RESEARCH ARTICLE

High order moment closure for Vlasov-Maxwell equations

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Abstract   The extended magnetohydrodynamic models are derived based on the moment closure of the Vlasov-Maxwell (VM) equations. We adopt the Grad type moment expansion which was firstly proposed for the Boltzmann equation. A new regularization method for the Grad's moment system was recently proposed to achieve the globally hyperbolicity so that the local well-posedness of the moment system is attained. For the VM equations, the moment expansion of the convection term is exactly the same as that in the Boltzmann equation, thus the new developed regularization applies. The moment expansion of the electromagnetic force term in the VM equations turns out to be a linear source term, which can preserve the conservative properties of the distribution function in the VM equations perfectly.

Keywords   Moment closure, Vlasov-Maxwell (VM) equations, Boltzmann equation, extended magnetohydrodynamics
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1   Introduction

Collisionless plasmas have been studied in a wide variety of fields, such as in laboratory plasma physics, space physics, and astrophysics. Evolution of collisionless plasmas and self-consistent electromagnetic fields are fully described by the Vlasov-Maxwell (VM) equations. In this paper, we study the evolution

of a single species of nonrelativistic electrons under the self-consistent electro-magnetic fields while the ions are treated as uniform fixed background. Under the scaling of the characteristic time by the inverse of the plasma frequency ω_p^{-1}, length by the Debye length λ_D, and electric and magnetic fields by $-mc\omega_p/e$ (with m the electron mass, c the speed of light, and e the electron charge), the evolution of the distribution function $f(t, \boldsymbol{x}, \boldsymbol{v})$ of electrons is described by the dimensionless form of the Vlasov equation

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f + (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \nabla_{\boldsymbol{v}} f = 0, \quad \boldsymbol{x}, \boldsymbol{v} \in \mathbb{R}^3. \tag{1}$$

The acceleration term in the phase space is the sum of the Coulomb force and the Lorentz force, by which the particles interact via the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. The charge and current densities, ρ and $\boldsymbol{j}$, act as sources of self-consistent electromagnetic fields according to the Maxwell equations:

$$\nabla_{\boldsymbol{x}} \cdot \boldsymbol{E} = \rho - \rho_i, \quad \nabla_{\boldsymbol{x}} \cdot \boldsymbol{B} = 0,$$
$$\nabla_{\boldsymbol{x}} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \nabla_{\boldsymbol{x}} \times \boldsymbol{B} = \boldsymbol{j} + \frac{\partial \boldsymbol{E}}{\partial t}. \tag{2}$$

And they can be obtained by the moments of the particle distribution function,

$$\rho(t, \boldsymbol{x}) = \int f(t, \boldsymbol{x}, \boldsymbol{v}) \mathrm{d}\boldsymbol{v}, \quad \boldsymbol{j}(t, \boldsymbol{x}) = \rho(t, \boldsymbol{x})\boldsymbol{u}(t, \boldsymbol{x}) = \int \boldsymbol{v} f(t, \boldsymbol{x}, \boldsymbol{v}) \mathrm{d}\boldsymbol{v}, \tag{3}$$

respectively. The charge density of background ions is denoted by ρ_i, which is chosen to satisfy total charge neutrality:

$$\int (\rho(t, \boldsymbol{x}) - \rho_i) \mathrm{d}\boldsymbol{x} = 0.$$

The computation of the initial boundary value problem associated to the VM equations is quite challenging, due to the high-dimensionality of the Vlasov equation, multiple temporal and spatial scales associated with various physical phenomena, nonlinearity, and the conservation of physical quantities. Particle-In-Cell (PIC) methods (comprehensively reviewed in [19]) have been very popular for a wide variety of plasma phenomena. Though the PIC method can often give satisfying results, it inherently has the large statistical noise which makes it difficult to study such as particle accelerations and thermal transport processes, where a small number of high energy particles play an important role. An alternative is represented by solving the Vlasov equation directly in phase-space (namely, with a sixth dimensional computational grid in space and velocity). It has been widely known that a numerical solution of the advection equation suffers from spurious oscillations and numerical diffusion, while a highly accurate scheme is required to preserve characteristics of the Vlasov equation, saying the Liouville theorem, as much as possible. Though there are a lot of studies [10,11,13,16,18] on the direct simulation, no standard scheme for the Vlasov simulation has been established so far.

In this paper, we focus on the moment method where the distribution function is expanded using Hermite functions in velocity space. The moment method can be tracked back to Grad's work in 1949 [6] for the Boltzmann equation, where a 13-moment model was given as an extension of the classic Euler equations. In [7], its major deficiencies were found soon, including the appearance of subshocks in the structure of a strong shock wave and the loss of global hyperbolicity. In the later study, a number of regularizations were attempted to solve or alleviate these problems, such as Levermore's work [9]. Jin and Slemrod [8] gave a regularization of the Burnett equations via relaxation, which resulted in a set of equations containing the same variables with Grad's 13-moment theory, and no subshocks appeared in the structure of shock waves. By integrating the moment method with the Chapman-Enskog expansion, Struchtrup and Torrilhon [14,15] regularized Grad's system to give the R13 equations. The R13 system removes the discontinuities in the shock wave and extends the region of hyperbolicity considerably [17].

