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RESEARCH ARTICLE

Lie bialgebra structure on cyclic cohomology of Fukaya categories

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Abstract Let M be an exact symplectic manifold with contact type boundary such that $c_1(M) = 0$. Motivated by noncommutative symplectic geometry and string topology, we show that the cyclic cohomology of the Fukaya category of M has an involutive Lie bialgebra structure.

Keywords Fukaya category, cyclic cohomology, Lie bialgebra
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1 Introduction

During the past two decades, great achievements have been obtained in the understanding of the geometry and topology of symplectic manifolds. Among them is the Fukaya category of symplectic manifolds, which is originated by Fukaya [7]. In this paper, we study some algebraic structures on the Fukaya category of an exact symplectic manifold with contact type boundary. Let us start with some backgrounds and motivations.

Roughly speaking, Fukaya category is an algebraic structure arising in the study of symplectic manifolds, where the objects are the Lagrangian submanifolds and the morphisms are the Lagrangian intersection Floer cochain complexes. As has been shown by Fukaya [7], the composition of two morphisms is in general not associative, but associative up to homotopy. There are homotopy of homotopies, and homotopy of homotopies of homotopies, etc., thus forming an A_∞ category, which is a categorical generalization of Stasheff's A_∞ algebra.

Ever since its first appearance, Fukaya categories have been extensively

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studied in the literature. It is an important class of noncommutative symplectic spaces in Kontsevich's homological mirror symmetry program. In fact, Kontsevich [20] conjectured that the Fukaya category of a Calabi-Yau manifold is a Calabi-Yau A_∞ category, which means that there is a non-degenerate symmetric bilinear pairing on the morphism space,

$$\langle \cdot, \cdot \rangle \colon \mathrm{Hom}(A, B) \otimes \mathrm{Hom}(B, A) \to k$$

for any objects A and B, such that it is cyclically invariant:

$$\langle a_0, m_n(a_1, \cdots, a_n) \rangle = \pm \langle a_n, m_n(a_0, \ldots, a_{n-1}) \rangle. \tag{1}$$

See also Costello [6]. Kontsevich's homological mirror symmetry conjecture says that, (the derived category of) the Fukaya category of a Calabi-Yau manifold should be equivalent, as Calabi-Yau categories, to the (derived category of) coherent sheaves of its mirror pair, and vice versa.

While the existence of the cyclically invariant inner product on the Fukaya category is still under verification (for some partial results, see [9]), some algebraic structures arising from Calabi-Yau categories may still exist on a general Fukaya category. Our main theorem in the following says that there is a graded involutive Lie bialgebra structure on the cyclic cohomology of the Fukaya category (no non-degenerate pairing is assumed).

Theorem A *Let M^{2d} be an exact symplectic manifold (possibly with contact type boundary) such that $c_1(M) = 0$. Denote by* $\mathrm{Fuk}(M)$ *the Fukaya category of M, and grade the corresponding Floer cochain complex negatively. Then the cyclic cochain complex of* $\mathrm{Fuk}(M)$ *has a structure of differential graded involutive Lie bialgebra of degree $(2-d, 2-d)$, and thus induces a structure of involutive Lie bialgebra of the same degree on the cyclic cohomology of* $\mathrm{Fuk}(M)$.

The Lie bialgebra structure is motivated by the following several results. In two very inspiring papers [18,19], Kontsevich first proposed his theory of noncommutative symplectic geometry, and proved that for a noncommutative symplectic space, the space of noncommutative functions on it forms a Lie algebra. Such a Lie algebra is nowadays called the Kontsevich bracket, and has motivated the work of Ginzburg [15] and Bocklandt-Le Bruyn [2], where the commutator quotient space of the path algebra of a doubled quiver (the space of closed paths) has been proved to be a Lie algebra. This Lie algebra is in the same form as that of Kontsevich and is now called the 'necklace' Lie algebra. Both the Kontsevich Lie algebra and the necklace Lie algebra were later improved to be an involutive Lie bialgebra, by Hamilton [16] and Schedler [26], respectively.

From these works, one also obtains that for a Calabi-Yau A_∞ category, there is an involutive Lie bialgebra structure on its cyclic cohomology. A natural question is then whether such a Lie bialgebra exists on the cyclic cohomology of a Fukaya category or not, even though the Calabi-Yau property (1) is to be verified. This question leads to our discovery of Theorem A. In fact, the Lie

algebra in Theorem A is very much related to Kontsevich's noncommutative symplectic geometry, and in a separate paper, we will study this Lie algebra from the noncommutative Poisson geometry point of view. The key point is that, in the construction of the Fukaya category, the Floer cochain complex between two objects is finite dimensional with a canonical basis, and counting the pseudo-holomorphic disks is *a priori* cyclically invariant.

The rest of this paper is devoted to the proof of Theorem A. It is organized as follows. In Section 2, we collect some basic facts on the Fukaya category of exact symplectic manifolds. In Section 3, we prove Theorem A. In Section 4, we study the example of cotangent bundles and relate it with string topology, which is another motivation of the present work.

2 Fukaya category of exact symplectic manifolds

We collect some necessary facts about Fukaya categories from [8,10,27]. Most statements are cited without proof, but are given with precise reference. Since some key facts will be used later, it seems appropriate to repeat here to fix notations.

Throughout the paper, we assume that k is a field of characteristic zero. All vector spaces and their tensor products are over k.

2.1 A_∞ categories and their homologies

A_∞ categories are categorical generalization of A_∞ algebras; they may be viewed as A_∞ algebras with 'several objects'. It is introduced by Fukaya in [7]; one may also refer to Seidel [27, Chapter 1] for more details.

Definition 1 (A_∞ category) An A_∞ *category* $\mathscr{A}$ consists of a set of objects $\mathrm{Ob}(\mathscr{A})$, a graded vector space $\mathrm{Hom}(A_1, A_2)$ for each pair of objects $A_1, A_2 \in \mathrm{Ob}(\mathscr{A})$, and a sequence of operators:

$$m_n \colon \mathrm{Hom}(A_1, A_2) \otimes \mathrm{Hom}(A_2, A_3) \otimes \cdots \otimes \mathrm{Hom}(A_n, A_{n+1}) \to \mathrm{Hom}(A_1, A_{n+1}),$$

where the degree $|m_n| = 2 - n$ for $n = 1, 2, \ldots$, satisfying the following A_∞ relations:

$$\sum_{p=1}^{n} \sum_{k=1}^{n-p+1} (-1)^{\mu_{npk}} m_{n-k+1}(a_1, \ldots, a_{p-1}, m_k(a_p, \ldots, a_{p+k-1}), a_{p+k}, \ldots, a_n) = 0, \tag{2}$$

where $a_i \in \mathrm{Hom}(A_i, A_{i+1})$ for $i = 1, 2, \ldots, n$,

$$\mu_{npk} = \sum_{r=1}^{n} (r-1)|a_r| + \sum_{s=1}^{p-1} k|a_s| + p(2-k),$$

and $|a_r|$ and $|a_s|$ are the gradings of the corresponding elements.

If an A_∞ category has one object, say A, then $\mathrm{Hom}(A,A)$ is an A_∞ *algebra*; and if furthermore, all m_i, $i \geqslant 3$, vanish, then $\mathscr{A}$ is the usual differential graded (DG) algebra.

Convention (The signs) The sign in equation (2) is given as follows. First, for a graded vector space V, let $\overline{V}$ be the desuspension of V, that is, $(\overline{V})_i = V_{i+1}$. Let $\Sigma\colon V \to \overline{V}$ be the identity map which maps v to $\overline{v}$, and let

$$\begin{aligned}\Sigma^{\otimes n}\colon\ & V\otimes\cdots\otimes V \to \overline{V}\otimes\cdots\otimes\overline{V},\\ & v_1\otimes\cdots\otimes v_n \mapsto (-1)^{(n-1)|v_1|+(n-2)|v_2|+\cdots+|v_{n-1}|}\overline{v}_1\otimes\cdots\otimes\overline{v}_n,\end{aligned}$$

be the n-fold tensor of Σ. Let $\overline{m}_n\colon (\overline{V})^{\otimes n} \to \overline{V}$ be the degree 1 map such that the diagram

$$\begin{array}{ccc} V\otimes\cdots\otimes V & \xrightarrow{m_n} & V \\ \big\downarrow{\scriptstyle \Sigma^{\otimes n}} & & \big\downarrow{\scriptstyle \Sigma} \\ \overline{V}\otimes\cdots\otimes\overline{V} & \xrightarrow{\overline{m}_n} & \overline{V} \end{array} \tag{3}$$

commutes. Then equation (2) is nothing but

$$\begin{aligned}\sum_{p=1}^{n}\sum_{k=1}^{n-p+1}(-1)^{|\overline{a}_1|+\cdots+|\overline{a}_{p-1}|}\overline{m}_{n-k+1}(\overline{a}_1,\ldots,\overline{a}_{p-1},\\ \overline{m}_k(\overline{a}_p,\ldots,\overline{a}_{p+k-1}),\overline{a}_{p+k},\ldots,\overline{a}_n)=0.\end{aligned} \tag{4}$$

The sign that appears in equation (4) follows from the usual Koszul convention rule. Namely, the canonical isomorphism

$$V\otimes W \xrightarrow{\cong} W\otimes V$$

is given by

$$a\otimes b \mapsto (-1)^{|a|\,|b|}b\otimes a.$$

One then obtains equation (2) by converting equation (4) via diagram (3). In the following, all signs are assigned in this way, and therefore, we will just simply write $\pm$, without specifying their particular value.

Definition 2 (Hochschild homology) Let $\mathscr{A}$ be an A_∞ category. The *Hochschild chain complex* $\mathrm{CH}_\bullet(\mathscr{A})$ of $\mathscr{A}$ is the chain complex whose underlying vector space is

$$\bigoplus_{n=1}^{\infty}\bigoplus_{A_1,A_2,\ldots,A_{n+1}\in\mathscr{O}b(\mathscr{A})}\overline{\mathrm{Hom}}(A_1,A_2)\otimes\overline{\mathrm{Hom}}(A_2,A_3)\otimes\cdots\otimes\overline{\mathrm{Hom}}(A_{n+1},A_1)$$

with differential $b = b' + b''$, where

$$\begin{aligned}&b'(\overline{a}_1,\overline{a}_2,\ldots,\overline{a}_{n+1})\\ &\quad=\sum_{k=1}^{n+1}\sum_{i=1}^{n-k+2}\pm(\overline{a}_1,\ldots,\overline{a}_{k-1},\overline{m}_i(\overline{a}_k,\ldots,\overline{a}_{k+i-1}),\overline{a}_{k+i},\ldots,\overline{a}_{n+1}),\end{aligned}$$

$$b''(\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_{n+1}) = \sum_{j=0}^{n-1} \sum_{i=1}^{n-j} \pm(\overline{m}_{i+j+1}(\overline{a}_{n-j+1}, \ldots, \overline{a}_{n+1}, \overline{a}_1, \ldots, \overline{a}_i), \overline{a}_{i+1} \ldots, \overline{a}_{n-j}).$$

The associated homology is called the *Hochschild homology* of $\mathscr{A}$, and is denoted by $\mathrm{HH}_\bullet(\mathscr{A})$.

We are also interested in the dual complex of the Hochschild b-complex $(\mathrm{CH}_\bullet(\mathscr{A}), b)$, denoted by $(\mathrm{CH}^\bullet(\mathscr{A}), b)$. This is related to the cyclic (co)homology of $\mathscr{A}$. Let

$$\begin{aligned} t_n \colon & \overline{\mathrm{Hom}}(A_1, A_2) \otimes \overline{\mathrm{Hom}}(A_2, A_3) \otimes \cdots \otimes \overline{\mathrm{Hom}}(A_{n+1}, A_1) \\ & \to \overline{\mathrm{Hom}}(A_{n+1}, A_1) \otimes \overline{\mathrm{Hom}}(A_1, A_2) \otimes \cdots \otimes \overline{\mathrm{Hom}}(A_n, A_{n+1}) \end{aligned}$$

for $n = 0, 1, 2, \ldots$, be the multilinear map

$$t_n(\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_{n+1}) := (-1)^{|\overline{a}_{n+1}|(|\overline{a}_1|+\cdots+|\overline{a}_n|)}(\overline{a}_{n+1}, \overline{a}_1, \ldots, \overline{a}_n). \tag{5}$$

Let

$$N_n = 1 + t_n + t_n^2 + \cdots + t_n^n.$$

Extend t_n and N_n to other elements in $\mathrm{CH}_\bullet(\mathscr{A})$ trivially, and let

$$T = t_0 + t_1 + t_2 + \cdots, \quad N = N_0 + N_1 + N_2 + \cdots.$$

Lemma 3 *Let $\mathscr{A}$ be an A_∞ category, and let T and N be as above. We have the following commutative diagram:*

$$\begin{array}{ccccc} \overline{\mathrm{Hom}}(\mathscr{A}, \mathscr{A})^{\otimes n} & \xrightarrow{N} & \overline{\mathrm{Hom}}(\mathscr{A}, \mathscr{A})^{\otimes n} & \xrightarrow{1-T} & \overline{\mathrm{Hom}}(\mathscr{A}, \mathscr{A})^{n} \\ \downarrow b & & \downarrow b' & & \downarrow b \\ \bigoplus_{m=1}^{n} \overline{\mathrm{Hom}}(\mathscr{A}, \mathscr{A})^{\otimes m} & \xrightarrow{N} & \bigoplus_{m=1}^{n} \overline{\mathrm{Hom}}(\mathscr{A}, \mathscr{A})^{\otimes m} & \xrightarrow{1-T} & \bigoplus_{m=1}^{n} \overline{\mathrm{Hom}}(\mathscr{A}, \mathscr{A})^{\otimes m}. \end{array}$$

Proof This is the A_∞ version of cyclic bicomplex (c.f. [22, §2.1.2]), and the proof is left for the interested reader. □

Definition 4 (Cyclic (co)homology) Suppose that $\mathscr{A}$ is an A_∞ category. The cokernel $\mathrm{CH}_\bullet(\mathscr{A})/(1-T)$ of $1-T$ forms a chain complex with the induced differential from the Hochschild b-complex (still denoted by b). Such a chain complex is denoted by $\mathrm{CC}_\bullet(\mathscr{A})$, and is called the *Connes cyclic complex* of $\mathscr{A}$. Its homology is called the *cyclic homology* of $\mathscr{A}$, and is denoted by $\mathrm{HC}_\bullet(\mathscr{A})$.

