

# Maximal function characterizations of Musielak-Orlicz-Hardy spaces associated with magnetic Schrödinger operators

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**Abstract** Let  $\varphi$  be a growth function, and let  $A := -(\nabla - i\mathbf{a}) \cdot (\nabla - i\mathbf{a}) + V$  be a magnetic Schrödinger operator on  $L^2(\mathbb{R}^n)$ ,  $n \geq 2$ , where  $\mathbf{a} := (a_1, a_2, \dots, a_n) \in L^2_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$  and  $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$ . We establish the equivalent characterizations of the Musielak-Orlicz-Hardy space  $H_{A,\varphi}(\mathbb{R}^n)$ , defined by the Lusin area function associated with  $\{e^{-t^2 A}\}_{t>0}$ , in terms of the Lusin area function associated with  $\{e^{-t\sqrt{A}}\}_{t>0}$ , the radial maximal functions and the non-tangential maximal functions associated with  $\{e^{-t^2 A}\}_{t>0}$  and  $\{e^{-t\sqrt{A}}\}_{t>0}$ , respectively. The boundedness of the Riesz transforms  $L_k A^{-1/2}$ ,  $k \in \{1, 2, \dots, n\}$ , from  $H_{A,\varphi}(\mathbb{R}^n)$  to  $L^\varphi(\mathbb{R}^n)$  is also presented, where  $L_k$  is the closure of  $\frac{\partial}{\partial x_k} - ia_k$  in  $L^2(\mathbb{R}^n)$ . These results are new even when  $\varphi(x, t) := \omega(x)t^p$  for all  $x \in \mathbb{R}^n$  and  $t \in (0, +\infty)$  with  $p \in (0, 1]$  and  $\omega \in A_\infty(\mathbb{R}^n)$  (the class of Muckenhoupt weights on  $\mathbb{R}^n$ ).

**Keywords** Magnetic Schrödinger operator, Musielak-Orlicz-Hardy space, Lusin area function, growth function, maximal function, Riesz transform

**MSC** 42B25, 42B20, 42B30, 42B35

## 1 Introduction

The development of the theory of Hardy spaces  $H^p(\mathbb{R}^n)$ ,  $p \in (0, 1]$ , was initiated by Stein and Weiss [37], and was originally tied to harmonic functions. In 1972, real variable methods were introduced into this subject by Fefferman and Stein [17]. Later, the advent of their atomic or molecular characterizations enabled the extension of  $H^p(\mathbb{R}^n)$  to far more general settings such as spaces

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of homogeneous type in the sense of Coifman and Weiss [8]. Nowadays, the theory of Hardy spaces has played an important role in analysis and partial differential equations; see, for example, [19,36]. It is known that  $H^p(\mathbb{R}^n)$  is essentially related to the Laplacian  $\Delta$  and there are many settings in which these classical spaces are not applicable. For instance, the Riesz transforms  $\nabla L^{-1/2}$  may not be bounded from  $H^1(\mathbb{R}^n)$  to  $L^1(\mathbb{R}^n)$  when  $L$  is a second order divergence form elliptic operator with complex bounded measurable coefficients; see [21].

Recently, the study of the theory of Hardy spaces associated with operators has been paid a lot of attention; see, for example, [1,4,7,12,14,15,20,22,25,40] and references therein. In particular, let

$$A := \sum_{k=1}^n L_k^* L_k + V$$

be a magnetic Schrödinger operator, where  $L_k$  is the closure in  $L^2(\mathbb{R}^n)$  of  $\frac{\partial}{\partial x_k} - ia_k$ ,  $L_k^*$  the adjoint operator of  $L_k$  in  $L^2(\mathbb{R}^n)$ ,  $k \in \{1, 2, \dots, n\}$ ,  $\mathbf{a} := (a_1, a_2, \dots, a_n): \mathbb{R}^n \rightarrow \mathbb{R}^n$  the magnetic potential, and  $V: \mathbb{R}^n \rightarrow \mathbb{R}$  the electrical potential. Auscher et al. [1] first investigated the theory of Hardy spaces  $H_L^p(\mathbb{R}^n)$ , defined by the Lusin area function associated with the semigroup  $\{e^{-tL}\}_{t>0}$ , where the infinitesimal generator  $L$  satisfies that the kernels of  $\{e^{-tL}\}_{t>0}$  have a Gaussian upper bound, and includes  $A$  as a special case. Duong and Yan [15] further showed that the dual space of  $H_A^1(\mathbb{R}^n)$  is  $\text{BMO}_A(\mathbb{R}^n)$  associated with  $A$ . Duong et al. [13] established the boundedness of the Riesz transforms  $L_k A^{-1/2}$  with  $k \in \{1, 2, \dots, n\}$  from the Hardy space  $H_A^1(\mathbb{R}^n)$  to  $L^1(\mathbb{R}^n)$ . Let  $\mathcal{X}$  be a metric space, and let  $L$  be a nonnegative self-adjoint operator satisfying the so-called Davies-Gaffney estimate. Hofmann et al. [20] introduced and characterized the space  $H_L^1(\mathcal{X})$  in terms of atoms, molecules and the Lusin area function associated with the semigroup  $\{e^{-t\sqrt{L}}\}_{t>0}$ . These characterizations were, in [20], applied to the Schrödinger operator  $A$  on  $\mathbb{R}^n$  with  $\mathbf{a} = 0$  to establish the equivalent characterizations of  $H_A^1(\mathbb{R}^n)$  in terms of the non-tangential maximal functions and the radial maximal functions associated with  $\{e^{-t^2 A}\}_{t>0}$  and  $\{e^{-t\sqrt{A}}\}_{t>0}$ , respectively. All these results were further generalized to Orlicz-Hardy spaces in [6,25], which include the Hardy spaces  $H_A^p(\mathbb{R}^n)$  with  $p \in (0, 1]$  as special cases. Inspired by [20,25], for general  $\mathbf{a}$ , the equivalent characterizations of  $H_A^p(\mathbb{R}^n)$  in terms of the non-tangential maximal functions and the radial maximal functions associated with  $\{e^{-t^2 A}\}_{t>0}$  and  $\{e^{-t\sqrt{A}}\}_{t>0}$  were established in [26].