The moment expansion for the Boltzmann equation can be extended to the VM equations. The resulting convective term in the moment system expanded from the drift term of the Vlasov equation has exactly the same format as that of the Boltzmann equation. Thus, the method of the hyperbolic regularization in [2] can be applied to the VM equations to achieve the global hyperbolicity. The major difference of the VM equations from the Boltzmann equation is the acceleration to the particles due to the electromagnetic fields. However, the moment expansion of the acceleration term turns out to be a linear source term, which can be expressed as a compact sparse coefficient matrix. Furthermore, such a coefficient matrix is block diagonal for the moments with the same order, which means that the evolutions of the moments with different orders are separated. It is shown that a weighted l_2 norm of the moments of the same order is invariant in time accelerated by the magnetic field alone. As a result, the high order moments are not growing at all due to the source term. So the derived moment system is formulated as a quasi-linear system, plus a linear source term which induces no growth of the high order moments. Since the convection term in the system is guaranteed to be globally hyperbolic by the regularization, the local well-posedness of the system is partially achieved.

The rest of this paper is arranged as follows. In Section 2, we present the moment expansion of VM equations and regularize it to achieve the final hyperbolic moment system. In Section 3, we discuss the conservation properties of the moment system. And an exact VM equilibrium with sheath-like magnetic field will be given in Section 4 for better understanding of the structure of the derived moment system. Concluding remarks are in the last section.

2 Grad moment system

When the density of electrons is not extremely high, we can assume that the

equilibrium distribution is a Maxwellian distribution:

$$f_{\mathrm{eq}}(t, \boldsymbol{x}, \boldsymbol{v}) = \frac{\rho(t, \boldsymbol{x})}{(2\pi k_B T(t, \boldsymbol{x}))^{3/2}} \exp\Big(-\frac{(\boldsymbol{v} - \boldsymbol{u}(t, x))^2}{2k_B T(t, \boldsymbol{x})} \Big), \qquad (4)$$

where k_B is the Boltzmann constant, and $T(t, \boldsymbol{x})$ is the particle temperature, which is related with the distribution function as follows:

$$3\rho(t, \boldsymbol{x}) k_B T(t, \boldsymbol{x}) = \int |\boldsymbol{v} - \boldsymbol{u}(t, \boldsymbol{x})|^2 f(t, \boldsymbol{x}, \boldsymbol{v}) \mathrm{d}\boldsymbol{v}. \qquad (5)$$

In this section, we derive the moment system of the Vlasov equation using the Grad type moment expansion. The basic idea is to expand the distribution function into an infinite series using Hermite functions as basis functions, and write down the equation for each coefficient. By cut-off and closure, we obtain a system with finite equations to approximate the Vlasov equation.

2.1 Hermite expansion of distribution function

Following the method in [3,4], we expand the distribution function into Hermite series as

$$f(t, \boldsymbol{x}, \boldsymbol{v}) = \sum_{\alpha \in \mathbb{N}^3} f_\alpha(t, \boldsymbol{x}) \mathscr{H}_{\mathscr{T}, \alpha}\Big(\frac{\boldsymbol{v} - \boldsymbol{u}(t, \boldsymbol{x})}{\sqrt{\mathscr{T}(t, \boldsymbol{x})}} \Big), \qquad (6)$$

where $f_\alpha(t, \boldsymbol{x})$ are the coefficients. The basis function $\mathscr{H}_{\mathscr{T}, \alpha}$ is an exponentially decaying function multiplied by a multi-dimensional Hermite polynomial shifted by the local macroscopic momentum $\boldsymbol{u}(t, \boldsymbol{x})$ and scaled by the square root of the local temperature $\mathscr{T}(t, \boldsymbol{x}) = k_B T(t, \boldsymbol{x})$:

$$\mathscr{H}_{\mathscr{T}, \alpha}(\boldsymbol{\xi}) = \prod_{d=1}^{3} \frac{1}{\sqrt{2\pi}} \, \mathscr{T}(t, \boldsymbol{x})^{-(\alpha_d + 1)/2} He_{\alpha_d}(\xi_d) \exp\Big(-\frac{\xi_d^2}{2} \Big), \qquad (7)$$

where α is considered as a multi-index. The Hermite polynomials $He_n(x)$ can be obtained by successive differentiation with respect to the Gaussian function:

$$He_n(x) = (-1)^n \exp\Big(\frac{x^2}{2} \Big) \frac{\mathrm{d}^n}{\mathrm{d}x^n} \exp\Big(-\frac{x^2}{2} \Big). \qquad (8)$$

For convenience, $He_n(x)$ is taken as zero if $n < 0$, and thus, $\mathscr{H}_{\mathscr{T}, \alpha}(\boldsymbol{\xi})$ is zero when any component of α is negative. Naturally, the equilibrium distribution $f_{\mathrm{eq}}(t, \boldsymbol{x}, \boldsymbol{v})$ in (4) is coincidently equal to the first term of expansion. Using the orthogonality of the Hermite polynomials, some simple relations can be obtained:

$$\rho = f_0, \quad f_{e_i} \equiv 0, \quad i = 1, 2, 3, \quad \sum_{d=1}^{3} f_{2e_d} = 0, \qquad (9)$$

$$q_i = 2f_{3e_i} + \sum_{d=1}^{3} f_{2e_d + e_i}, \quad p_{ij} = \delta_{ij}\rho\mathscr{T} + (1 + \delta_{ij})f_{e_i + e_j}, \qquad (10)$$

where e_d is the unit vector with its d-th entry to be 1, q_i is the heat flux, and p_{ij} is the pressure tensor.