The *cyclic cochain complex* of $\mathscr{A}$ is the dual cochain complex of $\mathrm{CC}_\bullet(\mathscr{A})$, and is denoted by $\mathrm{CC}^\bullet(\mathscr{A})$. Namely, suppose $f \in \mathrm{CH}^\bullet(\mathscr{A})$. Then $f \in \mathrm{CC}^\bullet(\mathscr{A})$ if and only if for all $\alpha \in \mathrm{CH}_\bullet(\mathscr{A})$,

$$f(\alpha) = f(T(\alpha)).$$

The corresponding cohomology is called the *cyclic cohomology* of $\mathscr{A}$ and is denoted by $\mathrm{HC}^\bullet(\mathscr{A})$.

One may have noticed that from the above definition, the degree of the Hochschild and cyclic homology differs from the usual one by one. It is convenient for us to keep track of the signs.

2.2 Construction of Fukaya category

In this subsection, we briefly recall the construction of the Fukaya category for *exact* symplectic manifolds. The complete treatment can be found in Seidel [27]. The construction of the Fukaya category of a general symplectic manifold was given by Fukaya [8, Chapter 1], largely based on [10].

Intuitively, the Fukaya category $\mathrm{Fuk}(M)$ of M is defined as follows: the objects are Lagrangian submanifolds in M; suppose that L_1 and L_2 are two transversal objects, $\mathrm{Hom}(L_1, L_2)$, called the *Floer cochain complex*, is spanned by the transversal intersection points of L_1 and L_2, and for $n+1$ objects $L_1, L_2, \ldots, L_{n+1}$,

$$\overline{m}_n \colon \overline{\mathrm{Hom}}(L_1, L_2) \otimes \overline{\mathrm{Hom}}(L_2, L_3) \otimes \cdots \otimes \overline{\mathrm{Hom}}(L_n, L_{n+1}) \to \overline{\mathrm{Hom}}(L_1, L_{n+1})$$

is given by counting pseudo-holomorphic disks whose boundary lies in $L_1, L_2, \ldots, L_{n+1}$. More precisely, if $a_i \in \mathrm{Hom}(L_i, L_{i+1})$, $i = 1, 2, \ldots, n$, then

$$\overline{m}_n(\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n) = \sum_{a \in L_1 \cap L_{n+1}} \#(\mathscr{M}(a, a_1, \ldots, a_n)) \cdot \overline{a},$$

where $\mathscr{M}(a, a_1, \ldots, a_n)$ is the moduli space of pseudo-holomorphic disks with $n+1$ (anti-clockwise) cyclically ordered marked points in its boundary, such that these marked points are mapped onto $a, a_n, \ldots, a_1$ and that the rest of the boundary lie in $L_1, L_2, \ldots, L_{n+1}$. The A_∞ relations (equation (2)) follow from the compactification of $\mathscr{M}(a, a_1, \ldots, a_n)$, where those pseudo-holomorphic disks with all possible 'bubbling-off' disks are added.

This is a very rough description of the construction of the Fukaya category. It is only partially defined in the sense that we have assumed that all Lagrangian submanifolds are transversal; also, the Floer cochains thus described are only $\mathbb{Z}_2$ graded. To make the Fukaya category be fully defined and be graded over $\mathbb{Z}$, we have to introduce the following concepts.

2.2.1 Exact symplectic manifolds and admissible Lagrangian submanifolds

A symplectic manifold (M^{2d}, ω) is said to be *exact* if $\omega = \mathrm{d}\eta$ for some 1-form η. An *exact symplectic manifold with boundary* is a quadruple (M, ω, η, J), where M is a compact $2d$-dimensional manifold with boundary, ω is a symplectic 2-form on M, η is a 1-form such that $\omega = \mathrm{d}\eta$, and J is an ω-compatible almost complex structure. These data also satisfy the following two convexity conditions:

- the negative Liouville vector field defined by $\omega(\cdot, X_\eta) = \eta$ points strictly inwards along the boundary of M;

• the boundary of M is weakly J-convex, which means that any pseudo-holomorphic curves cannot touch the boundary unless they are completely contained in it.

In the following, for a symplectic manifold (M^{2d}, ω) with or without boundary, we shall always assume $c_1(M) = 0$.

A d-dimensional submanifold $L \subset M$ is called *Lagrangian* if $\omega|_L = 0$. In the following, we will assume that L is *closed* and is disjoint from the boundary of M. L is called *exact* if $\eta|_L$ is an exact 1-form. In the following, we shall always assume that L is *admissible*, namely,

(i) $\eta|_L$ is exact;

(ii) L has vanishing Maslov class;

(iii) L is spin.

These three conditions guarantee that the Floer cochain complex is defined over a field of characteristic zero and is $\mathbb{Z}$-graded, and that the moduli spaces involved are oriented in a coherent way such that the A_∞ hierarchy equations are satisfied. For more details, see [8,10,27].

Example 5 (Cotangent bundles) Let N be a simply connected, compact spin manifold. Let T^*N be the cotangent bundle of N with the canonical symplectic structure. The cotangent bundle of N is an exact symplectic. In particular, N, viewed as the zero section of T^*N, is an admissible Lagrangian submanifold.

2.2.2 Pointed-boundary Riemann surfaces

Suppose that $\widehat{S}$ is a Riemann surface with boundary, and Σ is a set of boundary points in $\widehat{S}$. Σ is divided into two subsets Σ^+ and Σ^-, called the *output* subset and *input* subset. Now, to each $\zeta \in \Sigma^\pm$, one associates the following submanifolds and end.

• Two admissible Lagrangian submanifolds $(L_{\zeta 0}, L_{\zeta 1})$, where $L_{\zeta 0}$ is uniquely attached to the boundary component (with induced orientation) of $S := \widehat{S} \backslash \Sigma$ that comes before ζ and $L_{\zeta 1}$ is uniquely attached to the boundary component that comes after ζ if $\zeta \in \Sigma^+$; otherwise, if $\zeta \in \Sigma^-$, then $L_{\zeta 1}$ comes before $L_{\zeta 0}$. These Lagrangian submanifolds are called the *Lagrangian labels*.

• A *strip-like end* for each $\zeta \in \Sigma^\pm$, which is a proper holomorphic embedding $\varepsilon_\zeta \colon Z^\pm \to S$ satisfying

$$\varepsilon_\zeta^{-1}(\partial S) = \mathbb{R}^\pm \times \{0, 1\}, \quad \lim_{s \to \pm\infty} \varepsilon_\zeta(s, \cdot) = \zeta, \tag{6}$$

where $Z^\pm = \mathbb{R}^\pm \times [0, 1]$ denotes the semi-infinite strips. If Σ consists of more than one point, we also need the additional requirement that the images of the ε_ζ are pairwise disjoint. Such an S is called a *pointed-boundary Riemann surface with strip-like ends*.

2.2.3 Floer data and perturbation data

Suppose that (M, ω, J) is an exact symplectic manifold (with or without boundary). Let $\mathscr{J}$ be the space of all ω-compatible almost complex structures

on M which agree with the given J near the boundary; and let

$$\mathscr{H} = C_c^\infty(\mathrm{int}(M), \mathbb{R})$$

be the space of smooth functions on M vanishing near the boundary.

Definition 6 (Floer datum) For each *ordered* pair of Lagrangian submanifolds $L_0, L_1 \subset M$, a *Floer datum* consists of $H_{L_0,L_1} \in C^\infty([0,1], \mathscr{H})$ and $J_{L_0,L_1} \in C^\infty([0,1], \mathscr{J})$, with the following property: if X is the time-dependent Hamiltonian vector field of H and ϕ is its flow, then $\phi^1(L_0)$ intersects L_1 transversally.

Definition 7 (Perturbation datum) Let S be a pointed-boundary Riemann surface with Lagrangian labels. Suppose that we have chosen strip-like ends for it, and also a Floer datum (H_ζ, J_ζ) for each of the pairs of submanifolds $(L_{\zeta 0}, L_{\zeta 1})$ associated to the points at infinity $\zeta \in \Sigma$. A *perturbation datum* for S is a pair (K, J), where

- $K \in \Omega^1(S, \mathscr{H})$ satisfies $K(\xi)|L_C = 0$, for all $\xi \in TC$, where C is a component of ∂S, and
- J is a family of almost complex structures $J \in C^\infty(S, \mathscr{J})$,

such that they are compatible with the chosen strip-like ends and Floer data, in the sense that

$$\varepsilon_\zeta^* K = H_\zeta(t)\mathrm{d}t, \quad J(\varepsilon_\zeta(s,t)) = J_\zeta(t)$$

for each $\zeta \in \Sigma^\pm$ and $(s,t) \in Z^\pm$.

For convenience, we call a Floer datum together with a perturbation datum the *analytic data*.

2.2.4 Grading of Lagrangian submanifolds

Let $(\mathbb{R}^{2n}, \omega)$ be the standard symplectic vector space. Denote by

$$\mathrm{Lag}_n = \mathrm{Lag}(\mathbb{R}^{2n}, \omega)$$

the set of all linear Lagrangian subspaces. It is well known that $\pi_1(\mathrm{Lag}_n) \cong \mathbb{Z}$ (c.f. [23, Theorem 2.31]). Denote by $\widetilde{\mathrm{Lag}}_n$ the universal covering of Lag_n.

Now, suppose that (M^{2n}, ω) is a symplectic manifold. Then to each $p \in M$ is associated $\mathrm{Lag}(T_pM)$, and one obtains a fiber bundle, denoted by $\mathrm{Lag}(M) \to M$.

Lemma 8 [8, Lemma 2.6] *There exists a covering* $\widetilde{\mathrm{Lag}}(M)$ *of* $\mathrm{Lag}(M)$ *such that its restriction to each fiber is identified with* $\widetilde{\mathrm{Lag}}_n \to \mathrm{Lag}_n$ *if and only if* $c_1(M) = 0$.

From now on, we fix a covering $\widetilde{\mathrm{Lag}}(M)$ as in Lemma 8. Now, suppose that L is a Lagrangian submanifold. Then there is a canonical section s of the restriction of $\mathrm{Lag}(M)$ to L, which is given by $s(p) = T_pL \subset T_pM$.

Definition 9 (Grading of Lagrangian submanifolds) A *graded Lagrangian submanifold* of $(M, \widetilde{\mathrm{Lag}}(M))$ is an oriented Lagrangian submanifold L and a lift of s to $\widetilde{s}\colon L \to \widetilde{\mathrm{Lag}}(M)$. The lifting $\widetilde{s}$ is called the *grading* of L, and denote by $\widetilde{L}$ the pair $(L, \widetilde{s})$.

The grading of a Lagrangian submanifold is related to its Maslov class as follows. Let $\phi\colon (D^2, \partial D^2) \to (M, L)$ be a map representing $\pi_2(M, L)$. For each $t \in \partial D^2$, we have a Lagrangian subspace $T_{\phi(t)}L \subset T_{\phi(t)}M$, which gives a map $S^1 \to \mathrm{Lag}_n$. It determines an element in $\pi_1(\mathrm{Lag}_n) \cong \mathbb{Z}$, which is called the *Maslov index* of $[\phi]$, and is denoted by $\mu([\phi])$. Under the assumption that $c_1(M) = 0$, μ can be extended to $\pi_1(L)$ as follows. By Lemma 8, there exists a lift $\widetilde{\mathrm{Lag}}(M) \to M$. Let $\gamma\colon S^1 \to L$ be a representation of an element of $\pi_1(L)$. Define a map $\gamma^+\colon S^1 \to \mathrm{Lag}(M)$ by

$$\gamma^+(t) = T_{\gamma(t)}L \in \mathrm{Lag}(T_pM).$$

Since $\widetilde{\mathrm{Lag}}(M) \to \mathrm{Lag}(M)$ is a covering, we have a lift $\widetilde{\gamma}^+\colon [0,1] \to \widetilde{\mathrm{Lag}}(M)$ of γ^+. By the fact that $\widetilde{\mathrm{Lag}}_n/\mathbb{Z} = \mathrm{Lag}_n$, there exists $\overline{\mu}(\gamma) \in \mathbb{Z}$ such that

$$\overline{\mu}(\gamma) \cdot \widetilde{\gamma}^+(0) = \widetilde{\gamma}^+(1).$$

The map $\overline{\mu}\colon \pi_1(L) \to \mathbb{Z}$ is called the *Maslov class* of L.

Lemma 10 [8, Lemma 2.14] *Suppose $c_1(M) = 0$. Then there exists a lift $\widetilde{s}$ of $s\colon L \to \mathrm{Lag}(M)$ if and only if the Maslov class $\overline{\mu}\colon \pi_1(L) \to \mathbb{Z}$ is zero.*

2.2.5 Definition of a Floer cochain

Suppose that two graded $\widetilde{L}_1$ and $\widetilde{L}_2$ intersect transversally and $p \in \widetilde{L}_1 \cap \widetilde{L}_2$. We next define a grading $\eta_{\widetilde{L}_1, \widetilde{L}_2}(p)$ for p. Let

$$Y := D^2 \cup \{x + y\sqrt{-1} \mid x \geqslant 0,\ y \in [-1, 1]\} \subset \mathbb{C}.$$

The boundary ∂Y is identified with $\mathbb{R}$, where $\infty - \sqrt{-1}$ corresponds to $-\infty$ and $\infty + \sqrt{-1}$ corresponds to $+\infty$. Define a path $\widetilde{l}\colon \mathbb{R} \to \widetilde{\mathrm{Lag}}(T_pM)$ such that

$$\widetilde{l}(-\infty) = \widetilde{s}_1(p), \quad \widetilde{l}(+\infty) = \widetilde{s}_2(p),$$

where $\widetilde{s}_1$ and $\widetilde{s}_2$ are the gradings of the Lagrangian submanifolds L_1 and L_2. Assume that $\widetilde{l}(t)$ is locally constant if $|t| > T$.

