher [9] studied the universal central extension of Lie superalgebras.

We have known that

$$\phi\colon \mathfrak{st}_n(R,\varepsilon)\to\mathfrak{sl}_n(R,\varepsilon) \tag{19}$$

is a central extension. And both $\mathfrak{st}_n(R,\varepsilon)$ and $\mathfrak{sl}_n(R,\varepsilon)$ are perfect when $n \geqslant 3$. We will see that ϕ determines a universal central extension except in a few cases. On one hand, we will prove that

$$\phi\colon \mathfrak{st}_n(R,\varepsilon) \to \mathfrak{sl}_n(R,\varepsilon)$$

is a universal central extension if $n \geqslant 5$. On the other hand, we will use color 2-cocycle to construct the universal central extensions for the rest of two cases $n = 3$ and $n = 4$.

Definition 18 Let ξ be a color 2-cocycle on $\mathfrak{st}_n(R,\varepsilon)$ with value in $\mathscr{C}$ as above. Now, we define a Lie algebra $\mathfrak{st}_n(R,\varepsilon)^\sharp$ to be the Lie color algebra generated by the symbols $X^\sharp_{ij}(a)$, $a \in R$, $1 \leqslant i \neq j \leqslant n$, and the vector space $\mathscr{C}$, satisfying the following relations:

$$a \mapsto X^\sharp_{ij}(a) \text{ is a } K\text{-linear map}, \tag{20}$$

$$[\mathscr{C},\mathscr{C}] = [X^\sharp_{ij}(a),\mathscr{C}] = 0, \tag{21}$$

$$[X^\sharp_{ij}(a), X^\sharp_{jk}(b)] = X^\sharp_{ik}(ab) + \xi(X_{ij}(a), X_{jk}(b)), \quad i,j,k \text{ distinct}, \tag{22}$$

$$[X^\sharp_{ij}(a), X^\sharp_{kl}(b)] = \xi(X_{ij}(a), X_{kl}(b)), \quad j \neq k,\ i \neq l, \tag{23}$$

where $a, b \in R$, $1 \leqslant i,j,k,l \leqslant n$.

The following lemma shows that this construction of a central extension of $\mathfrak{st}_n(R,\varepsilon)$ is actually the same as the classical construction if ξ is surjective.

Lemma 19 *If*

$$0 \longrightarrow \mathscr{C} \longrightarrow \mathfrak{st}'_n(R,\varepsilon) \xrightarrow{\pi} \mathfrak{st}_n(R,\varepsilon) \longrightarrow 0$$

is a central extension of $\mathfrak{st}_n(R,\varepsilon)$ *constructed from a surjective super* 2*-cocycle* ξ, *then there is an isomorphism*

$$\rho\colon \mathfrak{st}_n(R,\varepsilon)^\sharp \to \mathfrak{st}'_n(R,\varepsilon)$$

with

$$\rho(X^\sharp_{ij}(a)) = X_{ij}(a), \quad \rho(c) = c, \quad a \in R,\ c \in \mathscr{C}.$$

Proof The proof is similar to [2,4]. □

4 Central extension of $\mathfrak{st}_4(R,\varepsilon)$

We define a color 2-cocycle on $\mathfrak{st}_4(R,\varepsilon)$ and construct the Lie color algebra $\widehat{\mathfrak{st}}_4(R,\varepsilon)$ as a covering (eventually a universal covering) of $\mathfrak{st}_4(R,\varepsilon)$. For any positive integer m, let $\mathscr{I}_m$ be the 2-sided Γ-graded ideal of R generated by the elements mc and $ab-\varepsilon(a,b)ba$ for any homogeneous elements $a,b\in S(R)$, $c\in R$. Then, we have the following lemma.