2.2 Moment expansion of Vlasov equation

The general method to get the moment system is to multiply the Vlasov equation (1) by polynomials of momentum $\boldsymbol{v}$ and integrate both sides over $\boldsymbol{v}$ on $\mathbb{R}^3$. One equivalent way is as follows. We substitute the expansion of the distribution function (6) into the Vlasov equation (1), then collect the coefficients of the same order on both sides, and finally equate them to yield the derived moment system. It should be noted that the Hermite function (7) used in this paper depends also on the time t and position $\boldsymbol{x}$ through $\boldsymbol{u}(t, \boldsymbol{x})$ and $\mathscr{T}(t, \boldsymbol{x})$, which is different from the general expansion using the Hermite functions depending only on the momentum $\boldsymbol{v}$ [12]. For convenience, we list some useful relations of Hermite polynomials as follows [1]:

(i) orthogonality:

$$\int_{\mathbb{R}} He_l(x) He_n(x) \exp\left(-\frac{x^2}{2}\right) dx = l! \sqrt{2\pi}\, \delta_{l,n};$$

(ii) recursion relation:

$$He_{n+1}(x) = x He_n(x) - n He_{n-1}(x);$$

(iii) differential relation:

$$He_n'(x) = n He_{n-1}(x).$$

And the following equality can be derived from the last two relations:

$$\frac{\partial}{\partial v_j} \mathscr{H}_{\mathscr{T},\alpha}\left(\frac{\boldsymbol{v}-\boldsymbol{u}}{\sqrt{\mathscr{T}}}\right) = -\mathscr{H}_{\mathscr{T},\alpha+e_j}\left(\frac{\boldsymbol{v}-\boldsymbol{u}}{\sqrt{\mathscr{T}}}\right). \tag{11}$$

With these relations, the general moment equations can be obtained with a slight rearrangement by matching the coefficients of the same weight function:

$$\frac{\partial f_\alpha}{\partial t} + \sum_{d=1}^{3}\left[\frac{\partial u_d}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_d}{\partial x_j} - E_d - \sum_{k,m=1}^{3} \varepsilon_{dkm} u_k B_m\right] f_{\alpha-e_d}$$

$$- \sum_{d,k,m=1}^{3} \varepsilon_{dkm}(\alpha_k+1) B_m f_{\alpha-e_d+e_k}$$

$$+ \frac{1}{2}\left(\frac{\partial \mathscr{T}}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial \mathscr{T}}{\partial x_j}\right) \sum_{d=1}^{3} f_{\alpha-2e_d}$$

$$+ \sum_{j,d=1}^{3}\left[\frac{\partial u_d}{\partial x_j}\left(\mathscr{T} f_{\alpha-e_d-e_j} + (\alpha_j+1) f_{\alpha-e_d+e_j}\right)\right.$$

$$+ \frac{1}{2} \frac{\partial \mathscr{T}}{\partial x_j} \left(\mathscr{T} f_{\alpha - 2e_d - e_j} + (\alpha_j + 1) f_{\alpha - 2e_d + e_j} \right) \Bigg]$$

$$+ \sum_{j=1}^{3} \left(\mathscr{T} \frac{\partial f_{\alpha - e_j}}{\partial x_j} + u_j \frac{\partial f_\alpha}{\partial x_j} + (\alpha_j + 1) \frac{\partial f_{\alpha + e_j}}{\partial x_j} \right) = 0, \qquad (12)$$

where the Levi-Civita symbols ε_{dkm} are defined as

$$\varepsilon_{dkm} = \begin{cases} 1, & d \neq k \neq m \text{ cyclic permutation of 1,2,3,} \\ -1, & d \neq k \neq m \text{ anti-cyclic permutation of 1,2,3,} \\ 0, & (d-k)(k-m)(m-d) = 0. \end{cases}$$

By setting $\alpha = 0$ in (12), we deduce the mass conservation:

$$\frac{\partial \rho}{\partial t} + \sum_{j=1}^{3} \left(u_j \frac{\partial \rho}{\partial x_j} + \rho \frac{\partial u_j}{\partial x_j} \right) = 0. \qquad (13)$$

By setting $\alpha = e_d$, with $d = 1, 2, 3$ and noting that $f_{e_d} = 0$ in (12), we obtain

$$\frac{\partial u_d}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_d}{\partial x_j} + \frac{1}{\rho} \sum_{j=1}^{3} \frac{\partial p_{jd}}{\partial x_j} = E_d + \sum_{k,m=1}^{3} \varepsilon_{dkm} u_k B_m. \qquad (14)$$