Lemma 11 [8, Lemma 3.9] *Let $l = \pi \circ \widetilde{l}$ and*

$$W^{1,k}(Y, T_pM; l) := \{u \in W^{1,k}(Y, T_pM) \mid u(x) \in l(x) \text{ if } x \in \partial Y \cong \mathbb{R}\}.$$

Then

$$\overline{\partial}\colon W^{1,k}(Y, T_pM; l) \to W^{0,k}(Y, T_pM \otimes \Lambda^{0,1}Y) \tag{7}$$

is a Fredholm operator.

Definition 12 (Floer cochain) If $\widetilde{L}_1$ and $\widetilde{L}_2$ intersect transversally, then the *grading* $\eta_{\widetilde{L}_1,\widetilde{L}_2}(p)$ of p is defined to be the index of $\overline{\partial}$ in Lemma 11. More generally, for two arbitrary $\widetilde{L}_1$ and $\widetilde{L}_2$ with analytic data, let

$$\mathrm{Hom}(\widetilde{L}_1,\widetilde{L}_2) := \mathrm{Span}\Big\{y\colon [0,1]\to M \mid y(0)\in L_1,\ y(1)\in L_2,\ \frac{\mathrm{d}y}{\mathrm{d}t}=X(t,y(t))\Big\}, \tag{8}$$

where the grading of y, when viewed as the intersection point p of $\widetilde{L}_1$ and $\phi(\widetilde{L}_2)$, is defined to be the grading $\eta_{\widetilde{L}_1,\phi(\widetilde{L}_2)}(p)$. An element in $\mathrm{Hom}(\widetilde{L}_1,\widetilde{L}_2)$ is called a *Floer cochain* of $\widetilde{L}_1$ and $\widetilde{L}_2$, and y is sometimes called a *Hamiltonian chord*.

Lemma 13 [8, Lemma 2.27] *If $\widetilde{L}_1$ and $\widetilde{L}_2$ intersect transversally, then one may choose H_{L_0,L_1} to be zero, and $p\in\mathrm{Hom}(\widetilde{L}_1,\widetilde{L}_2)$ implies the same p lies in $\mathrm{Hom}(\widetilde{L}_2,\widetilde{L}_1)$; to distinguish them, we write $p^*\in\mathrm{Hom}(\widetilde{L}_2,\widetilde{L}_1)$. We have*

$$\eta_{\widetilde{L}_0,\widetilde{L}_1}(p)+\eta_{\widetilde{L}_1,\widetilde{L}_0}(p^*)=\frac{1}{2}\dim M.$$

2.2.6 Moduli space of pseudo-holomorphic disks

Take a pointed-boundary disk S with Lagrangian labels. Equip it with strip-like ends, Floer data (H_ζ,J_ζ) for each point at infinity, and a compatible perturbation datum (K,J). K determines a vector-field-valued 1-form $Y\in\Omega^1(S,C^\infty(TM))$: for each $\xi\in TS$, $Y(\xi)$ is the Hamiltonian vector field of $K(\xi)$. The *inhomogeneous pseudo-holomorphic map* equation for $u\in C^\infty(S,M)$ is

$$\begin{cases} Du(z)+J(z,u)\circ Du(z)\circ I_S = Y(z,u)+J(z,u)\circ Y(z,u)\circ I_S, \\ u(C)\subset L_C, \quad \forall\, C\subset\partial S, \end{cases} \tag{9}$$

where I_S is the complex structure on S.

By varying the complex structures on S (we require that, at infinity, the complex structures is fixed), one obtains a universal family of pointed-boundary disks with strip-like ends, equipped with Lagrangian labels. For such a family, one may choose a family of consistent perturbation data (for the existence, see Seidel [27, §9i]). Now, suppose

$$a_1\in\mathrm{Hom}(\widetilde{L}_1,\widetilde{L}_2),\ \ldots,\ a_n\in\mathrm{Hom}(\widetilde{L}_n,\widetilde{L}_{n+1}),\ a_{n+1}\in\mathrm{Hom}(\widetilde{L}_1,\widetilde{L}_{n+1}).$$

Let

$$\mathscr{M}(a_1,a_2,\ldots,a_{n+1}) := \{u\in C^\infty(S,M) \mid u \text{ satisfies (9) and the strip-like ends converge to } a_1,\ldots,a_{n+1}, \text{ respectively}\}$$

be the moduli space of solutions to (9).

2.2.7 Compactification and orientation of moduli spaces

Theorem 14 [27, §9i] *$\mathscr{M}$ admits a natural compactification and coherent orientation.*

The compactification of $\mathscr{M}(a_1, a_2, \ldots, a_{n+1})$ is a smooth stratified space (manifold with corners), where the corners consist of all possible pseudo-holomorphic disks with 'bubbling-off' disks. Its codimension one strata consist of

$$\bigcup_{1\leqslant i<j\leqslant n+1} \bigcup_{b\in \mathrm{Hom}(\widetilde{L}_i, \widetilde{L}_{j-1})} \mathscr{M}(b, a_i, \ldots, a_{j-1}) \times \mathscr{M}(a_1, \ldots, a_{i-1}, b, a_j, \ldots, a_{n+1}).$$

The orientation is signed in the following way. For each a_i, let o_{a_i} be the determinant bundle $\det \overline{\partial}$ of equation (7). Then the orientation bundle of $\mathscr{M}(a_{n+1}, a_1, \ldots, a_n)$ is

$$o_{a_{n+1}} \otimes o_{a_1}^- \otimes \cdots \otimes o_{a_n}^-,$$

where $o_{a_i}^-$ is the dual bundle of o_{a_i}. Note that $\mathscr{M}(a_{n+1}, a_1, \ldots, a_n)$ and $\mathscr{M}(a_1, \ldots, a_n, a_{n+1})$ count the same set of pseudo-holomorphic disks, however, their orientations agree if and only if $|a_{n+1}|(|a_1| + \cdots + |a_n|)$ is even. Being spin for the Lagrangian submanifolds guarantees that there is a coherent orientation for the moduli spaces (for more details, see [10]).

2.2.8 Construction of Fukaya category

Theorem 15 *Suppose that M is an exact symplectic manifold with $c_1(M) = 0$, and possibly with contact type boundary. Suppose that $\widetilde{L}_1, \widetilde{L}_2, \ldots, \widetilde{L}_{n+1}$ are admissible graded Lagrangian submanifolds, and $a_i \in \mathrm{Hom}(\widetilde{L}_i, \widetilde{L}_{i+1})$, $i = 1, 2, \ldots, n$. Define*

$$\overline{m}_n \colon \overline{\mathrm{Hom}}(\widetilde{L}_1, \widetilde{L}_2) \otimes \cdots \otimes \overline{\mathrm{Hom}}(\widetilde{L}_n, \widetilde{L}_{n+1}) \to \overline{\mathrm{Hom}}(\widetilde{L}_1, \widetilde{L}_{n+1})$$

by

$$\overline{m}_n(\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n) = \sum_{a\in \mathrm{Hom}(\widetilde{L}_1, \widetilde{L}_{n+1})} \#\mathscr{M}(a, a_1, \ldots, a_n) \cdot \overline{a}$$

for $n = 1, 2, \ldots$. Then the set of admissible Lagrangian submanifolds and the Floer cochain complex among them together with $\{\overline{m}_n\}$ defined above form an A_∞ category, called the Fukaya category of M, and is denoted by $\mathrm{Fuk}(M)$.

Proof This is proved by Seidel [27, Proposition 12.3]. For the general case, one may refer to Fukaya [8, Theorem 4.17] □

The (ir)relevance of the construction to the choice of the analytic data is also completely discussed in [27, §12]. Such a technical problem will also appear in our case when counting the pseudo-holomorphic disks with punctures. However, all Seidel's argument can be applied to our case, and we will not address this issue in the current paper.

2.2.9 Cyclicity and analytic data

From Seidel's original definition, one sees that

- if L_0 and L_1 intersect transversally, then one may choose $H_{L_0,L_1} = 0$, and up to a degree shifting,

$$\mathrm{Hom}(\widetilde{L}_0, \widetilde{L}_1) \cong \mathrm{Hom}(\widetilde{L}_1, \widetilde{L}_0)$$

(see Lemma 13); however,

- if L_0 and L_1 do not intersect transversally, then H_{L_0,L_1} does not vanish, and therefore, $\mathrm{Hom}(\widetilde{L}_0, \widetilde{L}_1)$ is by no means the same as $\mathrm{Hom}(\widetilde{L}_1, \widetilde{L}_0)$.

To overcome this inconsistency, i.e., to make Lemma 13 hold even for non transversal pair of Lagrangian submanifolds, we make the following additional assumption for the Floer data.

Assumption 16 For each *ordered* pair of Lagrangian submanifolds $L_0, L_1 \subset M$, a *Floer datum* consists of $H_{L_0,L_1} \in C^\infty([0,1], \mathscr{H})$ and $J \in C^\infty([0,1], \mathscr{J})$, besides the conditions of Definition 6, satisfies the following additional properties:

- for the opposite ordered pair (L_1, L_0), $H_{L_1,L_0}(t) = -H_{L_0,L_1}(1-t)$;
- if X is the time-dependent Hamiltonian vector field of H_{L_0,L_1} and ϕ_X its flow, then $\phi^{1/2}_{X(t)}(L_0)$ intersects $\phi^{1/2}_{-X(1-t)}(L_1)$ transversally. (Note that $-X(1-t)$ is the Hamiltonian vector field of $H_{L_1,L_0}(t)$.)

The perturbation data will be changed accordingly. With such an assumption, one sees that in the case when L_0 and L_1 do not intersect transversally, the generators of $\mathrm{Hom}(\widetilde{L}_0, \widetilde{L}_1)$ and $\mathrm{Hom}(\widetilde{L}_1, \widetilde{L}_0)$ may both be identified with the intersection points of $\phi^{1/2}_X(L_0)$ and $\phi^{1/2}_{-X}(L_1)$, and the degrees at each point add up to n.

In the following, if $p \in \mathrm{Hom}(\widetilde{L}_0, \widetilde{L}_1)$, we will use notation p^* for the same Hamiltonian chord but with an opposite orientation when viewed as an element in $\mathrm{Hom}(\widetilde{L}_1, \widetilde{L}_0)$.

3 Lie bialgebra structure

From now on, we grade the Floer cochains negatively. Such a convention is usually adopted in algebraic topology when studying Hochschild/cyclic homology of the cochain complex of a topological space.

Recall that a graded vector space L with a linear map $\delta\colon L \to L \otimes L$ is called a *Lie coalgebra* if the images of δ are graded skew-symmetric, and satisfy the following *co-Jacobi identity*:

$$(\tau^2 + \tau + id) \circ (id \otimes \delta) \circ \delta(a) = 0 \in L^{\otimes 3}, \quad \forall\, a \in L,$$

where

$$\tau\colon x \otimes y \otimes z \mapsto (-1)^{|z|(|x|+|y|)} z \otimes x \otimes y$$

is the cyclic permutation.

Definition 17 (Lie bialgebra) Let L be a (graded) vector space over k. A Lie bialgebra on L is the triple $(L, [\cdot,\cdot], \delta)$ such that

- $(L, [\cdot,\cdot])$ is a Lie algebra with Lie bracket $[\cdot,\cdot]$;
- (L, δ) is a Lie coalgebra with Lie cobracket δ;

• the Lie algebra and coalgebra satisfy the following identity, called the *Drinfeld compatibility*:

$$\begin{aligned}\delta[a,b] = &\sum_{(a)}((-1)^{|a''|\,|b|}[a',b]\otimes a'' + a'\otimes[a'',b])\\ &+\sum_{(b)}([a,b']\otimes b'' + (-1)^{|a|\,|b'|}b'\otimes[a,b'']),\quad \forall\, a,b\in L,\end{aligned}$$

where we write

$$\delta(a)=\sum_{(a)} a'\otimes a'',\quad \delta(b)=\sum_{(b)} b'\otimes b''.$$

If moreover,

$$[\cdot,\cdot]\circ\delta(a)=0,\quad \forall\, a\in L,$$

then $(L,[\cdot,\cdot],\delta)$ is said to be *involutive*. When the Lie bracket has degree r and the Lie cobracket has degree s, we say that L is a Lie bialgebra with degree (r,s).

Proof of Theorem A The proof consists of the rest of the section. Since it is pretty long, we split it into several parts. The Lie algebra and coalgebra structures are proved in Lemmas 18 and 19, respectively, and the Drinfeld compatibility and involutivity are proved in § 3.1 and § 3.2, respectively.

Lemma 18 (Lie algebra structure) *Let M be as above, and let* $\mathrm{Fuk}(M)$ *be the Fukaya category of M. Define*

$$[\cdot,\cdot]\colon \mathrm{CC}^\bullet(\mathrm{Fuk}(M))\otimes\mathrm{CC}^\bullet(\mathrm{Fuk}(M))\to\mathrm{CC}^\bullet(\mathrm{Fuk}(M))$$

by

$$\begin{aligned}&[f,g](\overline{a}_1,\overline{a}_2,\dots,\overline{a}_n)\\ &:=\sum_{\substack{i<j\\ p\in\mathrm{Hom}(\widetilde{L}_j,\widetilde{L}_i)}}\pm f(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p})\cdot g(\overline{p}^*,\overline{a}_j,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{i-1})\\ &\quad-\sum_{\substack{i<j\\ p\in\mathrm{Hom}(\widetilde{L}_j,\widetilde{L}_i)}}\pm g(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p})\cdot f(\overline{p}^*,\overline{a}_j,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{i-1}).\end{aligned}$$

Then $(\mathrm{CC}^\bullet(\mathrm{Fuk}(M)),[\cdot,\cdot],b)$ *forms a differential graded Lie algebra of degree* $2-d$.