Lemma 20
$$\mathscr{I}_m = mR + R[R,R], \quad [R,R]R = R[R,R].$$

Let $R_m := R/\mathscr{I}_m$ be the quotient color algebra over K; it is color commutative. Write
$$\overline{a} = a + \mathscr{I}_m, \quad a\in R.$$
If $m=2$ and 2 is invertible in R, then $R_2=0$. For $\{i,j,k,l\}=\{1,2,3,4\}$, let $\epsilon_{ijkl}(R_2)$ denote a copy of R_2 and identify $\epsilon_{ijkl}(\overline{r})$, $\epsilon_{ilkj}(\overline{r})$, $\epsilon_{kjil}(\overline{r})$, $\epsilon_{klij}(\overline{r})$ for $\overline{r}\in R_2$. Thus, we have six distinct copies of R_2 whose direct sum is denote by $\mathscr{W}$. Using decomposition (12) of $\mathfrak{st}_4(R,\varepsilon)$, we define a K-bilinear map
$$\psi\colon \mathfrak{st}_4(R,\varepsilon)\times\mathfrak{st}_4(R,\varepsilon)\to\mathscr{W}$$
by
$$\psi(X_{ij}(a),X_{kl}(b)) = \epsilon_{ijkl}(\overline{ab}), \quad \{i,j,k,l\}=\{1,2,3,4\},\ a,b\in R,$$
and by $\psi(x,y)=0$ for all other pairs of elements from the summand of (12). Note that, if $m=2$, then $\overline{a}=-\overline{a}$ in R_2.

Lemma 21 *The bilinear map ψ is a color 2-cocycle.*

Proof Since there is a Lie color algebra homomorphism
$$a\colon \mathfrak{st}_4(R,\varepsilon)\to\mathfrak{st}_4(R_2,\varepsilon)$$
with
$$a(X_{ij}(a)) = X_{ij}(\overline{a})$$
and
$$\psi(a(x),a(y)) = \psi(x,y)$$
is well defined for $x,y\in\mathfrak{st}_4(R,\varepsilon)$, it suffices to verify the lemma for R_2. Since
$$\psi(X_{kl}(b),X_{ij}(a)) = \epsilon_{klij}(\overline{ba}), \quad ab = \pm\varepsilon(a,b)ba,$$
we have
$$\psi(X_{ij}(a),X_{kl}(b)) = \epsilon_{ijkl}(\overline{ab}) = -\varepsilon(a,b)\epsilon_{klij}(\overline{ba}) = -\varepsilon(a,b)\psi(X_{kl}(b),X_{ij}(a)),$$
where $a,b\in S(R)$. Now, let
$$J(x,y,z) = \varepsilon(z,x)\psi(x,[y,z]) + \varepsilon(x,y)\psi(y,[z,x]) + \varepsilon(y,z)\psi(z,[x,y])$$
for homogeneous elements $x,y,z\in\mathfrak{st}_4(R,\varepsilon)$. We will show $J(x,y,z)=0$ by taking homogeneous elements x,y,z in summands of (12). If a term $J(x,y,z)$ is not 0, then we can reorder to assume that $z=X_{kl}(d)$ and $0\neq[x,y]\in X_{ij}(R)$ with $\{i,j,k,l\}=\{1,2,3,4\}$.

Case 1 x or y is in $\mathfrak{T}$.

Without loss of generality, we assume $x = T_{pq}(a,b)$ and $y = X_{ij}(c)$. Since $R\ (= R_2)$ is commutative with $2R = 0$, $[x,y] \neq 0$ forces precisely one of p or q to be in $\{i,j\}$ (so the other to be in $\{k,l\}$ by equations (7)–(9)). Moveover, in this case,

$$[x,y] = X_{ij}(abc), \quad [y,z] = 0, \quad [z,x] = X_{kl}(dab),$$

and so

$$\begin{aligned}J(x,y,z) &= \varepsilon(a+b,c)\psi(X_{ij}(c),[X_{kl}(d),T_{pq}(a,b)]) + \varepsilon(c,d)\psi(X_{kl}(d),X_{ij}(abc))\\ &= \varepsilon(a,c)\varepsilon(b,c)\varepsilon(d,a)\varepsilon(d,b)\varepsilon(a,b)\epsilon_{ijkl}(cbad) + \varepsilon(c,d)\epsilon_{klij}(dabc)\\ &= 0.\end{aligned}$$

Case 2 Neither x nor y is in $\mathfrak{T}$.