By setting $\alpha = 2e_d$, we obtain

$$\frac{\partial f_{2e_d}}{\partial t} + \frac{\rho}{2} \left(\frac{\partial \mathscr{T}}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial \mathscr{T}}{\partial x_j} \right) + \rho \mathscr{T} \frac{\partial u_d}{\partial x_d}$$

$$+ \sum_{j,l=1}^{3} (1 + 2\delta_{jd}) f_{2e_d - e_l + e_j} \frac{\partial u_l}{\partial x_j} - \sum_{k,m=1}^{3} \varepsilon_{dkm} B_m f_{e_d + e_k}$$

$$+ \sum_{j=1}^{3} u_j \frac{\partial f_{2e_d}}{\partial x_j} + (1 + 2\delta_{jd}) \frac{\partial f_{2e_d + e_j}}{\partial x_j} = 0. \qquad (15)$$

Noting that

$$\sum_{d=1}^{3} f_{2e_d} = 0,$$

we sum the upper equations over d to get

$$\rho \left(\frac{\partial \mathscr{T}}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial \mathscr{T}}{\partial x_j} \right) + \frac{2}{3} \sum_{j=1}^{3} \left(\frac{\partial q_j}{\partial x_j} + \sum_{d=1}^{3} p_{jd} \frac{\partial u_d}{\partial x_j} \right) = 0. \qquad (16)$$

Since

$$\rho \mathscr{T} = \frac{1}{3} \sum_{d=1}^{3} p_{dd},$$

we have

$$\frac{\partial \mathscr{T}}{\partial x_j} = \frac{1}{3\rho} \sum_{d=1}^{3} \frac{\partial p_{dd}}{\partial x_j} - \frac{\mathscr{T}}{\rho} \frac{\partial \rho}{\partial x_j}, \quad j = 1, 2, 3. \tag{17}$$

Substituting (14), (16), and (17) into (12), we eliminate the time derivatives of $\boldsymbol{u}$ and $\mathscr{T}$ and the spatial derivatives of $\mathscr{T}$. Then the quasi-linear form of the moment system reads

$$\frac{\partial f_\alpha}{\partial t} + \sum_{j=1}^{3} \left(\mathscr{T} \frac{\partial f_{\alpha-e_j}}{\partial x_j} + u_j \frac{\partial f_\alpha}{\partial x_j} + (\alpha_j+1) \frac{\partial f_{\alpha+e_j}}{\partial x_j} \right)$$

$$+ \sum_{j=1}^{3} \sum_{d=1}^{3} \left(\mathscr{T} f_{\alpha-e_d-e_j} + (\alpha_j+1) f_{\alpha-e_d+e_j} - \frac{p_{jd}}{3\rho} \sum_{k=1}^{3} f_{\alpha-2e_k} \right) \frac{\partial u_d}{\partial x_j}$$

$$- \sum_{j=1}^{3} \sum_{d=1}^{3} \frac{f_{\alpha-e_d}}{\rho} \frac{\partial p_{jd}}{\partial x_j} - \frac{1}{3\rho} \sum_{k=1}^{3} f_{\alpha-2e_k} \sum_{j=1}^{3} \frac{\partial q_j}{\partial x_j}$$

$$+ \sum_{j=1}^{3} \left(\sum_{k=1}^{3} (\mathscr{T} f_{\alpha-2e_k-e_j} + (\alpha_j+1) f_{\alpha-2e_k+e_j}) \left(-\frac{\mathscr{T}}{2\rho} \frac{\partial \rho}{\partial x_j} + \frac{1}{6\rho} \sum_{d=1}^{3} \frac{\partial p_{dd}}{\partial x_j} \right) \right)$$

$$= \sum_{d,k,m=1}^{3} \varepsilon_{dkm} (\alpha_k+1) B_m f_{\alpha-e_d+e_k}, \quad \forall |\alpha| \geqslant 2. \tag{18}$$

Remark 1 Taking ρ, u_d ($d = 1, 2, 3$), p_{ij} ($1 \leqslant i \leqslant j \leqslant 3$), and f_α ($|\alpha| \geqslant 3$) as unknowns, equations (13), (14), and (18) are collected to obtain a moment system with an infinite number of equations. By the relation between u_d and f_{e_d} given in (9) and the definition of q_i and p_{ij} in (10), one can see that the obtained system is quasi-linear for the unknowns.

2.3 Moment closure with global hyperbolicity

For a positive integer $M \geqslant 3$, expansion (6) is truncated as

$$f_h(t, \boldsymbol{x}, \boldsymbol{v}) = \sum_{|\alpha| \leqslant M} f_\alpha(t, \boldsymbol{x}) \mathscr{H}_{\mathscr{T}, \alpha} \left(\frac{\boldsymbol{v} - \boldsymbol{u}(t, \boldsymbol{x})}{\sqrt{\mathscr{T}(t, \boldsymbol{x})}} \right). \tag{19}$$