Proof First, we show that $[\cdot,\cdot]$ has degree $2-d$. Note that $(\overline{a}_1,\dots,\overline{a}_n)$ has degree

$$|a_1|+\cdots+|a_n|+n,$$

(recall that we grade $a_i \in \mathrm{Hom}(\widetilde{L}_i, \widetilde{L}_{i+1})$ negatively). And the sum of the degrees of $(\overline{a}_i, \ldots, \overline{a}_{j-1}, \overline{p})$ and $(\overline{p}^*, \overline{a}_j, \ldots, \overline{a}_n, \ldots, \overline{a}_{i-1})$ is

$$\begin{aligned}
&|a_1| + \cdots + |a_n| + |p| + |p^*| + (n+2)\\
&\quad = |a_1| + \cdots + |a_n| + |p| + (-d - |p|) + (n+2) \quad (|p^*| + |p| = -d)\\
&\quad = |a_1| + \cdots + |a_n| + n + (2-d).
\end{aligned}$$

The difference of these two degrees is exactly $d-2$. Going to the cyclic cochain (i.e., the dual space) level, the degree of the bracket $[\cdot,\cdot]$ becomes $2-d$.

Second, we show that $[\cdot,\cdot]$ is graded skew-symmetric. Observe that in the definition of $[\cdot,\cdot]$, if we switch f and g, then we get exactly the opposite sign.

Third, we show that the graded Jacobi identity holds: for any $f, g, h \in \mathrm{CC}^\bullet(\mathrm{Fuk}(M))$ and any $(a_1, a_2, \ldots, a_n) \in \mathrm{CH}^\bullet(\mathrm{Fuk}(M))$,

$$\begin{aligned}
&[[f,g],h](\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n)\\
&\quad = \sum_{i<j} \sum_{p \in \mathrm{Hom}(\widetilde{L}_j, \widetilde{L}_i)} \pm [f,g](\overline{a}_i, \ldots, \overline{a}_{j-1}, \overline{p}) \cdot h(\overline{p}^*, \overline{a}_j, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1})\\
&\qquad - \sum_{i<j} \sum_{p \in \mathrm{Hom}(\widetilde{L}_j, \widetilde{L}_i)} \pm h(\overline{a}_i, \ldots, \overline{a}_{j-1}, \overline{p}) \cdot [f,g](\overline{p}^*, \overline{a}_j, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1})\\
&\quad = \sum_{i<k<\ell<j} \sum_{p,q} \pm f(\overline{a}_k, \ldots, \overline{a}_{\ell-1}, \overline{q}) g(\overline{q}^*, \overline{a}_\ell, \ldots, \overline{a}_{j-1}, \overline{p}, \overline{a}_i, \ldots, \overline{a}_{k-1})\\
&\qquad \cdot h(\overline{p}^*, \overline{a}_j, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1}) \qquad (10a)\\
&\qquad - \sum_{i<k<\ell<j} \sum_{p,q} \pm f(\overline{q}^*, \overline{a}_\ell, \ldots, \overline{a}_{j-1}, \overline{p}, \overline{a}_i, \ldots, \overline{a}_{k-1}) g(\overline{a}_k, \ldots, \overline{a}_{\ell-1}, \overline{q})\\
&\qquad \cdot h(\overline{p}^*, \overline{a}_j, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1}) \qquad (10b)\\
&\qquad - \sum_{i<j} \sum_{j \leqslant k<\ell<n} \sum_{p,q} \pm f(\overline{a}_k, \ldots, \overline{a}_{\ell-1}, \overline{q}) g(\overline{q}^*, \overline{a}_\ell, \ldots, \overline{a}_{i-1}, \overline{p}^*, \overline{a}_j, \ldots, \overline{a}_{k-1})\\
&\qquad \cdot h(\overline{a}_i, \ldots, \overline{a}_{j-1}, \overline{p}) \qquad (10c)\\
&\qquad - \sum_{i<j} \sum_{1 \leqslant k<\ell<i-1} \sum_{p,q} \pm f(\overline{a}_k, \ldots, \overline{a}_{\ell-1}, \overline{q})\\
&\qquad \cdot g(\overline{q}^*, \overline{a}_\ell, \ldots, \overline{a}_{i-1}, \overline{p}^*, \overline{a}_j, \ldots, \overline{a}_{k-1}) h(\overline{a}_i, \ldots, \overline{a}_{j-1}, \overline{p}) \qquad (10d)\\
&\qquad - \sum_{i<j} \sum_{k \leqslant n, \ell<i-1} \sum_{p,q} \pm f(\overline{a}_k, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{\ell-1}, \overline{q})\\
&\qquad \cdot g(\overline{q}^*, \overline{a}_\ell, \ldots, \overline{a}_{i-1}, \overline{p}^*, \overline{a}_j, \ldots, \overline{a}_{k-1}) h(\overline{a}_i, \ldots, \overline{a}_{j-1}, \overline{p}) \qquad (10e)\\
&\qquad + \sum_{i<j} \sum_{j<k<\ell<n} \sum_{p,q} \pm f(\overline{q}^*, \overline{a}_\ell, \ldots, \overline{a}_{i-1}, \overline{p}^*, \overline{a}_j, \ldots, \overline{a}_{k-1}) g(\overline{a}_k, \ldots, \overline{a}_{\ell-1}, \overline{q})\\
&\qquad \cdot h(\overline{a}_i, \ldots, \overline{a}_{j-1}, \overline{p}) \qquad (10f)\\
&\qquad + \sum_{i<j} \sum_{1 \leqslant k<\ell<i-1} \sum_{p,q} \pm f(\overline{q}^*, \overline{a}_\ell, \ldots, \overline{p}^*, \overline{a}_j, \ldots, \overline{a}_{k-1}) g(\overline{a}_k, \ldots, \overline{a}_{\ell-1}, \overline{q})\\
&\qquad \cdot h(\overline{a}_i, \ldots, \overline{a}_{j-1}, \overline{p}) \qquad (10g)
\end{aligned}$$

$$+\sum_{i<j}\sum_{k\leqslant n,\ell<i-1}\sum_{p,q}\pm f(\overline{q}^*,\overline{a}_\ell,\dots,\overline{a}_{i-1},\overline{p}^*,\overline{a}_j,\dots,\overline{a}_{k-1})$$
$$\cdot g(\overline{a}_k,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{\ell-1},\overline{q})h(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p}). \tag{10h}$$

Similarly,

$$[[h,f],g](\overline{a}_1,\overline{a}_2,\dots,\overline{a}_n)$$
$$=\sum_{i<k<\ell<j}\sum_{p,q}\pm h(\overline{a}_k,\dots,\overline{a}_{\ell-1},\overline{q})f(\overline{q}^*,\overline{a}_\ell,\dots,\overline{a}_{j-1},\overline{p},\overline{a}_i,\dots,\overline{a}_{k-1})$$
$$\cdot g(\overline{p}^*,\overline{a}_j,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{i-1}) \tag{11a}$$
$$-\sum_{i<k<\ell<j}\sum_{p,q}\pm h(\overline{q}^*,\overline{a}_\ell,\dots,\overline{a}_{j-1},\overline{p},\overline{a}_i,\dots,\overline{a}_{k-1})$$
$$\cdot f(\overline{a}_k,\dots,\overline{a}_{\ell-1},\overline{q})g(\overline{p}^*,\overline{a}_j,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{i-1}) \tag{11b}$$
$$-\sum_{i<j}\sum_{j\leqslant k<\ell<n}\sum_{p,q}\pm h(\overline{a}_k,\dots,\overline{a}_{\ell-1},\overline{q})$$
$$\cdot f(\overline{q}^*,\overline{a}_\ell,\dots,\overline{a}_{i-1},\overline{p}^*,\overline{a}_j,\dots,\overline{a}_{k-1})g(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p}) \tag{11c}$$
$$-\sum_{i<j}\sum_{1\leqslant k<\ell<i-1}\sum_{p,q}\pm h(\overline{a}_k,\dots,\overline{a}_{\ell-1},\overline{q})$$
$$\cdot f(\overline{q}^*,\overline{a}_\ell,\dots,\overline{p}^*,\overline{a}_j,\dots,\overline{a}_{k-1})g(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p}) \tag{11d}$$
$$-\sum_{i<j}\sum_{k\leqslant n,\ell<i-1}\sum_{p,q}\pm h(\overline{a}_k,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{\ell-1},\overline{q})$$
$$\cdot f(\overline{q}^*,\overline{a}_\ell,\dots,\overline{a}_{i-1},\overline{p}^*,\overline{a}_j,\dots,\overline{a}_{k-1})g(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p}) \tag{11e}$$
$$+\sum_{i<j}\sum_{j<k<\ell<n}\sum_{p,q}\pm h(\overline{q}^*,\overline{a}_\ell,\dots,\overline{a}_{i-1},\overline{p}^*,\overline{a}_j,\dots,\overline{a}_{k-1})$$
$$\cdot f(\overline{a}_k,\dots,\overline{a}_{\ell-1},\overline{q})g(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p}) \tag{11f}$$
$$+\sum_{i<j}\sum_{1\leqslant k<\ell<i-1}\sum_{p,q}\pm h(\overline{q}^*,\overline{a}_\ell,\dots,\overline{p}^*,\overline{a}_j,\dots,\overline{a}_{k-1})$$
$$\cdot f(\overline{a}_k,\dots,\overline{a}_{\ell-1},\overline{q})g(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p}) \tag{11g}$$
$$+\sum_{i<j}\sum_{k\leqslant n,\ell<i-1}\sum_{p,q}\pm h(\overline{q}^*,\overline{a}_\ell,\dots,\overline{a}_{i-1},\overline{p}^*,\overline{a}_j,\dots,\overline{a}_{k-1})$$
$$\cdot f(\overline{a}_k,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{\ell-1},\overline{q})g(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p}). \tag{11h}$$

And

$$[[g,h],f](\overline{a}_1,\overline{a}_2,\dots,\overline{a}_n)$$
$$=\sum_{i<k<\ell<j}\sum_{p,q}\pm g(\overline{a}_k,\dots,\overline{a}_{\ell-1},\overline{q})h(\overline{q}^*,\overline{a}_\ell,\dots,\overline{a}_{j-1},\overline{p},\overline{a}_i,\dots,\overline{a}_{k-1})$$
$$\cdot f(\overline{p}^*,\overline{a}_j,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{i-1}) \tag{12a}$$
$$-\sum_{i<k<\ell<j}\sum_{p,q}\pm g(\overline{q}^*,\overline{a}_\ell,\dots,\overline{a}_{j-1},\overline{p},\overline{a}_i,\dots,\overline{a}_{k-1})$$

$$\cdot h(\overline{a}_k,\dots,\overline{a}_{\ell-1},\overline{q})f(\overline{p}^*,\overline{a}_j,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{i-1}) \tag{12b}$$

$$-\sum_{i<j}\sum_{j\leqslant k<\ell<n}\sum_{p,q}\pm g(\overline{a}_k,\dots,\overline{a}_{\ell-1},\overline{q})$$

$$\cdot h(\overline{q}^*,\overline{a}_\ell,\dots,\overline{a}_{i-1},\overline{p}^*,\overline{a}_j,\dots,\overline{a}_{k-1})f(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p}) \tag{12c}$$

$$-\sum_{i<j}\sum_{1\leqslant k<\ell<i-1}\sum_{p,q}\pm g(\overline{a}_k,\dots,\overline{a}_{\ell-1},\overline{q})$$

$$\cdot h(\overline{q}^*,\overline{a}_\ell,\dots,\overline{p}^*,\overline{a}_j,\dots,\overline{a}_{k-1})f(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p}) \tag{12d}$$

$$-\sum_{i<j}\sum_{k\leqslant n,\ell<i-1}\sum_{p,q}\pm g(\overline{a}_k,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{\ell-1},\overline{q})$$

$$\cdot h(\overline{q}^*,\overline{a}_\ell,\dots,\overline{a}_{i-1},\overline{p}^*,\overline{a}_j,\dots,\overline{a}_{k-1})f(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p}) \tag{12e}$$

$$+\sum_{i<j}\sum_{j<k<\ell<n}\sum_{p,q}\pm g(\overline{q}^*,\overline{a}_\ell,\dots,\overline{a}_{i-1},\overline{p}^*,\overline{a}_j,\dots,\overline{a}_{k-1})$$

$$\cdot h(\overline{a}_k,\dots,\overline{a}_{\ell-1},\overline{q})f(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p}) \tag{12f}$$

$$+\sum_{i<j}\sum_{1\leqslant k<\ell<i-1}\sum_{p,q}\pm g(\overline{q}^*,\overline{a}_\ell,\dots,\overline{p}^*,\overline{a}_j,\dots,\overline{a}_{k-1})$$

$$\cdot h(\overline{a}_k,\dots,\overline{a}_{\ell-1},\overline{q})f(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p}) \tag{12g}$$

$$+\sum_{i<j}\sum_{k\leqslant n,\ell<i-1}\sum_{p,q}\pm g(\overline{q}^*,\overline{a}_\ell,\dots,\overline{a}_{i-1},\overline{p}^*,\overline{a}_j,\dots,\overline{a}_{k-1})$$

$$\cdot h(\overline{a}_k,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{\ell-1},\overline{q})f(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p}). \tag{12h}$$

One sees that

$$\begin{aligned}
&(10a)=-(11e),\quad (10b)=-(12h),\quad (10c)=-(12g),\quad (10d)=-(12f),\\
&(10e)=-(12b),\quad (10f)=-(11d),\quad (10g)=-(11c),\quad (10h)=-(11a),\\
&(11b)=-(12e),\quad (11f)=-(12d),\quad (11g)=-(12c),\quad (11h)=-(12a),
\end{aligned}$$

and therefore, the sum of the above three items is identically zero, which proves the (graded) Jacobi identity.