We can assume that $x = X_{ip}(a)$ and $y = X_{pj}(b)$ with $p \in \{k,l\}$, and so

$$[x,y] = X_{ij}(ab).$$

For $p = k$, we have

$$[y,z] = 0, \quad [z,x] = -\varepsilon(d,a)X_{kl}(ad),$$

and so

$$\begin{aligned}J(x,y,z) &= \varepsilon(a,b)\psi(X_{kj}(b),[X_{kl}(d),X_{ik}(a)]) + \varepsilon(b,d)\psi(X_{kl}(d),X_{ij}(ab))\\ &= -\varepsilon(a,b)\varepsilon(d,a)\epsilon_{kjil}(bad) + \varepsilon(b,d)\epsilon_{klij}(dab)\\ &= 0.\end{aligned}$$

For $p = l$, we have

$$[y,z] = -\varepsilon(b,d)X_{kj}(db), \quad [z,x] = 0,$$

and so

$$\begin{aligned}J(x,y,z) &= -\varepsilon(d,a)\varepsilon(b,d)\psi(X_{il}(a),X_{kj}(db)) + \varepsilon(b,d)\psi(X_{kl}(d),X_{ij}(ab))\\ &= -\varepsilon(d,a)\varepsilon(b,d)\epsilon_{ilkj}(adb) + 0 + \varepsilon(b,d)\epsilon_{klij}(dab)\\ &= 0.\end{aligned}$$

□

Therefore, we obtain a central extension of Lie color algebra $\mathfrak{st}_4(R,\varepsilon)$:

$$0 \longrightarrow \mathscr{W} \longrightarrow \widehat{\mathfrak{st}}_4(R,\varepsilon) \xrightarrow{\pi} \mathfrak{st}_4(R,\varepsilon) \longrightarrow 0,$$

i.e.,

$$\widehat{\mathfrak{st}}_4(R,\varepsilon) = \mathfrak{st}_4(R,\varepsilon) \oplus \mathscr{W},$$

with Lie color bracket

$$[(x,c),(y,c')] = ([x,y],\psi(x,y)), \quad \forall\, x,y \in \mathfrak{st}_4(R,\varepsilon),\ \forall\, c,c' \in \mathscr{W}.$$

Here, π is the projection on the first summand:

$$\pi\colon \mathfrak{st}_4(R,\varepsilon)\oplus\mathscr{W}\to\mathfrak{st}_4(R,\varepsilon).$$

We can now apply Definition 18 with $\mathscr{C}=\mathscr{W}$ and $\xi=\psi$ to obtain the Lie color algebra $\mathfrak{st}_4^\sharp(R,\varepsilon)$. Since $1\in R$ and $\mathfrak{st}_4^\sharp(R,\varepsilon)$ is perfect, by Lemma 19, there exists a unique Lie color algebra isomorphism

$$\rho\colon \mathfrak{st}_4^\sharp(R,\varepsilon)\to\widehat{\mathfrak{st}}_4(R,\varepsilon)$$

such that

$$\rho(X_{ij}^\sharp(a))=X_{ij}(a),\quad \rho|_{\mathscr{W}}=id.$$

5 Central extension of $\mathfrak{st}_3(R,\varepsilon)$

In this section, we will handle $\mathfrak{st}_3(R,\varepsilon)$. Recall that

$$\mathscr{I}_3=3R+R[R,R]$$

and $R_3=R/\mathscr{I}_3$ is an associative commutative K-algebra. For $\{i,j,k\}=\{1,2,3\}$, let $\epsilon_{ijpq}(R_3)$ for $(p,q)=(i,k)$ or (k,j) denote a copy of R_3 and identify $\epsilon_{ijpq}(\overline{r})$ with $\epsilon_{pqij}(-\overline{r})$. Thus, we have six distinct copies of R_3 whose direct sum is denoted by $\mathscr{U}$. Using decomposition (12) of $\mathfrak{st}_3(R,\varepsilon)$, we define a K-bilinear map

$$\psi\colon \mathfrak{st}_3(R,\varepsilon)\times\mathfrak{st}_3(R,\varepsilon)\to\mathscr{U}$$

by

$$\psi(X_{ij}(r),X_{pq}(s))=\epsilon_{ijpq}(\overline{rs})$$

for $(p,q)=(i,k)$ or (k,j) with $\{i,j,k\}=\{1,2,3\}$ and $r,s\in R$, and by $\psi(x,y)=0$ for all other pairs of elements from the summands of (12).