For any

$$\alpha \in \mathscr{S}_M = \{\alpha \in \mathbb{N}^3 \mid |\alpha| \leqslant M\},$$

let

$$\mathscr{N}(\alpha) = \sum_{i=1}^{3} \binom{\sum_{k=4-i}^{3} \alpha_k + i - 1}{i} + 1 \tag{20}$$

be the ordinal number of α in $\mathscr{S}_M$. Then the total number of the set $\mathscr{S}_M$ will be

$$N = \mathscr{N}(Me_3) = \binom{M+3}{3}.$$

Let $\boldsymbol{w} = (w_1, \ldots, w_N)^{\mathrm{T}} \in \mathbb{R}^N$, for each $i, j \in \{1, 2, 3\}$ and $i < j$,

$$w_1 = \rho, \quad w_{\mathcal{N}(e_i)} = u_i, \tag{21a}$$

$$w_{\mathcal{N}(2e_i)} = \frac{p_{ii}}{2}, \quad w_{\mathcal{N}(e_i+e_j)} = p_{ij}, \tag{21b}$$

$$w_{\mathcal{N}(\alpha)} = f_\alpha, \quad 3 \leqslant |\alpha| \leqslant M. \tag{21c}$$

The moment system (13), (14), and (18) is collected in a quasi-linear format as

$$\frac{\partial \boldsymbol{w}}{\partial t} + \sum_{j=1}^{3} \mathbf{M}_j(w) \frac{\partial \boldsymbol{w}}{\partial x_j} = \mathbf{G}\boldsymbol{w} + \boldsymbol{g}, \tag{22}$$

where $\mathbf{M}_j$, $\mathbf{G}$ are $N \times N$ matrices, and $\boldsymbol{g}$ is N vector, corresponding to the terms with derivatives of $\boldsymbol{w}$, the magnetic force term, and the electronic force term, respectively. Precisely, from (14) and (18), we have

$$\boldsymbol{g}_{\mathcal{N}(e_i)} = E_i, \quad \mathbf{G}_{\mathcal{N}(\alpha), \mathcal{N}(\alpha - e_d + e_k)} = \sum_{m=1}^{3} \varepsilon_{dkm}(\alpha_k + 1)B_m, \tag{23}$$

while all other entries of $\mathbf{G}$ and $\boldsymbol{g}$ vanish.

It has been pointed out in [2] that it is not appropriate to set

$$\frac{\partial f_{\alpha+e_j}}{\partial x_j} = 0 \quad (|\alpha| = M)$$

as the closure since the system is lack of hyperbolicity if the distribution function is far away from the equilibrium. For any α with $|\alpha| = M$, we define

$$\mathscr{R}_M{}^j(\alpha) = (\alpha_j + 1)\left[\sum_{d=1}^{3} f_{\alpha - e_d + e_j} \frac{\partial u_d}{\partial x_j} + \frac{1}{2}\left(\sum_{d=1}^{3} f_{\alpha - 2e_d + e_j} \right) \frac{\partial \mathscr{T}}{\partial x_j} \right], \tag{24}$$

and

$$\hat{\mathbf{M}}_j \frac{\partial \boldsymbol{w}}{\partial x_j} = \mathbf{M}_j \frac{\partial \boldsymbol{w}}{\partial x_j} - \sum_{|\alpha|=M} \mathscr{R}_M{}^j(\alpha) I_{\mathcal{N}(\alpha)} \tag{25}$$

for any admissible $\boldsymbol{w}$, where I_k is the k-th column of the $N \times N$ identity matrix. We regularize system (22) as

$$\frac{\partial \boldsymbol{w}}{\partial t} + \sum_{j=1}^{3} \hat{\mathbf{M}}_j(w) \frac{\partial \boldsymbol{w}}{\partial x_j} = \mathbf{G}\boldsymbol{w} + \boldsymbol{g}. \tag{26}$$

We refer the readers to [2] for more details. The global hyperbolicity has been rigorously proven in [2]. For convenience, the most important result is given in the following lemma.

Lemma 1 *The regularized moment system* (26) *is hyperbolic for any $\boldsymbol{w}$ with positive temperature. Precisely, for a given unit vector $\boldsymbol{n} = (n_1, n_2, n_3)$, the matrix*

$$\sum_{j=1}^{3} n_j \hat{\mathbf{M}}_j(\boldsymbol{w}) \tag{27}$$

is diagonalizable with real eigenvalues as

$$\boldsymbol{u} \cdot \boldsymbol{n} + C_{k,m}\sqrt{\mathscr{T}}, \quad 1 \leqslant k \leqslant m \leqslant M+1, \tag{28}$$

where $C_{k,m}$ is a root of m-order Hermite polynomial, and satisfies

$$C_{1,m} < C_{2,m} < \cdots < C_{m,m}.$$

The structure of the N eigenvectors can be fully clarified.

Remark 2 Based on Lemma 1, the regularized moment system (26) is locally well-posed due to the hyperbolicity. We would like to mention that the regularization here actually does not add any new terms to system (22). On the contrary, it has erased the terms in (18) with a factor $\alpha_j + 1$ in its coefficient for the equations of f_α with $|\alpha| = M$ only.