Finally, we show that the bracket commutes with the boundary:

$$\begin{aligned}
&(b[f,g])(\overline{a}_1,\overline{a}_2,\dots,\overline{a}_n)\\
&\quad=[f,g](b(\overline{a}_1,\overline{a}_2,\dots,\overline{a}_n))\\
&\quad=[f,g]\Big(\sum_i\sum_k\pm(\overline{a}_1,\dots,\overline{m}_k(\overline{a}_i,\dots,\overline{a}_{i+k-1}),\dots,\overline{a}_n)\Big) &(13a)\\
&\qquad+[f,g]\Big(\sum_j\sum_k\pm(\overline{m}_k(\overline{a}_{n-j},\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_i),\dots,\overline{a}_{n-j-1})\Big), &(13b)
\end{aligned}$$

while

$$\begin{aligned}
&([bf,g]+(-1)^{|f|}[f,bg])(\overline{a}_1,\overline{a}_2,\dots,\overline{a}_n)\\
&\quad=\sum_{i<j}\sum_{p}\pm f(b(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p}))\cdot g(\overline{p}^*,\overline{a}_j,\dots,\overline{a}_{i-1}) &\text{(14a)}\\
&\qquad-\sum_{i<j}\sum_{p}\pm g(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p})\cdot f(b(\overline{p}^*,\overline{a}_j,\dots,\overline{a}_{i-1})) &\text{(14b)}\\
&\qquad+\sum_{i<j}\sum_{p}\pm f(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p})\cdot g(b(\overline{p}^*,\overline{a}_j,\dots,\overline{a}_{i-1})) &\text{(14c)}\\
&\qquad-\sum_{p}\pm g(b(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p}))\cdot f(\overline{p}^*,\overline{a}_j,\dots,\overline{a}_{i-1}). &\text{(14d)}
\end{aligned}$$

From the definition of the bracket $[\cdot,\cdot]$ and the boundary b, one sees that (14a)–(14d) altogether contains more terms than (13a) and (13b), namely, those terms involving $\overline{m}_k$ acting on $\overline{p}$ and $\overline{p}^*$. For example, the extra terms coming from (14a) are

$$\begin{aligned}
&\sum_{p\in\mathrm{Hom}(\widetilde{L}_j,\widetilde{L}_i)}\sum_{k}\sum_{r} f(\overline{m}_r(\overline{a}_k,\dots,\overline{a}_{j-1},\overline{p},\overline{a}_i,\dots,\overline{a}_l),\overline{a}_{l+1},\dots,\overline{a}_{k-1})\\
&\qquad\cdot g(\overline{p}^*,\overline{a}_j,\dots,\overline{a}_{i-1}),
\end{aligned}\tag{15}$$

and the ones from (14c) are

$$\begin{aligned}
&\sum_{p\in\mathrm{Hom}(\widetilde{L}_j,\widetilde{L}_i)}\sum_{k}\sum_{r} f(\overline{a}_i,\dots,\overline{a}_j,\overline{p})\\
&\qquad\cdot g(\overline{m}_r(\overline{a}_k,\dots,\overline{a}_{i-1},\overline{p}^*,\overline{a}_j,\dots,\overline{a}_l),\overline{a}_{l+1},\dots,\overline{a}_{k-1}).
\end{aligned}\tag{16}$$

However, these two groups of terms cancel with each other because

$$\begin{aligned}
&\sum_{p\in\mathrm{Hom}(\widetilde{L}_j,\widetilde{L}_i)}\overline{m}_r(\overline{a}_k,\dots,\overline{a}_j,\overline{p},\overline{a}_i,\dots,\overline{a}_l)\otimes\overline{p}^*\\
&\quad=\sum_{p\in\mathrm{Hom}(\widetilde{L}_j,\widetilde{L}_i)}\sum_{q\in\mathrm{Hom}(\widetilde{L}_k,\widetilde{L}_{l+1})}\#\mathscr{M}(q,a_k,\dots,a_j,p,a_i,\dots,a_l)\overline{q}\otimes\overline{p}^*\\
&\quad=\sum_{q\in\mathrm{Hom}(\widetilde{L}_k,\widetilde{L}_{l+1})}\sum_{p\in\mathrm{Hom}(\widetilde{L}_j,\widetilde{L}_i)}\overline{q}\otimes\#\mathscr{M}(q,a_k,\dots,a_j,p,a_i,\dots,a_l)\overline{p}^*\\
&\quad\overset{\text{cyclicity}}{=}\sum_{q\in\mathrm{Hom}(\widetilde{L}_k,\widetilde{L}_{l+1})}\sum_{p\in\mathrm{Hom}(\widetilde{L}_j,\widetilde{L}_i)}\overline{q}\otimes\#\mathscr{M}(p,a_i,\dots,a_l,q,a_k,\dots,a_j)\overline{p}^*\\
&\quad\overset{\S2.2.9}{=}\sum_{q\in\mathrm{Hom}(\widetilde{L}_k,\widetilde{L}_{l+1})}\sum_{p^*\in\mathrm{Hom}(\widetilde{L}_i,\widetilde{L}_j)}\overline{q}\otimes\#\mathscr{M}(p^*,a_i,\dots,a_l,q^*,a_k,\dots,a_j)\overline{p}^*\\
&\quad=\sum_{q\in\mathrm{Hom}(\widetilde{L}_k,\widetilde{L}_{l+1})}\overline{q}\otimes\overline{m}_r(\overline{a}_i,\dots,\overline{a}_l,\overline{q}^*,\overline{a}_k,\dots,\overline{a}_j).
\end{aligned}$$

By substituting the above identity into (15), we get exactly (16). Similarly, the extra terms in (14b) and in (14d) cancel with each other. □

Lemma 19 (Lie coalgebra structure) *Define*

$$\mathrm{CC}^\bullet(\mathrm{Fuk}(M)) \to \mathrm{CC}^\bullet(\mathrm{Fuk}(M)) \otimes \mathrm{CC}^\bullet(\mathrm{Fuk}(M))$$

by

$$\begin{aligned}&(\delta f)((\overline{a}_1, \overline{a}_2, \dots, \overline{a}_n) \otimes (\overline{b}_1, \overline{b}_2, \dots, \overline{b}_m))\\ &:= \sum_{i=1}^{n}\sum_{j=1}^{m} \sum_{p\in \mathrm{Hom}(\widetilde{L}_i, \widetilde{L}_j)} \pm f(\overline{a}_1, \dots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \dots, \overline{b}_m, \overline{b}_1, \dots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \dots, \overline{a}_n).\end{aligned}$$

Then $(\mathrm{CC}^\bullet(\mathrm{Fuk}(M)), \delta, b)$ *forms a differential graded Lie coalgebra of degree* $2-d$.

Proof From the definition of δ, the following two statements are obvious:

(i) δf is well defined, namely, the value of δf is invariant under the cyclic permutations of $(\overline{a}_1, \dots, \overline{a}_n)$ and $(\overline{b}_1, \dots, \overline{b}_m)$;

(ii) δf is (graded) skew-symmetric, namely, if we switch $(\overline{a}_1, \dots, \overline{a}_n)$ and $(\overline{b}_1, \dots, \overline{b}_m)$, the sign of the value of δf changes.

The co-Jacobi identity can be proved in a similar way to the proof of Jacobi identity. For any $f \in \mathrm{CC}^\bullet(\mathrm{Fuk}(M))$ and $(\overline{a}_1, \dots, \overline{a}_n)$, $(\overline{b}_1, \dots, \overline{b}_m)$, $(\overline{c}_1, \dots, \overline{c}_\ell) \in \mathrm{CH}_\bullet(\mathrm{Fuk}(M))$, we have

$$\begin{aligned}&((id\otimes\delta)\circ\delta f)((\overline{a}_1, \dots, \overline{a}_n), (\overline{b}_1, \dots, \overline{b}_m), (\overline{c}_1, \dots, \overline{c}_\ell))\\ &= \sum \pm \delta f((\overline{a}_1, \dots, \overline{a}_n), (\overline{b}_1, \dots, \overline{b}_{j-1}, \overline{p}, \overline{c}_k, \dots, \overline{c}_\ell, \overline{c}_1, \dots, \overline{c}_{k-1}, \overline{p}^*, \overline{b}_j, \dots, \overline{b}_m))\\ &= \sum \pm f(\overline{a}_1, \dots, \overline{a}_{i-1}, \overline{q}, \overline{b}_r, \dots, \overline{b}_{j-1}, \overline{p}, \overline{c}_k, \dots, \overline{c}_\ell,\\ &\quad \overline{c}_1, \dots, \overline{c}_{k-1}, \overline{p}^*, \overline{b}_j, \dots, \overline{b}_{r-1}, \overline{q}^*, \overline{a}_i, \dots, \overline{a}_n) \qquad (17a)\\ &\quad + \sum \pm f(\overline{a}_1, \dots, \overline{a}_{i-1}, \overline{q}, \overline{c}_s, \dots, \overline{c}_{k-1}, \overline{p}^*, \overline{b}_j, \dots, \overline{b}_m,\\ &\quad \overline{b}_1, \dots, \overline{b}_{j-1}, \overline{p}, \overline{c}_k, \dots, \overline{c}_{s-1}, \overline{q}^*, \overline{a}_i, \dots, \overline{a}_n). \qquad (17b)\end{aligned}$$

Let $\tau\colon x\otimes y\otimes z \mapsto \pm z\otimes x\otimes y$ be the cyclic permutation of three elements. Then

$$\begin{aligned}&(\tau\circ(id\otimes\delta)\circ\delta f)((\overline{a}_1, \dots, \overline{a}_n), (\overline{b}_1, \dots, \overline{b}_m), (\overline{c}_1, \dots, \overline{c}_\ell))\\ &\quad = ((id\otimes\delta)\circ\delta f)((\overline{c}_1, \dots, \overline{c}_\ell), (\overline{a}_1, \dots, \overline{a}_n), (\overline{b}_1, \dots, \overline{b}_m))\\ &\quad = \sum \pm f(\overline{c}_1, \dots, \overline{c}_{k-1}, \overline{q}, \overline{a}_r, \dots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \dots, \overline{b}_m,\\ &\qquad \overline{b}_1, \dots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \dots, \overline{a}_{r-1}, \overline{q}^*, \overline{c}_k, \dots, \overline{c}_\ell) \qquad (18a)\\ &\quad + \sum \pm f(\overline{c}_1, \dots, \overline{c}_{k-1}, \overline{q}, \overline{b}_s, \dots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \dots, \overline{a}_n,\\ &\qquad \overline{a}_1, \dots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \dots, \overline{b}_{s-1}, \overline{q}^*, \overline{c}_k, \dots, \overline{c}_\ell). \qquad (18b)\end{aligned}$$

Similarly,

$$\begin{aligned}
&(\tau^2 \circ (id \otimes \delta) \circ \delta f)((\overline{a}_1, \ldots, \overline{a}_n), (\overline{b}_1, \ldots, \overline{b}_m), (\overline{c}_1, \ldots, \overline{c}_\ell)) \\
&\quad = ((id \otimes \delta) \circ \delta f)((\overline{b}_1, \ldots, \overline{b}_m), (\overline{c}_1, \ldots, \overline{c}_\ell), (\overline{a}_1, \ldots, \overline{a}_n)) \\
&\quad = \sum \pm f(\overline{b}_1, \ldots, \overline{b}_{j-1}, \overline{q}, \overline{c}_r, \ldots, \overline{c}_{k-1}, \overline{p}, \overline{a}_i, \ldots, \overline{a}_n, \\
&\qquad \overline{a}_1, \ldots, \overline{a}_{i-1}, \overline{p}^*, \overline{c}_k, \ldots, \overline{c}_{k-1}, \overline{q}^*, \overline{b}_j, \ldots, \overline{b}_m) \qquad (19a) \\
&\qquad + \sum \pm f(\overline{b}_1, \ldots, \overline{b}_{j-1}, \overline{q}, \overline{a}_s, \ldots, \overline{a}_{i-1}, \overline{p}^*, \overline{c}_k, \ldots, \overline{c}_\ell, \\
&\qquad \overline{c}_1, \ldots, \overline{c}_{k-1}, \overline{p}, \overline{a}_i, \ldots, \overline{a}_{s-1}, \overline{q}^*, \overline{b}_j, \ldots, \overline{b}_m). \qquad (19b)
\end{aligned}$$

One sees that (17a) cancels with (18b), so do (17b) with (19a) and (18a) with (19b), and we obtain the co-Jacobi identity.