Lemma 22 *The bilinear map ψ is a color 2-cocycle.*

Proof As in the proof of Lemma 21, we can assume $R=R_3$; i.e., R is commutative and $3R=0$. By the definition, we know that ψ is skew-symmetric and $\psi(x,x)=0$ for x in a summand of (12), and hence, for all $x\in\mathfrak{st}_3(R,\varepsilon)$. We will show

$$J(x,y,z)=\varepsilon(z,x)\psi(x,[y,z])+\varepsilon(x,y)\psi(y,[z,x])+\varepsilon(y,z)\psi(z,[x,y])$$

for homogeneous elements $x,y,z\in\mathfrak{st}_3(R,\varepsilon)$. We will show $J(x,y,z)=0$ by taking homogeneous elements x,y,z in summands of (12). If a term $J(x,y,z)$ is not 0, then we can reorder to assume that $z=X_{pq}(d)$ and $0\neq[x,y]\in X_{st}(R)$ with $(p,q)=(s,u)$ or (u,t) and $\{i,j,k\}=\{1,2,3\}$.

Case 1 x or y is in $\mathfrak{T}$.

Without loss of generality, we assume $x = T_{ij}(a, b)$ and $y = X_{st}(c)$. From equations (7)–(9) and $R = R_3$, we have

$$J(x, y, z) = \epsilon_{stpq}(\theta abcd) = 0$$

with $\theta = 0, 3, -3$, and thus, $J(x, y, z) = 0$.

Case 2 Neither x nor y is in $\mathfrak{T}$.

We can assume that $x = X_{su}(a)$ and $y = X_{ut}(b)$. If $(p, q) = (s, u)$, then we have

$$\begin{aligned} J(x, y, z) &= -\varepsilon(d, a)\varepsilon(b, d)\psi(X_{su}(a), X_{st}(db)) + \varepsilon(b, d)\psi(X_{su}(d), X_{st}(ab)) \\ &= -\varepsilon(d, a)\varepsilon(b, d)\epsilon_{sust}(adb) + \varepsilon(b, d)\epsilon_{sust}(dab) \\ &= 0. \end{aligned}$$

While, if $(p, q) = (u, t)$, then we have

$$\begin{aligned} J(x, y, z) &= -\varepsilon(a, b)\varepsilon(d, a)\psi(X_{ut}(b), X_{st}(ad)) + \varepsilon(b, d)\psi(X_{ut}(d), X_{st}(ab)) \\ &= -\varepsilon(a, b)\varepsilon(d, a)\epsilon_{utst}(bad) + \varepsilon(b, d)\epsilon_{utst}(dab) \\ &= 0. \end{aligned}$$

□

As in the $\mathfrak{st}_4(R, \varepsilon)$ case, we have a central extension of $\mathfrak{st}_3(R, \varepsilon)$:

$$0 \longrightarrow \mathscr{U} \longrightarrow \widehat{\mathfrak{st}}_3(R, \varepsilon) \xrightarrow{\pi} \mathfrak{st}_3(R, \varepsilon) \longrightarrow 0,$$

i.e.,

$$\widehat{\mathfrak{st}}_3(R, \varepsilon) = \mathfrak{st}_3(R, \varepsilon) \oplus \mathscr{U},$$

with Lie color bracket

$$[(x, c), (y, c')] = ([x, y], \psi(x, y)), \quad \forall\, x, y \in \mathfrak{st}_3(R, \varepsilon), \forall\, c, c' \in \mathscr{W}.$$

Here, π is the projection on the first summand:

$$\pi\colon \mathfrak{st}_3(R, \varepsilon) \oplus \mathscr{W} \to \mathfrak{st}_3(R, \varepsilon).$$

We can now apply Definition 18 with $\mathscr{C} = \mathscr{U}$ and $\xi = \psi$ to obtain the Lie color algebra $\mathfrak{st}_3^\sharp(R, \varepsilon)$. Since $1 \in R$ and $\mathfrak{st}_3^\sharp(R, \varepsilon)$ is perfect, by Lemma 19, there exists a unique Lie color algebra isomorphism

$$\rho\colon \mathfrak{st}_3^\sharp(R, \varepsilon) \to \widehat{\mathfrak{st}}_3(R, \varepsilon)$$

such that

$$\rho(X_{ij}^\sharp(a)) = X_{ij}(a), \quad \rho|_{\mathscr{U}} = id.$$

Theorem 23 *The universal covering of* $\mathfrak{st}_n(R, \varepsilon)$ *is* $(\mathfrak{st}_n(R, \varepsilon), id)$ *if* $n \geqslant 5$ *and* $(\widehat{\mathfrak{st}}_n(R, \varepsilon), \pi)$ *if* $n = 3$ *or* 4.