3   Conservations

It is well known that the Vlasov-Maxwell equations, (1) and (2), conserve total number of particles, momentum, and energy, which are given, respectively, by

$$\mathscr{N} = \iint f \mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{x}, \tag{29}$$

$$\mathscr{P} = \int \left[\int \boldsymbol{v} f \mathrm{d}\boldsymbol{v} + \boldsymbol{E} \times \boldsymbol{B} \right] \mathrm{d}\boldsymbol{x}, \tag{30}$$

$$\mathscr{E} = \frac{1}{2} \int \left[\int \boldsymbol{v}^2 f \mathrm{d}\boldsymbol{v} + (\boldsymbol{E}^2 + \boldsymbol{B}^2) \right] \mathrm{d}\boldsymbol{x}. \tag{31}$$

Due to the truncation (19), all the above conservations can be expressed by the moments of first three orders:

$$\mathscr{N}_h = \iint f_h \mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{x} = \int \rho \mathrm{d}\boldsymbol{x}, \tag{32}$$

$$\mathscr{P}_h = \int \left(\int \boldsymbol{v} f_h \mathrm{d}\boldsymbol{v} + \boldsymbol{E} \times \boldsymbol{B} \right) \mathrm{d}\boldsymbol{x} = \int (\rho \boldsymbol{u} + \boldsymbol{E} \times \boldsymbol{B}) \mathrm{d}\boldsymbol{x}, \tag{33}$$

$$\mathscr{E}_h = \frac{1}{2} \int \left[\int \boldsymbol{v}^2 f_h \mathrm{d}\boldsymbol{v} + (\boldsymbol{E}^2 + \boldsymbol{B}^2) \right] \mathrm{d}\boldsymbol{x}$$

$$= \frac{1}{2} \int [\rho(3\mathscr{T} + \boldsymbol{u}^2) + (\boldsymbol{E}^2 + \boldsymbol{B}^2)] \mathrm{d}\boldsymbol{x}. \tag{34}$$

Actually, using equations (13), (14), and (16), it is straightforward to obtain the following proposition.

Proposition 1 *The semi-discrete moment expansion $f_h(t, \boldsymbol{x}, \boldsymbol{v})$ has the following conservation properties*:

$$\frac{\mathrm{d}\mathscr{N}_h}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\mathscr{P}_h}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\mathscr{E}_h}{\mathrm{d}t} = 0. \tag{35}$$

Denoting by $s \rightarrow (t(s), \boldsymbol{x}(s))$ a parametric representation of the characteristic integral curves associating with eigenvalue λ and left eigenvector $\boldsymbol{l}$, the characteristic equations become

$$\boldsymbol{l}^{\mathrm{T}}\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}s} = \boldsymbol{l}^{\mathrm{T}}(\mathbf{G}\boldsymbol{w} + \boldsymbol{g}). \tag{36}$$

So omitting the convective term $\sum_{j=1}^{3}\mathbf{M}_j(\boldsymbol{w})\mathrm{d}\boldsymbol{w}/\mathrm{d}x_j$ in (22) temporarily, let us consider the system with the source term only

$$\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}s} = \mathbf{G}\boldsymbol{w} + \boldsymbol{g}. \tag{37}$$

We first point out that the matrix $\mathbf{G}$ is block diagonal as

$$\mathbf{G} = \mathrm{diag}\{0, \mathbf{G}_1, \mathbf{G}_2, \dots, \mathbf{G}_M\},$$

where

$$\mathbf{G}_m = [\mathbf{G}_{\mathscr{N}(\alpha), \mathscr{N}(\beta)}], \quad |\alpha| = |\beta| = m, \, 1 \leqslant m \leqslant M,$$

and the nonzero entries are given by (23). We define a partition diagonal matrix

$$\mathbf{D} = \mathrm{diag}\{1, \mathbf{D}_1, \mathbf{D}_2, \dots, \mathbf{D}_M\},$$

where

$$\mathbf{D}_m = \mathrm{diag}\{\mathbf{D}_{\mathscr{N}(\alpha), \mathscr{N}(\alpha)}\}_{|\alpha|=m}, \quad \mathbf{D}_{\mathscr{N}(\alpha), \mathscr{N}(\alpha)} = \alpha\,! := \prod_{d=1}^{3}\alpha_d!. \tag{38}$$

Correspondingly, the vector $\boldsymbol{w}$ is divided into

$$\boldsymbol{w} = [\rho, \boldsymbol{u}^{\mathrm{T}}, \hat{\boldsymbol{w}}_2^{\mathrm{T}}, \dots, \hat{\boldsymbol{w}}_M^{\mathrm{T}}]^{\mathrm{T}},$$

where

$$\hat{\boldsymbol{w}}_m = [w_{\mathscr{N}(\alpha)}]^{\mathrm{T}}, \quad |\alpha| = m.$$

We have the following properties.