Next, we show that b respects the cobracket. This is also similar to the Lie case. By definition,

$$\begin{aligned}
&((b \otimes id \pm id \otimes b)\delta(f))((\overline{a}_1, \ldots, \overline{a}_n) \otimes (\overline{b}_1, \ldots, \overline{b}_m)) \\
&\quad = \delta f(b(\overline{a}_1, \ldots, \overline{a}_n) \otimes (\overline{b}_1, \ldots, \overline{b}_m) \pm (\overline{a}_1, \ldots, \overline{a}_n) \otimes b(\overline{b}_1, \ldots, \overline{b}_m)), \qquad (20)
\end{aligned}$$

while

$$\begin{aligned}
&\delta(b(f))((\overline{a}_1, \ldots, \overline{a}_n) \otimes (\overline{b}_1, \ldots, \overline{b}_m)) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{p} \pm b(f)(\overline{a}_1, \ldots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \ldots, \overline{b}_m, \overline{b}_1, \ldots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \ldots, \overline{a}_n). \qquad (21)
\end{aligned}$$

Compared with (20), the right-hand side of (21) has extra terms

$$\begin{aligned}
&\sum_{p} \pm f(\overline{a}_1, \ldots, \overline{m}_r(\overline{a}_k, \ldots, \overline{p}, \ldots, \overline{b}_{l-1}), \overline{b}_l, \ldots, \overline{p}^*, \ldots, \overline{a}_n) \qquad (22a) \\
&\quad + \sum_{p} \pm f(\overline{a}_1, \ldots, \overline{e}_p, \ldots, \overline{b}_{j-1}, \overline{m}_r(\overline{b}_j, \ldots, \overline{p}^*, \ldots, \overline{a}_l), \ldots, \overline{a}_n) \qquad (22b) \\
&\quad + \sum_{p} \pm f(\overline{a}_1, \ldots, \overline{m}_r(\overline{a}_k, \ldots, \overline{p}, \overline{b}_j, \ldots, \overline{p}^*, \ldots, \overline{a}_{l-1}), \overline{a}_l, \ldots, \overline{a}_n). \qquad (22c)
\end{aligned}$$

However, a similar argument as in the Lie case, (22a) + (22b) vanishes, and the terms in (22c) come in pairs (counting p and p^*), which cancel within themselves. This proves that the differential commutes with the cobracket. □

3.1 Proof of Drinfeld compatibility

For any two homogeneous $f, g \in \mathrm{CC}^\bullet(\mathrm{Fuk}(M))$ and any two Hochschild chains $(a_1, \ldots, a_n), (b_1, \ldots, b_m) \in \mathrm{CH}_\bullet(\mathrm{Fuk}(M))$, we have

$$\begin{aligned}
&\delta \circ [f, g]((a_1, \ldots, a_n) \otimes (b_1, \ldots, b_m)) \\
&\quad = \sum_{i,j} \pm [f, g](\overline{a}_1, \ldots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \ldots, \overline{b}_m, \overline{b}_1, \ldots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \ldots, \overline{a}_n). \qquad (23)
\end{aligned}$$

The expansion of (23) is pretty long, and in the following, we collect them into several groups. Denote $(\overline{a}_1,\dots,\overline{a}_{i-1},\overline{p},\overline{b}_j,\dots,\overline{b}_m,\overline{b}_1,\dots,\overline{b}_{j-1},\overline{p}^*,\overline{a}_i,\dots,\overline{a}_n)$ by $(c_1,\dots,c_{m+n+2})$. Then

$$\begin{aligned}&[f,g](c_1,\dots,c_{m+n+2})\\&\quad=\sum_{r<s}(\pm f(c_r,\dots,c_{s-1},q)g(q^*,c_s,\dots,c_{m+n+2},c_1,\dots,c_{r-1})\\&\qquad\pm g(c_r,\dots,c_{s-1},q)f(q^*,c_s,\dots,c_{m+n+2},c_1,\dots,c_{r-1})).\end{aligned}\tag{24}$$

The right-hand side of (24) consists of summands grouped according to the values of r and s:

(i) when $c_r,c_{s-1}\in\{\overline{a}_1,\dots,\overline{a}_{i-1}\}$, the corresponding summand is

$$\begin{aligned}&\sum\pm f(\overline{a}_r,\dots,\overline{a}_{s-1},\overline{q})\\&\cdot g(\overline{q}^*,\overline{a}_s,\dots,\overline{a}_{i-1},\overline{p},\overline{b}_j,\dots,\overline{b}_m,\overline{b}_1,\dots,\overline{b}_{j-1},\overline{p}^*,\overline{a}_i,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{r-1})\qquad(25a)\\&-\sum\pm g(\overline{a}_r,\dots,\overline{a}_{s-1},\overline{q})\\&\cdot f(\overline{q}^*,\overline{a}_s,\dots,\overline{a}_{i-1},\overline{p},\overline{b}_j,\dots,\overline{b}_m,\overline{b}_1,\dots,\overline{b}_{j-1},\overline{p}^*,\overline{a}_i,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{r-1});\qquad(25b)\end{aligned}$$

(ii) when $c_r\in\{\overline{a}_1,\dots,\overline{a}_{i-1}\}$ and $c_{s-1}=\overline{p}$, the corresponding summand is

$$\begin{aligned}&\sum\pm f(a_r,\dots,\overline{a}_{i-1},\overline{p},\overline{q})\\&\quad\cdot g(\overline{q}^*,\overline{b}_j,\dots,\overline{b}_m,\overline{b}_1,\dots,\overline{b}_{j-1},\overline{p}^*,\overline{a}_i,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{r-1})\qquad(26a)\\&\quad-\sum\pm g(a_r,\dots,\overline{a}_{i-1},\overline{p},\overline{q})\\&\quad\cdot f(\overline{q}^*,\overline{b}_j,\dots,\overline{b}_m,\overline{b}_1,\dots,\overline{b}_{j-1},\overline{p}^*,\overline{a}_i,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{r-1});\qquad(26b)\end{aligned}$$

(iii) when $c_r\in\{\overline{a}_1,\dots,\overline{a}_{i-1}\}$ and $c_{s-1}\in\{\overline{b}_j,\dots,\overline{b}_m\}$, the corresponding summand is

$$\begin{aligned}&\sum\pm f(a_r,\dots,\overline{a}_{i-1},\overline{p},\overline{b}_j,\dots,\overline{b}_{s-1},\overline{q})\\&\quad\cdot g(\overline{q}^*,\overline{b}_s,\dots,\overline{b}_m,\overline{b}_1,\dots,\overline{b}_{j-1},\overline{p}^*,\overline{a}_i,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{r-1})\qquad(27a)\\&\quad-\sum\pm g(a_r,\dots,\overline{a}_{i-1},\overline{p},\overline{b}_j,\dots,\overline{b}_{s-1},\overline{q})\\&\quad\cdot f(\overline{q}^*,\overline{b}_s,\dots,\overline{b}_m,\overline{b}_1,\dots,\overline{b}_{j-1},\overline{p}^*,\overline{a}_i,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{r-1});\qquad(27b)\end{aligned}$$

(iv) when $c_r\in\{\overline{a}_1,\dots,\overline{a}_{i-1}\}$ and $c_{s-1}\in\{\overline{b}_1,\dots,\overline{b}_{j-1}\}$, the corresponding summand is

$$\begin{aligned}&\sum\pm f(a_r,\dots,\overline{a}_{i-1},\overline{p},\overline{b}_j,\dots,\overline{b}_m,\overline{b}_1,\dots,\overline{b}_{s-1},\overline{q})\\&\quad\cdot g(\overline{q}^*,\overline{b}_s,\dots,\overline{b}_{j-1},\overline{p}^*,\overline{a}_i,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{r-1})\qquad(28a)\\&\quad-\sum\pm g(a_r,\dots,\overline{a}_{i-1},\overline{p},\overline{b}_j,\dots,\overline{b}_m,\overline{b}_1,\dots,\overline{b}_{s-1},\overline{q})\\&\quad\cdot f(\overline{q}^*,\overline{b}_s,\dots,\overline{b}_{j-1},\overline{p}^*,\overline{a}_i,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{r-1});\qquad(28b)\end{aligned}$$

(v) when $c_r \in \{\overline{a}_1, \dots, \overline{a}_{i-1}\}$ and $c_{s-1} = \overline{p}^*$, the corresponding summand is

$$\begin{aligned}
&\sum \pm f(a_r, \dots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \dots, \overline{b}_m, \overline{b}_1, \dots, \overline{b}_{j-1}, \overline{p}^*, \overline{q}) \\
&\quad \cdot g(\overline{q}^*, \overline{a}_i, \dots, \overline{a}_n, \overline{a}_1, \dots, \overline{a}_{r-1}) && \text{(29a)} \\
&\quad - \sum \pm g(a_r, \dots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \dots, \overline{b}_m, \overline{b}_1, \dots, \overline{b}_{j-1}, \overline{p}^*, \overline{q}) \\
&\quad \cdot f(\overline{q}^*, \overline{a}_i, \dots, \overline{a}_n, \overline{a}_1, \dots, \overline{a}_{r-1}); && \text{(29b)}
\end{aligned}$$

(vi) when $c_r \in \{\overline{a}_1, \dots, \overline{a}_{i-1}\}$ and $c_{s-1} \in \{\overline{a}_i, \dots, \overline{a}_n\}$, the corresponding summand is

$$\begin{aligned}
&\sum \pm f(a_r, \dots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \dots, \overline{b}_m, \overline{b}_1, \dots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \dots, \overline{a}_{s-1}, \overline{q}) \\
&\quad \cdot g(\overline{q}^*, \overline{a}_s, \dots, \overline{a}_n, \overline{a}_1, \dots, \overline{a}_{r-1}) && \text{(30a)} \\
&\quad - \sum \pm g(a_r, \dots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \dots, \overline{b}_m, \overline{b}_1, \dots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \dots, \overline{a}_{s-1}, \overline{q}) \\
&\quad \cdot f(\overline{q}^*, \overline{a}_s, \dots, \overline{a}_n, \overline{a}_1, \dots, \overline{a}_{r-1}); && \text{(30b)}
\end{aligned}$$

(vii) when $c_r = \overline{p}$ and $c_{s-1} \in \{\overline{b}_j, \dots, \overline{b}_m\}$, the corresponding summand is

$$\begin{aligned}
&\sum \pm f(\overline{p}, \overline{b}_j, \dots, \overline{b}_{s-1}, \overline{q}) \\
&\quad \cdot g(\overline{q}^*, \overline{b}_s, \dots, \overline{b}_m, \overline{b}_1, \dots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \dots, \overline{a}_n, \overline{a}_1, \dots, \overline{a}_{i-1}) && \text{(31a)} \\
&\quad - \sum \pm g(\overline{p}, \overline{b}_j, \dots, \overline{b}_{s-1}, \overline{q}) \\
&\quad \cdot f(\overline{q}^*, \overline{b}_s, \dots, \overline{b}_m, \overline{b}_1, \dots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \dots, \overline{a}_n, \overline{a}_1, \dots, \overline{a}_{i-1}); && \text{(31b)}
\end{aligned}$$

(viii) when $c_r = \overline{p}$ and $c_{s-1} \in \{\overline{b}_1, \dots, \overline{b}_{j-1}\}$, the corresponding summand is

$$\begin{aligned}
&\sum \pm f(\overline{p}, \overline{b}_j, \dots, \overline{b}_m, \overline{b}_1, \dots, \overline{b}_{s-1}, \overline{q}) \\
&\quad \cdot g(\overline{q}^*, \overline{b}_s, \dots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \dots, \overline{a}_n, \overline{a}_1, \dots, \overline{a}_{i-1}) && \text{(32a)} \\
&\quad - \sum \pm g(\overline{p}, \overline{b}_j, \dots, \overline{b}_m, \overline{b}_1, \dots, \overline{b}_{s-1}, \overline{q}) \\
&\quad \cdot f(\overline{q}^*, \overline{b}_s, \dots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \dots, \overline{a}_n, \overline{a}_1, \dots, \overline{a}_{i-1}); && \text{(32b)}
\end{aligned}$$

(ix) when $c_r = \overline{p}$ and $c_{s-1} = \overline{p}^*$, the corresponding summand is

$$\begin{aligned}
&\sum \pm f(\overline{p}, \overline{b}_j, \dots, \overline{b}_m, \overline{b}_1, \dots, \overline{b}_{j-1}, \overline{p}^*, \overline{q}) g(\overline{q}^*, \overline{a}_i, \dots, \overline{a}_n, \overline{a}_1, \dots, \overline{a}_{i-1}) && \text{(33a)} \\
&- \sum \pm g(\overline{p}, \overline{b}_j, \dots, \overline{b}_m, \overline{b}_1, \dots, \overline{b}_{j-1}, \overline{p}^*, \overline{q}) f(\overline{q}^*, \overline{a}_i, \dots, \overline{a}_n, \overline{a}_1, \dots, \overline{a}_{i-1}); && \text{(33b)}
\end{aligned}$$

(x) when $c_r = \overline{p}$ and $c_{s-1} \in \{\overline{a}_i, \dots, \overline{a}_n\}$, the corresponding summand is

$$\begin{aligned}
&\sum \pm f(\overline{p}, \overline{b}_j, \dots, \overline{b}_m, \overline{b}_1, \dots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \dots, \overline{a}_{s-1}, \overline{q}) \\
&\quad \cdot g(\overline{q}^*, \overline{a}_s, \dots, \overline{a}_n, \overline{a}_1, \dots, \overline{a}_{i-1}) && \text{(34a)} \\
&\quad - \sum \pm g(\overline{p}, \overline{b}_j, \dots, \overline{b}_m, \overline{b}_1, \dots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \dots, \overline{a}_{s-1}, \overline{q}) \\
&\quad \cdot f(\overline{q}^*, \overline{a}_s, \dots, \overline{a}_n, \overline{a}_1, \dots, \overline{a}_{i-1}); && \text{(34b)}
\end{aligned}$$

(xi) when $c_r, c_{s-1} \in \{\overline{b}_j, \ldots, \overline{b}_m\}$ the corresponding summand is

$$\begin{aligned}
&\sum \pm f(\overline{b}_r, \ldots, \overline{b}_{s-1}, \overline{q}) \\
&\quad \cdot g(\overline{q}^*, \overline{b}_s, \ldots, \overline{b}_m, \overline{b}_1, \ldots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \ldots, \overline{b}_{r-1}) \quad (35a) \\
&- \sum \pm g(\overline{b}_r, \ldots, \overline{b}_{s-1}, \overline{q}) \\
&\quad \cdot f(\overline{q}^*, \overline{b}_s, \ldots, \overline{b}_m, \overline{b}_1, \ldots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \ldots, \overline{b}_{r-1}); \quad (35b)
\end{aligned}$$

(xii) when $c_r \in \{\overline{b}_j, \ldots, \overline{b}_m\}$ and $c_{s-1} \in \{\overline{b}_1, \ldots, \overline{b}_{j-1}\}$, the corresponding summand is

$$\begin{aligned}
&\sum \pm f(\overline{b}_r, \ldots, \overline{b}_m, \overline{b}_1, \ldots, \overline{b}_{s-1}, \overline{q}) \\
&\quad \cdot g(\overline{q}^*, \overline{b}_s, \ldots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \ldots, \overline{b}_{r-1}) \quad (36a) \\
&\quad - \sum \pm g(\overline{b}_r, \ldots, \overline{b}_m, \overline{b}_1, \ldots, \overline{b}_{s-1}, \overline{q}) \\
&\quad \cdot f(\overline{q}^*, \overline{b}_s, \ldots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \ldots, \overline{b}_{r-1}); \quad (36b)
\end{aligned}$$