Proof Suppose that

$$0 \longrightarrow \mathscr{V} \longrightarrow \widetilde{\mathfrak{st}_n}(R, \varepsilon) \xrightarrow{\chi} \mathfrak{st}_n(R, \varepsilon) \longrightarrow 0$$

is a central extension of $\mathfrak{st}_n(R,\varepsilon)$. We will show that we can choose a preimage $\widetilde{X}_{ij}(a)$ of $X_{ij}(a)$ under χ for $1 \leqslant i \neq j \leqslant n$, $a \in R$, and a linear map $\mu\colon \mathscr{C} \to \mathscr{V}$, which satisfy relations (20)–(23) for $\mathfrak{st}_n^\sharp(R,\varepsilon)$ with $\xi = 0$ and $\mathscr{C} = 0$ if $n \geqslant 5$ and with $\xi = \psi$ and $\mathscr{C} = \mathscr{U}, \mathscr{W}$ for $n = 3, 4$, respectively. Thus, we will have a homomorphism

$$\theta\colon \mathfrak{st}_n^\sharp(R,\varepsilon) \to \widetilde{\mathfrak{st}_n}(R,\varepsilon)$$

with $\theta(X_{ij}^\sharp(a)) = \widetilde{X}_{ij}(a)$ so $\chi \circ \theta = \pi \circ \rho$ as in Lemma 19. Thus, the Lie color homomorphism

$$\theta \circ \rho^{-1}\colon \widehat{\mathfrak{st}}_n(R,\varepsilon) \to \widetilde{\mathfrak{st}}_n(R,\varepsilon)$$

satisfies

$$\chi \circ (\theta \circ \rho^{-1}) = \pi$$

and $(\mathfrak{st}'_n(R,\varepsilon),\pi)$ is the universal covering. Since

$$(\mathfrak{st}'_n(R,\varepsilon),\pi) = \begin{cases} (\mathfrak{st}_n(R,\varepsilon), id), & n \geqslant 5, \\ (\widehat{\mathfrak{st}}_n(R,\varepsilon),\pi), & n = 3, 4, \end{cases}$$

the result will follow. We begin by choosing any preimage $\widetilde{X}_{ij}(a)$ for a in a K-basis of R and extend linearly to all $a \in R$. We observe that

$$\widetilde{T}_{ij}(a,b) := [\widetilde{X}_{ij}(a), \widetilde{X}_{ji}(b)]$$

is independently of the choice of $\widetilde{X}_{ij}(a)$. If

$$[T_{pq}(a,b), X_{ij}(c)] = X_{ij}(d)$$

in (6)–(10), then

$$[\widetilde{T}_{pq}(a,b), \widetilde{X}_{ij}(c)] \in \widetilde{X}_{ij}(d) + \mathscr{V}.$$

Also, we have

$$[T_{pq}(1,1), X_{ij}(c)] = X_{ij}(mc),$$

where m may be an integer in $\{0, \pm 1, \pm 2\}$. Nevertheless, we always have

$$[T_{pq}(1,1), X_{ji}(c)] = X_{ji}(-mc),$$

so

$$[\widetilde{T}_{pq}(1,1), \widetilde{T}_{ij}(a,b)] \in [\widetilde{X}_{ij}(ma) + \mathscr{V}, \widetilde{X}_{ji}(b)] + [\widetilde{X}_{ij}(a), \widetilde{X}_{ji}(-mb) + \mathscr{V}] = \{0\}.$$

Fix some $k \neq i, j$ and replace $\widetilde{X}_{ij}(a)$ by $[\widetilde{T}_{ik}(1,1), \widetilde{X}_{ij}(a)]$. Then

$$[\widetilde{T}_{ik}(1,1), \widetilde{X}_{ij}(a)] = \widetilde{X}_{ij}(a).$$

We see that

$$\begin{aligned}
[\widetilde{T}_{pq}(a,b), \widetilde{X}_{ij}(c)] &= [\widetilde{T}_{pq}(a,b), [\widetilde{T}_{ik}(1,1), \widetilde{X}_{ij}(c)]] \\
&= [[\widetilde{T}_{pq}(a,b), \widetilde{T}_{ik}(1,1)], \widetilde{X}_{ij}(c)] + [\widetilde{T}_{ik}(1,1), [\widetilde{T}_{pq}(a,b), \widetilde{X}_{ij}(c)]] \\
&= \widetilde{X}_{ij}(d)
\end{aligned}$$