Proposition 2 *The solution of system* (37) *satisfies*

$$\frac{\mathrm{d}\rho}{\mathrm{d}s} = 0, \tag{39}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{2}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}\right) = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{E}, \tag{40}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{2}\hat{\boldsymbol{w}}_m^{\mathrm{T}}\mathbf{D}_m\hat{\boldsymbol{w}}_m\right) = 0, \quad 2 \leqslant m \leqslant M. \tag{41}$$

Proof There is no source term for the density ρ, so (39) is obvious. In (23), it is noticed that

$$\sum_{i,k,m=1}^{3} \varepsilon_{ikm} u_i u_k B_m = 0,$$

and thus, (40) is obtained. As for (41), we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{2}\,\hat{\boldsymbol{w}}_m^{\mathrm{T}}\mathbf{D}_m\hat{\boldsymbol{w}}_m\right) = \hat{\boldsymbol{w}}_m^{\mathrm{T}}\mathbf{D}_m\frac{\partial\hat{\boldsymbol{w}}_m}{\partial s} = \hat{\boldsymbol{w}}_m^{\mathrm{T}}\mathbf{D}_m\mathbf{G}_m\hat{\boldsymbol{w}}_m.$$

Notice that the matrix $\mathbf{D}_m\mathbf{G}_m$ is a diagonal block of the matrix $\mathbf{DG}$, and from (23), each one of its nonzero entries satisfies

$$\begin{aligned}
(\mathbf{DG})_{\mathscr{N}(\alpha),\mathscr{N}(\alpha-e_d+e_k)} &= \alpha!\,\mathbf{G}_{\mathscr{N}(\alpha),\mathscr{N}(\alpha-e_d+e_k)}\\
&= \sum_{m=1,2,3}\varepsilon_{dkm}(\alpha_k+1)B_m\alpha_d!\,\alpha_k!\,\alpha_m!\\
&= \sum_{m=1,2,3}\varepsilon_{dkm}\alpha_d B_m(\alpha_d-1)!\,(\alpha_k+1)!\,\alpha_m!\\
&= -(\alpha-e_d+e_k)!\,\mathbf{G}_{\mathscr{N}(\alpha-e_d+e_k),\mathscr{N}(\alpha)}\\
&= -(\mathbf{DG})_{\mathscr{N}(\alpha-e_d+e_k),\mathscr{N}(\alpha)}.
\end{aligned}$$

It is turned out that $\mathbf{D}_m\mathbf{G}_m$ is a skew-symmetric matrix, and thus,

$$\hat{\boldsymbol{w}}_m^{\mathrm{T}}\mathbf{D}_m\mathbf{G}_m\hat{\boldsymbol{w}}_m = 0. \tag{42}$$

This ends the proof. $\qquad\square$

Remark 3 The Coulomb force, which is the term $\boldsymbol{g}$, provides an acceleration on the mean velocity only, and the Lorentz force in $\mathbf{G}$ exerting on the mean velocity is perpendicular to the mean velocity clearly. The result in Proposition 2 indicates that the Lorentz force will not change the magnitude of the high order moments for any order $m \geqslant 2$, too, taking the matrix $\mathbf{D}_m$ as the l_2 weight. One may observe that the Lorentz force in the Vlasov equation will rotate the distribution function in the velocity space only, and here we see such behavior is preserved in the moment system we derived.

4 Exact Vlasov-Maxwell equilibrium

Knowledge of the exact Vlasov-Maxwell equilibrium is often necessary when analyzing the stability of a plasma [5]. In this section, we present a simple example of exact Vlasov-Maxwell equilibrium. This makes it possible to examine the residual of the moment system if we substitute the exact solution into the moment system. The moment system we derived is then partially validated once it is observed that the residual of the system is small enough.

For simplicity, we consider a situation in which all quantities vary only in the x_1 direction and the magnetic field is unidirectional with one component B_3 in the x_3 direction. Then the magnetic field can be derived from a potential, A_2,

$$B_3 = \frac{\mathrm{d}A_2}{\mathrm{d}x_1}, \quad B_3(-\infty) = B_0. \tag{43}$$

The equilibrium is characterized by a zero electric field. The Vlasov equation (1) becomes

$$v_1 \frac{\partial f}{\partial x_1} + v_2 \frac{\mathrm{d}A_2}{\mathrm{d}x_1} \frac{\partial f}{\partial v_1} - v_1 \frac{\mathrm{d}A_2}{\mathrm{d}x_1} \frac{\partial f}{\partial v_2} = 0. \tag{44}$$

The exact distribution function can be obtained (more details in [5]),

$$f(x_1, \boldsymbol{v}) = \frac{\beta^2 B_0^2}{8\pi^2} \left(\frac{\pi\beta}{2\gamma\delta} \right)^{1/2} \exp\left(-\frac{(v_2 + A_2(x_1))^2}{4\delta} - \frac{\beta|\boldsymbol{v}|^2}{2} \right), \tag{45}$$

where β and γ are constants, and

$$\delta = \frac{1}{4\gamma} - \frac{1}{2\beta}.$$

Maxwell's equations (2) for the magnetic field become

$$\frac{\mathrm{d}^2 A_2}{\mathrm{d}x_1^2} = -B_0^2 \gamma A_2 \exp(-\gamma A_2^2). \tag{46}$$