(xiii) when $c_r \in \{\overline{b}_j, \ldots, \overline{b}_m\}$ and $c_{s-1} = \overline{p}^*$, the corresponding summand is

$$\begin{aligned}
&\sum \pm f(\overline{b}_r, \ldots, \overline{b}_m, \overline{b}_1, \ldots, \overline{b}_{j-1}, \overline{p}^*, \overline{q}) \\
&\quad \cdot g(\overline{q}^*, \overline{a}_i, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \ldots, \overline{b}_{r-1}) \quad (37a) \\
&\quad - \sum \pm g(\overline{b}_r, \ldots, \overline{b}_m, \overline{b}_1, \ldots, \overline{b}_{j-1}, \overline{p}^*, \overline{q}) \\
&\quad \cdot f(\overline{q}^*, \overline{a}_i, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \ldots, \overline{b}_{r-1}); \quad (37b)
\end{aligned}$$

(xiv) when $c_r \in \{\overline{b}_j, \ldots, \overline{b}_m\}$ and $c_{s-1} \in \{\overline{a}_i, \ldots, \overline{a}_n\}$, the corresponding summand is

$$\begin{aligned}
&\sum \pm f(\overline{b}_r, \ldots, \overline{b}_m, \overline{b}_1, \ldots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \ldots, \overline{a}_{s-1}, \overline{q}) \\
&\quad \cdot g(\overline{q}^*, \overline{a}_s, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \ldots, \overline{b}_{r-1}) \quad (38a) \\
&\quad - \sum \pm g(\overline{b}_r, \ldots, \overline{b}_m, \overline{b}_1, \ldots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \ldots, \overline{a}_{s-1}, \overline{q}) \\
&\quad \cdot f(\overline{q}^*, \overline{a}_s, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \ldots, \overline{b}_{r-1}); \quad (38b)
\end{aligned}$$

(xv) when $c_r, c_{s-1} \in \{\overline{b}_1, \ldots, \overline{b}_{j-1}\}$, the corresponding summand is

$$\begin{aligned}
&\sum \pm f(\overline{b}_r, \ldots, \overline{b}_{s-1}, \overline{q}) \\
&\quad \cdot g(\overline{q}^*, \overline{b}_s, \ldots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \ldots, \overline{b}_m, \overline{b}_1, \ldots, \overline{b}_{r-1}) \quad (39a) \\
&- \sum \pm g(\overline{b}_r, \ldots, \overline{b}_{s-1}, \overline{q}) \\
&\quad \cdot f(\overline{q}^*, \overline{b}_s, \ldots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \ldots, \overline{b}_m, \overline{b}_1, \ldots, \overline{b}_{r-1}); \quad (39b)
\end{aligned}$$

(xvi) when $c_r \in \{\overline{b}_1, \dots, \overline{b}_{j-1}\}$ and $c_{s-1} = \overline{p}^*$, the corresponding summand is

$$\begin{aligned}
&\sum \pm f(\overline{b}_r, \dots, \overline{b}_{j-1}, \overline{p}^*, \overline{q}) \\
&\quad \cdot g(\overline{q}^*, \overline{a}_i, \dots, \overline{a}_n, \overline{a}_1, \dots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \dots, \overline{b}_m, \overline{b}_1, \dots, \overline{b}_{r-1}) && \text{(40a)} \\
&\quad - \sum \pm g(\overline{b}_r, \dots, \overline{b}_{j-1}, \overline{p}^*, \overline{q}) \\
&\quad \cdot f(\overline{q}^*, \overline{a}_i, \dots, \overline{a}_n, \overline{a}_1, \dots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \dots, \overline{b}_m, \overline{b}_1, \dots, \overline{b}_{r-1}); && \text{(40b)}
\end{aligned}$$

(xvii) when $c_r \in \{\overline{b}_1, \dots, \overline{b}_{j-1}\}$ and $c_{s-1} \in \{\overline{a}_i, \dots, \overline{a}_n\}$, the corresponding summand is

$$\begin{aligned}
&\sum \pm f(\overline{b}_r, \dots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \dots, \overline{a}_{s-1}, \overline{q}) \\
&\quad \cdot g(\overline{q}^*, \overline{a}_s, \dots, \overline{a}_n, \overline{a}_1, \dots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \dots, \overline{b}_m, \overline{b}_1, \dots, \overline{b}_{r-1}) && \text{(41a)} \\
&\quad - \sum \pm g(\overline{b}_r, \dots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \dots, \overline{a}_{s-1}, \overline{q}) \\
&\quad \cdot f(\overline{q}^*, \overline{a}_s, \dots, \overline{a}_n, \overline{a}_1, \dots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \dots, \overline{b}_m, \overline{b}_1, \dots, \overline{b}_{r-1}); && \text{(41b)}
\end{aligned}$$

(xviii) when $c_r = \overline{p}^*$ and $c_{s-1} \in \{\overline{a}_i, \dots, \overline{a}_n\}$, the corresponding summand is

$$\begin{aligned}
&\sum \pm f(\overline{p}^*, \overline{a}_i, \dots, \overline{a}_{s-1}, \overline{q}) \\
&\quad \cdot g(\overline{q}^*, \overline{a}_s, \dots, \overline{a}_n, \overline{a}_1, \dots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \dots, \overline{b}_m, \overline{b}_1, \dots, \overline{b}_{j-1}) && \text{(42a)} \\
&\quad - \sum \pm g(\overline{p}^*, \overline{a}_i, \dots, \overline{a}_{s-1}, \overline{q}) \\
&\quad \cdot f(\overline{q}^*, \overline{a}_s, \dots, \overline{a}_n, \overline{a}_1, \dots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \dots, \overline{b}_m, \overline{b}_1, \dots, \overline{b}_{j-1}); && \text{(42b)}
\end{aligned}$$

(xix) when $c_r, c_{s-1}\{\overline{a}_i, \dots, \overline{a}_n\}$, the corresponding summand is

$$\begin{aligned}
&\sum \pm f(\overline{a}_r, \dots, \overline{a}_{s-1}, \overline{q}) \\
&\cdot g(\overline{q}^*, \overline{a}_s, \dots, \overline{a}_n, \overline{a}_1, \dots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \dots, \overline{b}_m, \overline{b}_1, \dots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \dots, \overline{a}_{r-1}) && \text{(43a)} \\
&- \sum \pm g(\overline{a}_r, \dots, \overline{a}_{s-1}, \overline{q}) \\
&\cdot f(\overline{q}^*, \overline{a}_s, \dots, \overline{a}_n, \overline{a}_1, \dots, \overline{a}_{i-1}, \overline{p}, \overline{b}_j, \dots, \overline{b}_m, \overline{b}_1, \dots, \overline{b}_{j-1}, \overline{p}^*, \overline{a}_i, \dots, \overline{a}_{r-1}). && \text{(43b)}
\end{aligned}$$

On the other hand, if we write

$$\delta(f) = \sum f' \otimes f'', \quad \delta(g) = \sum g' \otimes g'',$$

then

$$\begin{aligned}
&\sum \pm [f', g] \otimes f''((\overline{a}_1, \dots, \overline{a}_n) \otimes (\overline{b}_1, \dots, \overline{b}_m)) \\
&= \sum \pm f'(\overline{a}_i, \dots, \overline{a}_{j-1}, \overline{p}) g(\overline{p}^*, \overline{a}_j, \dots, \overline{a}_n, \overline{a}_1, \dots, \overline{a}_{i-1}) f''(\overline{b}_1, \dots, \overline{b}_m)
\end{aligned}$$

$$
\begin{aligned}
&\quad - \sum \pm g(\overline{a}_i, \ldots, \overline{a}_{j-1}, \overline{p}) f'(\overline{p}^*, \overline{a}_j, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1}) f''(\overline{b}_1, \ldots, \overline{b}_m) \\
&= \sum \pm f'(\overline{a}_i, \ldots, \overline{a}_{j-1}, \overline{p}) f''(\overline{b}_1, \ldots, \overline{b}_m) g(\overline{p}^*, \overline{a}_j, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1}) \\
&\quad - \sum \pm g(\overline{a}_i, \ldots, \overline{a}_{j-1}, \overline{p}) f'(\overline{p}^*, \overline{a}_j, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1}) f''(\overline{b}_1, \ldots, \overline{b}_m) \\
&= \sum \pm f(\overline{b}_1, \ldots, \overline{b}_{r-1}, \overline{q}, \overline{a}_s, \ldots, \overline{a}_{j-1}, \overline{p}, \overline{a}_i, \ldots, \overline{a}_{s-1}, \overline{q}^*, \overline{b}_r, \ldots, \overline{b}_m) \\
&\quad \cdot g(\overline{p}^*, \overline{a}_j, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1}) \qquad (44a) \\
&\quad - \sum \pm f(\overline{b}_1, \ldots, \overline{b}_{r-1}, \overline{q}, \overline{a}_s, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1}, \overline{p}^*, \\
&\quad \overline{a}_j, \ldots, \overline{a}_{s-1}, \overline{q}^*, \overline{b}_r, \ldots, \overline{b}_m) g(\overline{a}_i, \ldots, \overline{a}_{j-1}, \overline{p}) \qquad (44b) \\
&\quad - \sum \pm f(\overline{b}_1, \ldots, \overline{b}_{r-1}, \overline{q}, \overline{a}_s, \ldots, \overline{a}_{i-1}, \overline{p}^*, \overline{a}_j, \ldots, \overline{a}_n, \\
&\quad \overline{a}_1, \ldots, \overline{a}_{s-1}, \overline{q}^*, \overline{b}_r, \ldots, \overline{b}_m) g(\overline{a}_i, \ldots, \overline{a}_{j-1}, \overline{p}). \qquad (44c)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\sum \pm f' \otimes [f'', g]((\overline{a}_1, \ldots, \overline{a}_n) \otimes (\overline{b}_1, \ldots, \overline{b}_m)) \\
&= \sum \pm f'(\overline{a}_1, \ldots, \overline{a}_n) f''(\overline{b}_k, \ldots, \overline{b}_{\ell-1}, \overline{p}) g(\overline{p}^*, \overline{b}_\ell, \ldots, \overline{b}_m, \overline{b}_1, \ldots, \overline{b}_{k-1}) \\
&\quad - \sum \pm f'(\overline{a}_1, \ldots, \overline{a}_n) g(\overline{b}_k, \ldots, \overline{b}_{\ell-1}, \overline{p}) f''(\overline{p}^*, \overline{b}_\ell, \ldots, \overline{b}_m, \overline{b}_1, \ldots, \overline{b}_{k-1}) \\
&= \sum \pm f'(\overline{a}_1, \ldots, \overline{a}_n) f''(\overline{b}_k, \ldots, \overline{b}_{\ell-1}, \overline{p}) g(\overline{p}^*, \overline{b}_\ell, \ldots, \overline{b}_m, \overline{b}_1, \ldots, \overline{b}_{k-1}) \\
&\quad - \sum \pm g(\overline{b}_k, \ldots, \overline{b}_{\ell-1}, \overline{p}) f'(\overline{a}_1, \ldots, \overline{a}_n) f''(\overline{p}^*, \overline{b}_\ell, \ldots, \overline{b}_m, \overline{b}_1, \ldots, \overline{b}_{k-1}) \\
&= \sum \pm f(\overline{a}_1, \ldots, \overline{a}_{i-1}, \overline{q}, \overline{b}_r, \ldots, \overline{b}_{\ell-1}, \overline{p}, \overline{b}_k, \ldots, \overline{b}_{r-1}, \overline{q}^*, \overline{a}_i, \ldots, \overline{a}_n) \\
&\quad \cdot g(\overline{p}^*, \overline{b}_\ell, \ldots, \overline{b}_m, \overline{b}_1, \ldots, \overline{b}_{k-1}) \qquad (45a) \\
&\quad - \sum \pm f(\overline{a}_1, \ldots, \overline{a}_{r-1}, \overline{q}, \overline{b}_s, \ldots, \overline{b}_m, \overline{b}_1, \ldots, \overline{b}_{k-1}, \overline{p}^*, \\
&\quad \overline{b}_\ell, \ldots, \overline{b}_{s-1}, \overline{q}^*, \overline{a}_r, \ldots, \overline{a}_n) g(\overline{b}_k, \ldots, \overline{b}_{\ell-1}, \overline{p}) \qquad (45b) \\
&\quad - \sum \pm f(\overline{a}_1, \ldots, \overline{a}_{r-1}, \overline{q}, \overline{b}_s, \ldots, \overline{b}_{k-1}, \overline{p}^*, \overline{b}_\ell, \ldots, \overline{b}_m, \\
&\quad \overline{b}_1, \ldots, \overline{b}_{s-1}, \overline{q}^*, \overline{a}_r, \ldots, \overline{a}_n) g(\overline{b}_k, \ldots, \overline{b}_{\ell-1}, \overline{p}), \qquad (45c)
\end{aligned}
$$

$$
\begin{aligned}
&\sum \pm [f, g'] \otimes g''((\overline{a}_1, \ldots, \overline{a}_n) \otimes (\overline{b}_1, \ldots, \overline{b}_m)) \\
&= \sum \pm f(\overline{a}_i, \ldots, \overline{a}_{j-1}, \overline{p}) g'(\overline{p}^*, \overline{a}_j, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1}) g''(\overline{b}_1, \ldots, \overline{b}_m) \\
&\quad - \sum \pm g'(\overline{a}_i, \ldots, \overline{a}_{j-1}, \overline{p}) f(\overline{p}^*, \overline{a}_j, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1} g''(\overline{b}_1, \ldots, \overline{b}_m) \\
&= \sum \pm f(\overline{a}_i, \ldots, \overline{a}_{j-1}, \overline{p}) g'(\overline{p}^*, \overline{a}_j, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1}) g''(\overline{b}_1, \ldots, \overline{b}_m) \\
&\quad - \sum \pm f(\overline{p}^*, \overline{a}_j, \ldots, \overline{a}_n, \overline{a}_1, \ldots, \overline{a}_{i-1}) g'(\overline{a}_i, \ldots, \overline{a}_{j-1}, \overline{p}) g''(\overline{b}_1, \ldots, \overline{b}_m) \\
&= \sum \pm f(\overline{a}_i, \ldots, \overline{a}_{j-1}, \overline{p}) g(\overline{b}_1, \ldots, \overline{b}_{r-1}, \overline{q}, \overline{a}_s, \ldots, \overline{a}_n,
\end{aligned}
$$