if

$$[T_{pq}(a,b), X_{ij}(c)] = X_{ij}(d).$$

In particular, taking $(p,q) = (i,l)$, we can observe that $\widetilde{X}_{ij}(a)$ does not depend on the choice of $k = i$ or j. Applying $\mathrm{ad}(\widetilde{T}_{ij}(1,1))$ to

$$[\widetilde{X}_{ij}(a), \widetilde{X}_{jk}(b)] \in \widetilde{X}_{ik}(ab) + \mathscr{V}$$

gives

$$[\widetilde{X}_{ij}(2a), \widetilde{X}_{jk}(b)] + [\widetilde{X}_{ij}(a), \widetilde{X}_{jk}(-b)] = \widetilde{X}_{ik}(ab)$$

i.e.,

$$[\widetilde{X}_{ij}(a), \widetilde{X}_{jk}(b)] = \widetilde{X}_{ik}(ab).$$

Thus, $\widetilde{X}_{ij}(a)$, $\mathscr{V}$ satisfy relations (20)–(22).

For relation (23), we first apply $\mathrm{ad}(\widetilde{X}_{ik}(1))$ to $[\widetilde{X}_{kj}(a), \widetilde{X}_{ij}(b)] \in \mathscr{V}$ giving

$$[\widetilde{X}_{ij}(a), \widetilde{X}_{ij}(b)] + [\widetilde{X}_{kj}(b), \mathscr{V}] = 0$$

for i, j, k are distinct, i.e.,

$$[\widetilde{X}_{ij}(a), \widetilde{X}_{ij}(b)] = 0.$$

If there exist distinct i, j, k, l, m, then applying $\mathrm{ad}(\widetilde{T}_{im}(1,1))$ to $[\widetilde{X}_{ij}(a), \widetilde{X}_{kl}(b)] \in \mathscr{V}$ gives

$$[\widetilde{X}_{ij}(a), \widetilde{X}_{kl}(b)] + 0 = 0,$$

by $[\widetilde{T}_{im}(1,1), \widetilde{X}_{kl}(b)] \in \mathscr{V}$. If i, j, k, l are distinct, applying $\mathrm{ad}(\widetilde{T}_{st}(1,1))$ to $[\widetilde{X}_{ij}(a), \widetilde{X}_{pq}(b)] \in \mathscr{V}$, where $(p,q) = (i,k)$ and $(s,t) = (l,j)$ or $(p,q) = (k,j)$ and $(s,t) = (i,l)$, gives

$$[\widetilde{X}_{ij}(a), \widetilde{X}_{pq}(b)] = 0.$$

If $n \geqslant 5$, we have shown that $\widetilde{X}_{ij}(a)$, $\mathscr{V}$ satisfy relations (20)–(23) for $\xi = 0$ and $\mathscr{V} = 0$. Moreover, for $n = 3, 4$, all relations are satisfied, except possibly

$$[\widetilde{X}_{ij}(a), \widetilde{X}_{pq}(b)] = \mu(\epsilon_{ijpq}(\overline{ab})), \quad n = 3, \tag{24}$$

with $(p,q) = (i,k)$ or (k,j);

$$[\widetilde{X}_{ij}(a), \widetilde{X}_{kl}(b)] = \mu(\epsilon_{ijpq}(\overline{ab})), \quad n = 4, \tag{25}$$

where i, j, k, l are distinct.

For (24), we set

$$\mu_{ijpq}(c) = [\widetilde{X}_{ij}(c), \widetilde{X}_{pq}(1)]$$

for $(p,q) = (i,k)$ or (k,j). Applying $\mathrm{ad}(\widetilde{T}_{ij}(a,b))$ gives

$$\mu_{ijpq}(abc + \varepsilon(a,b)\varepsilon(a+b,c)cba) + \varepsilon(a+b,c)[\widetilde{X}_{ij}(c), \widetilde{X}_{pq}(d)] = 0, \tag{26}$$

where $d = ab$ if $(p,q) = (i,k)$ and $d = ba$ if $(p,q) = (k,j)$. Now, $a = b = 1$ gives