Let us calculate the moments and plug them into our moment equations. Then we find that the residual is going to zero as the order goes to infinity. With the expression (45), the density $\rho(x_1)$, the mean velocity $\boldsymbol{u}(x_1)$, and the temperature $\mathscr{T}(x_1)$ are calculated:

$$\rho(x_1) = \int_{\mathbb{R}^3} f(t, x_1, \boldsymbol{v})\mathrm{d}\boldsymbol{v} = \frac{\beta}{2} B_0^2 \exp(-\gamma A_2^2(x_1)), \tag{47}$$

$$\boldsymbol{u}(x_1) = \frac{1}{\rho} \int_{\mathbb{R}^3} \boldsymbol{v} f(t, x_1, \boldsymbol{v})\mathrm{d}\boldsymbol{v} = \left(0, \frac{-2\gamma}{\beta} A_2(x_1), 0 \right)^{\mathrm{T}}, \tag{48}$$

$$\mathscr{T}(x_1) = \frac{1}{3\rho} \int_{\mathbb{R}^3} (\boldsymbol{v} - \boldsymbol{u})^2 f(t, x_1, \boldsymbol{v})\mathrm{d}\boldsymbol{v} = \left(\frac{1}{\beta} - \frac{2\gamma}{3\beta^2} \right). \tag{49}$$

Then the equilibrium distribution function $f(\boldsymbol{x}, \boldsymbol{v})$ can be expanded into the Hermite series:

$$f(\boldsymbol{x}, \boldsymbol{v}) = \sum_{\alpha} f_\alpha(\boldsymbol{x}) \mathscr{H}_{\mathscr{T}, \alpha}\left(\frac{\boldsymbol{v} - \boldsymbol{u}(\boldsymbol{x})}{\sqrt{\mathscr{T}(\boldsymbol{x})}} \right), \tag{50}$$

where

$$f_\alpha(\boldsymbol{x}) = \begin{cases} \dfrac{\left(\frac{2\gamma}{3\beta^2}\right)^{\alpha_1/2}}{(\alpha_1)!!} \dfrac{\left(\frac{-4\gamma}{3\beta^2}\right)^{\alpha_2/2}}{(\alpha_2)!!} \dfrac{\left(\frac{2\gamma}{3\beta^2}\right)^{\alpha_3/2}}{(\alpha_3)!!} \dfrac{\beta}{2} B_0^2 \exp(-\gamma A_2^2(x_1)), & \alpha_i \text{ even,} \\ 0, & \text{otherwise.} \end{cases} \tag{51}$$

It is obvious that

$$\lim_{|\alpha|\to+\infty} f_\alpha(\boldsymbol{x}) = 0. \tag{52}$$

Actually, all the moment equations, which are not modified by the closure, are satisfied by the moments of $f(\boldsymbol{x}, \boldsymbol{v})$. In the regularized moment system with the truncation order M, it is only required to examine the moment equations of order $|\alpha| = M$, which has been modified due to the truncation and closure. Substituting the exact moments into the regularized moment system and calculating the residual yields

$$\mathrm{Res}(\alpha) = \sum_{j=1}^{3} \mathscr{R}_M^j(\alpha) + \sum_{j=1}^{3}(\alpha_j+1)\frac{\partial f_{\alpha+e_j}}{\partial x_j}, \tag{53}$$

where the closure term $\mathscr{R}_M^j(\alpha)$ is defined in (24) and the truncation term

$$\sum_{j=1}^{3}(\alpha_j+1)\frac{\partial f_{\alpha+e_j}}{\partial x_j}$$

is easy to be identified by observing the original moment equation (18). The residue (53) is reduced into

$$\mathrm{Res}(\alpha) = (\alpha_1+1)\Big(\frac{\mathrm{d}u_2}{\mathrm{d}x_1}f_{\alpha-e_2+e_1} + \frac{\partial f_{\alpha+e_1}}{\partial x_1}\Big)$$

$$= \begin{cases} \dfrac{\big(\frac{2\gamma}{3\beta^2}\big)^{(\alpha_1+1)/2}\big(\frac{-4\gamma}{3\beta^2}\big)^{(\alpha_2-1)/2}\big(\frac{2\gamma}{3\beta^2}\big)^{\alpha_3/2}}{(\alpha_1-1)!!\,(\alpha_2-1)!!\,\alpha_3!!}\,\rho\,\dfrac{\mathrm{d}u_2}{\mathrm{d}x_1}, \\ \qquad\qquad \alpha = (2k_1-1, 2k_2+1, 2k_3),\ k_i > 0, \\ \dfrac{(\mathscr{T}_1-\mathscr{T})^{(\alpha_1+1)/2}(\mathscr{T}_2-\mathscr{T})^{\alpha_2/2}(\mathscr{T}_3-\mathscr{T})^{\alpha_3/2}}{(\alpha_1-1)!!\,\alpha_2!!\,\alpha_3!!}\,\dfrac{\mathrm{d}\rho}{\mathrm{d}x_1}, \\ \qquad\qquad \alpha = (2k_1-1, 2k_2, 2k_3),\ k_i > 0, \\ 0, \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{otherwise.} \end{cases}$$

The residue goes to zero as the truncation order M going to infinity, i.e.,

$$\lim_{|\alpha|\to+\infty} \mathrm{Res}(\alpha) = 0. \tag{54}$$

5 Conclusion

In summary, the hyperbolic moment system has been derived for the Vlasov-Maxwell equations. We have proven that the moment method conserves charge, conserves momentum, conserves energy and is stable. The corresponding numerical method is developing for the moment system obtained, and will be applied to study important plasma physics problems in the future.

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