$$\overline{a}_1,\dots,\overline{a}_{i-1},\overline{p}^*,\overline{a}_j,\dots,\overline{a}_{s-1},\overline{q}^*,\overline{b}_r,\dots,\overline{b}_m) \tag{46a}$$
$$+\sum \pm f(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p})g(\overline{b}_1,\dots,\overline{b}_{r-1},\overline{q},\overline{a}_s,\dots,\overline{a}_{i-1},\overline{p}^*,$$
$$\overline{a}_j,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{s-1},\overline{q}^*,\overline{b}_r,\dots,\overline{b}_m) \tag{46b}$$
$$-\sum \pm f(\overline{p}^*,\overline{a}_j,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{i-1})$$
$$\cdot g(\overline{b}_1,\dots,\overline{b}_{r-1},\overline{q},a_s,\dots,\overline{a}_{j-1},\overline{p},\overline{a}_i,\dots,\overline{a}_{s-1},\overline{q}^*,\overline{b}_r,\dots,\overline{b}_m), \tag{46c}$$

and

$$\sum \pm g'\otimes[f,g''](({\overline{a}}_1,\dots,{\overline{a}}_n)\otimes({\overline{b}}_1,\dots,{\overline{b}}_m))$$
$$=\sum \pm g'(\overline{a}_1,\dots,\overline{a}_n)f(\overline{b}_i,\dots,\overline{b}_{j-1},\overline{p})g''(\overline{p}^*,\overline{b}_j,\dots,\overline{b}_m,\overline{b}_1,\dots,\overline{b}_{i-1})$$
$$-\sum \pm g'(\overline{a}_1,\dots,\overline{a}_n)g''(\overline{b}_i,\dots,\overline{b}_{j-1},\overline{p})f(\overline{p}^*,\overline{b}_j,\dots,\overline{b}_m,\overline{b}_1,\dots,\overline{b}_{i-1})$$
$$=\sum \pm g'(\overline{a}_1,\dots,\overline{a}_n)g''(\overline{p}^*,\overline{b}_j,\dots,\overline{b}_m,\overline{b}_1,\dots,\overline{b}_{i-1})f(\overline{b}_i,\dots,\overline{b}_{j-1},\overline{p})$$
$$-\sum \pm g'(\overline{a}_1,\dots,\overline{a}_n)g''(\overline{b}_i,\dots,\overline{b}_{j-1},\overline{p})f(\overline{p}^*,\overline{b}_j,\dots,\overline{b}_m,\overline{b}_1,\dots,\overline{b}_{i-1})$$
$$=\sum \pm f(\overline{b}_i,\dots,\overline{b}_{j-1},\overline{p})g(\overline{a}_1,\dots,\overline{a}_{r-1},\overline{q},\overline{b}_s,\dots,\overline{b}_m,$$
$$\overline{b}_1,\dots,\overline{b}_{i-1},\overline{p}^*,\overline{b}_j,\dots,\overline{b}_{s-1},\overline{q}^*,\overline{a}_r,\dots,\overline{a}_n) \tag{47a}$$
$$+\sum \pm f(\overline{b}_i,\dots,\overline{b}_{j-1},\overline{p})g(\overline{a}_1,\dots,\overline{a}_{r-1},\overline{q},\overline{b}_s,\dots,\overline{b}_{i-1},\overline{p}^*,$$
$$\overline{b}_j,\dots,\overline{b}_m,\overline{b}_1,\dots,\overline{b}_{s-1},\overline{q}^*,\overline{a}_r,\dots,\overline{a}_n) \tag{47b}$$
$$-\sum \pm f(\overline{p}^*,\overline{b}_j,\dots,\overline{b}_m,\overline{b}_1,\dots,\overline{b}_{i-1})$$
$$\cdot g(\overline{a}_1,\dots,\overline{a}_{r-1},\overline{q},\overline{b}_s,\dots,\overline{b}_{j-1},\overline{p},\overline{b}_i,\dots,\overline{b}_{s-1},\overline{q}^*,\overline{a}_r,\dots,\overline{a}_n). \tag{47c}$$

In the expansion for $\delta\circ[f,g]$, we have

$$(26a)+(26b)=-(42a)-(42b),\quad (27a)+(27b)=-(41a)-(41b),$$
$$(28a)+(28b)=-(38a)-(38b),\quad (31a)+(31b)=-(40a)-(40b),$$
$$(32a)+(32b)=-(37a)-(37b).$$

After these terms cancel with each other, we have

$$(25),\ (29),\ (30),\ (33),\ (34),\ (35),\ (36),\ (39),\ (43)$$

remained. Among these terms, we have

$$(44a)+(44b)+(44c)=(25b)+(29a)+(30a)+(33a)+(34a)+(43b),$$
$$(45a)+(45b)+(45c)=(33b)+(35b)+(36b)+(39b),$$
$$(46a)+(46b)+(46c)=(25a)+(29b)+(30b)+(34b)+(43a),$$
$$(47a)+(47b)+(47c)=(35a)+(36a)+(39a),$$

which means

$$\delta \circ [f,g] = [\delta(f),g] + [f,\delta(g)].$$

Thus, the Drinfeld compatibility follows.

3.2 Proof of involutivity

By definition, for any $f \in \mathrm{CC}^\bullet(\mathrm{Fuk}(M))$,

$$\begin{aligned}
&([\cdot,\cdot]\circ\delta(f))(\overline{a}_1,\overline{a}_2,\dots,\overline{a}_n)\\
&\quad=\delta(f)\bigg(\sum_{i<j}\sum_{p}\pm(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p})\otimes(\overline{a}_1,\dots,\overline{a}_{i-1},\overline{p}^*,\overline{a}_j,\dots,\overline{a}_n)\bigg) \qquad (48a)\\
&\quad-\delta(f)\bigg(\sum_{i<j}\sum_{p}\pm(\overline{a}_1,\dots,\overline{a}_{i-1},\overline{p}^*,\overline{a}_j,\dots,\overline{a}_n)\otimes(\overline{a}_i,\dots,\overline{a}_{j-1},\overline{p})\bigg). \qquad (48b)
\end{aligned}$$

The right-hand side of above equality vanishes because in (48a) and (48b), the two Hochschild chains on which $\delta(f)$ evaluates are graded symmetric while $\delta(f)$ is graded skew-symmetric, and henceforth the evaluation is zero. More directly, one sees that the value of (48a) equals the evaluation of f at

$$\begin{aligned}
&\sum_{\substack{i<j\\ i\leqslant k\leqslant j,l<i}}\sum_{p,q}\pm(\overline{a}_i,\dots,\overline{a}_{k-1},\overline{q},\overline{a}_l,\dots,\overline{a}_{i-1},\overline{p}^*,\overline{a}_j,\dots,\overline{a}_n,\\
&\quad\overline{a}_1,\dots,\overline{a}_{l-1},\overline{q}^*,\overline{a}_k,\dots,\overline{a}_{j-1},\overline{p})\\
&\quad+\sum_{\substack{i<j\\ i\leqslant k\leqslant j,j<l}}\sum_{p,q}\pm(\overline{a}_i,\dots,\overline{a}_{k-1},\overline{q},\overline{a}_l,\dots,\overline{a}_n,\overline{a}_1,\dots,\overline{a}_{i-1},\overline{p}^*,\\
&\quad\overline{a}_j,\dots,\overline{a}_{l-1},\overline{q}^*,\overline{a}_k,\dots,\overline{a}_{j-1},\overline{p}),
\end{aligned}$$

and the value of (48b) equals the evaluation of f at the same expression up to a cyclic order. This proves the involutivity.

In summary, we have shown that $(\mathrm{CC}^\bullet(\mathrm{Fuk})M),[\cdot,\cdot],\delta,b)$ forms a differential graded involutive Lie bialgebra of degree $(2-d,2-d)$.

4 Cotangent bundles

In this last section, we briefly discuss the cyclic cohomology of the Fukaya category of cotangent bundles. In general, the Fukaya category is very difficult to compute. However, in the case of cotangent bundles, the Fukaya category is strikingly simple, due to the following theorem.

Theorem 20 (Fukaya-Seidel-Smith and Nadler) *Let N be a simply-connected, compact spin manifold, and let T^*N be its cotangent bundle. Then there is a quasi-equivalence of A_∞ categories*

$$\Phi\colon \mathrm{Fuk}(T^*N)\to CF^\bullet(N),$$

where the latter is the Floer cochain complex of N, viewed as an A_∞ category with one object.

Proof See Fukaya et al. [11, Theorem 1] and Nadler [24, Theorem 1.3.1]. A further discussion can be found in Fukaya et al. [12]. A little more precisely, in [11, §1f] the authors showed that for any $K, L \in \mathrm{Fuk}(T^*N)$, there exists an element $\mu \in \mathrm{Hom}(N, L)$ such that

$$\begin{aligned} \mathrm{Hom}(L, K) &\to \mathrm{Hom}(N, K), \\ \alpha &\mapsto \alpha \circ \mu, \end{aligned}$$

extends to be a quasi-isomorphism. From this, they deduced that every $L \in \mathrm{Fuk}(T^*N)$ is quasi-isomorphic to N, and the theorem follows. □

We recommend the interested reader to Seidel [27, Chapter 1] for more details on A_∞ functors and their propositions. As a corollary to the above result, we obtain the following theorem.

Theorem 21 *Let N be a simply-connected, compact spin manifold, and let T^*N be the cotangent bundle of N. Then the cyclic homology of* $\mathrm{Fuk}(T^*N)$ *is isomorphic to the cyclic homology of the Floer cochain complex $CF^\bullet(N)$.*

Proof Getzler and Jones [13, Theorem 37] proved that the cyclic homology is invariant under quasi-isomorphisms of A_∞ algebras. On the category level, quasi-equivalent functors also induces isomorphic cyclic (co)homologies.

We first recall a special property of A_∞ categories: any quasi-equivalent A_∞ functor has an inverse. More precisely, suppose that $F\colon \mathscr{A} \to \mathscr{B}$ is a quasi-equivalence. Then there exists a quasi-equivalence functor $G\colon \mathscr{B} \to \mathscr{A}$ such that $G \circ F \cong \mathrm{Id}$ in $\mathrm{H}^0(fun(\mathscr{A}, \mathscr{A}))$ and $F \circ G \cong \mathrm{Id}$ in $\mathrm{H}^0(fun(\mathscr{B}, \mathscr{B}))$, where $\mathrm{H}^0(fun(\cdot, \cdot))$ is the homology category of A_∞ functors of the corresponding category (for a proof, see Seidel [27, Theorem 2.9]).

From this result, one deduces that for a quasi-equivalence $F\colon \mathscr{A} \to \mathscr{B}$, supposing that G is its inverse, by functoriality, we have maps

$$\mathrm{CC}(F)\colon \mathrm{CC}_\bullet(\mathscr{A}) \to \mathrm{CC}_\bullet(\mathscr{B}), \quad \mathrm{CC}(G)\colon \mathrm{CC}_\bullet(\mathscr{B}) \to \mathrm{CC}_\bullet(\mathscr{A}),$$

which are inverse to each other up to homotopy, and hence, they induce isomorphisms on homology. The proof for cyclic cohomology is the same, and the statement follows. □

On the other hand, the Floer cochain complex $CF^\bullet(N)$, and even its cyclic homology, is known for symplectic geometers by the following two theorems.

Theorem 22 (PSS isomorphism) *Let N be a simply-connected manifold as before. Then the Floer cochain complex $CF^\bullet(N)$ of N is quasi-isomorphic to its de Rham cochain complex $\Omega^\bullet(N)$.*

Proof This is the chain level statement of the famous Piunikhin-Salamon-Schwarz (PSS for short, [25]) isomorphism. A proof in the most general case

(without assuming the result of Fukaya-Seidel-Smith and Nadler cited above), can be found in Abouzaid [1, Theorem 1.1]. □

Theorem 23 [4, Theorem 4.3.1; 17, Theorem A] *Let N be a simply connected manifold, and let LN be its free loop space. Denote by $\Omega^\bullet(N)$ the de Rham cochain complex of N. Then the cyclic homology of $\Omega^\bullet(N)$ is isomorphic to the equivariant cohomology $\mathrm{H}^\bullet_{S^1}(LN)$.*

Thus, combining the above three theorems, we in fact have the following result.

Theorem 24 *Let N be a simply-connected compact spin manifold, and let T^*N be its cotangent bundle. Then the cyclic cohomology of $\mathrm{Fuk}(T^*N)$ is isomorphic to the equivariant homology $\mathrm{H}_\bullet^{S^1}(LN)$ of the free loop space LN, which induces on the latter an involutive Lie bialgebra structure.*

Proof The theorem follows from combining Theorems 21–23 and our main result Theorem A. □

The motivation of the above result is as follows: Chas and Sullivan [3] showed that the S^1-equivariant homology of the free loop space of a compact, smooth d-manifold M, modulo the constant loops, has a degree $(2-d, 2-d)$ involutive Lie bialgebra structure. Theorem 24 may be viewed in some sense a 'categorification' of their result. However, at present, we are not able to identify these two Lie bialgebras (for some partial results, see [5]); in particular, modulo the constant loops in Chas-Sullivan's construction of the Lie bialgebra is essential, and on the other hand, in our above theorem, there is no such issue.

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