$$\mu_{ijpq}(3c) = 0,$$

where $c = 1$ gives

$$\mu_{ijpq}(ab + \varepsilon(a,b)ba) + [\widetilde{X}_{ij}(1), \widetilde{X}_{pq}(d)] = 0,$$

i.e.,

$$\mu_{pqij}(d) = \mu_{ijpq}(ab + \varepsilon(a,b)ba).$$

In particular, $b = 1$ gives

$$\mu_{pqij}(a) = -\mu_{ijpq}(a),$$

so

$$\mu_{ijpq}(2d) = \mu_{pqij}(-d) = \mu_{ijpq}(ab + \varepsilon(a,b)ba),$$

which implies

$$\mu_{ijpq}(ab) = \varepsilon(a,b)\mu_{ijpq}(ba),$$

whenever $d = ab$ or $\varepsilon(a,b)ba$. Letting $b = 1$ in (26) gives

$$\varepsilon(a,c)[\widetilde{X}_{ij}(c), \widetilde{X}_{pq}(a)] = -\mu_{ijpq}(ac + \varepsilon(a,c)ca) = \mu_{ijpq}(ac).$$

Furthermore, (26) is equivalent to

$$\mu_{ijpq}(a[b,c]) = \mu_{ijpq}(abc - \varepsilon(b,c)acb) = 0.$$

Thus,

$$\mu_{ijpq}(\mathscr{I}_3) = 0.$$

Since $\mathscr{I}_3$ is linearly spanned by elements $3a$ and $a[b,c]$ for $a,b,c \in R$ by Lemma 20, we can define

$$\mu\colon \mathscr{U} \to \mathscr{V}$$

by

$$\mu(\epsilon_{ijpq}(\overline{a})) = \mu_{ijpq}(a)$$

such that (24) holds.

For (25), we set

$$\mu_{ijkl}(c) = [\widetilde{X}_{ij}(c), \widetilde{X}_{kl}(1)], \quad \{i,j,k,l\} = \{1,2,3,4\}.$$

Applying $\mathrm{ad}(\widetilde{T}_{ij}(a,b))$ gives

$$\mu_{ijkl}(abc + \varepsilon(a,b)\varepsilon(a,c)\varepsilon(b,c)cba) = 0.$$

Now, $b = c = 1$ gives

$$\mu_{ijkl}(2a) = 0,$$

and $c=1$ gives

$$\mu_{ijkl}(ab) = -\varepsilon(a,b)\mu_{ijkl}(ba) = \mu_{ijkl}(ba).$$

Moreover,

$$\begin{aligned}\mu_{ijkl}(a[b,c]) &= \mu_{ijkl}(abc) - \varepsilon(b,c)\mu_{ijkl}(acb)\\ &= \mu_{ijkl}(abc) - \varepsilon(b,c)\mu_{ijkl}(-\varepsilon(a,b+c)cba)\\ &= \mu_{ijkl}(abc + \varepsilon(a,b)\varepsilon(b,c)\varepsilon(a,c)cba)\\ &= 0.\end{aligned}$$

Thus,

$$\mu_{ijkl}(\mathscr{I}_2) = 0.$$

Applying $\mathrm{ad}(\widetilde{X}_{kl}(-b))$ to $[\widetilde{X}_{ik}(a), \widetilde{X}_{kj}(1)]$ gives

$$[\widetilde{X}_{ij}(a), \widetilde{X}_{kl}(b)] = [\widetilde{X}_{il}(ab), \widetilde{X}_{kj}(1)] = \mu_{ilkj}(ab).$$

In particular, $b=1$ gives

$$\mu_{ilkj}(a) = \mu_{ijkl}(a)$$

and

$$[\widetilde{X}_{ij}(a), \widetilde{X}_{kl}(b)] = \mu_{ijkl}(ab).$$

Clearly,

$$\mu_{klij}(a) = [\widetilde{X}_{kl}(1), \widetilde{X}_{ij}(a)] = [\widetilde{X}_{ij}(-a), \widetilde{X}_{kl}(1)] = \mu_{klij}(a)$$

and

$$\mu_{ilkj}(a) = \mu_{ijkl}(a) = \mu_{kjil}(a).$$

We can now define

$$\mu\colon \mathscr{W} \to \mathscr{V}$$

by

$$\mu(\epsilon_{ijkl}(\overline{a})) = \mu_{ijkl}(a),$$

and so (25) holds. □

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