On the other hand, Ky [29] studied Hardy spaces of Musielak-Orlicz type, which generalize the Orlicz-Hardy spaces introduced by Strömberg [38] and Janson [24] and the weighted Hardy spaces by García-Cuerva [18] and Strömberg and Torchinsky [39]. We point out that the motivation to study function spaces of Musielak-Orlicz type comes from applications to many fields of mathematics and physics; see [2,3,10,11,28,30]. Let  $L$  be a nonnegative self-adjoint operator

on a metric measure space  $\mathcal{X}$ , whose heat kernels satisfy Davies-Gaffney estimates, and let

$$\varphi: \mathcal{X} \times [0, +\infty) \rightarrow [0, +\infty)$$

be a function such that  $\varphi \in \mathbb{A}_\infty(\mathcal{X})$ , the class of uniformly Muckenhoupt weights (see Definition 1.1 below), its critical uniformly upper type index  $I(\varphi) \in (0, 1]$  and  $\varphi(\cdot, t) \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$ , namely,  $\varphi$  satisfies the uniformly reverse Hölder inequality of order  $2/[2 - I(\varphi)]$  (see Definition 2.2 below). In [42], the Musielak-Orlicz-Hardy space  $H_{\varphi, L}(\mathcal{X})$  was introduced and characterized in terms of atoms, molecules, and the Lusin-area function associated with the Poisson semigroup  $\{e^{-t\sqrt{A}}\}_{t>0}$ , and applied to the Schrödinger operator  $A$  on  $\mathbb{R}^n$  with  $\mathbf{a} = 0$  to obtain the equivalent characterizations of  $H_{\varphi, A}(\mathbb{R}^n)$  in terms of aforementioned four maximal functions. Recently, Bui et al. [5] further investigated  $H_{\varphi, L}(\mathcal{X})$  when  $L$  is a one-to-one operator of type  $\omega$ , has a bounded  $H_\infty$ -functional calculus in  $L^2(\mathcal{X})$  satisfying the reinforced  $(p_L, q_L)$  off-diagonal estimates on balls, and  $\varphi(\cdot, t) \in \mathbb{RH}_{(q_L/I(\varphi))'}(\mathcal{X})$ , where  $p_L \in [1, 2)$ ,  $q_L \in (2, +\infty]$ , and  $(q_L/I(\varphi))'$  denotes the conjugate exponent of  $q_L/I(\varphi)$ . Here and hereafter, for any index  $q \in [1, +\infty]$ ,  $q'$  denotes its *conjugate index*, that is,

$$\frac{1}{q} + \frac{1}{q'} = 1.$$

Let  $a_k \in L^2_{\text{loc}}(\mathbb{R}^n)$  be real-valued,  $k \in \{1, 2, \dots, n\}$ , and  $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$ . The aim of this article is to characterize the Musielak-Orlicz-Hardy space  $H_{\varphi, A}(\mathbb{R}^n)$  in terms of the Lusin-area function associated with the Poisson semigroup  $\{e^{-t\sqrt{A}}\}_{t>0}$  and aforementioned four maximal functions, and obtain the boundedness of the Riesz transforms  $L_k A^{-1/2}$  for  $k \in \{1, 2, \dots, n\}$  from the Musielak-Orlicz-Hardy space  $H_{\varphi, A}(\mathbb{R}^n)$  to the Musielak-Orlicz space  $L^\varphi(\mathbb{R}^n)$ .

To state our main results, we first recall some necessary notions and notation. In this article, for  $k \in \{1, 2, \dots, n\}$ ,  $L_k$  denotes the *closure* in  $L^2(\mathbb{R}^n)$  of  $\frac{\partial}{\partial x_k} - ia_k$  with domain  $C_c^\infty(\mathbb{R}^n)$  (the *set of  $C^\infty(\mathbb{R}^n)$  functions with compact support*). The corresponding *sesquilinear form*  $Q$  is defined by setting, for all  $f, g \in \mathcal{D}(Q)$ ,

$$Q(f, g) := \sum_{k=1}^n \int_{\mathbb{R}^n} L_k f(x) \overline{L_k(x)g(x)} dx + \int_{\mathbb{R}^n} V(x) f(x) \overline{g(x)} dx,$$

where

$$\mathcal{D}(Q) := \{f \in L^2(\mathbb{R}^n) : L_k f \in L^2(\mathbb{R}^n), k \in \{1, 2, \dots, n\}, \sqrt{V} f \in L^2(\mathbb{R}^n)\}.$$

The form  $Q$  is symmetric and closed. It was showed by Simon [35] that this form coincides with the minimal closure of the form given by the same expression but defined on  $C_c^\infty(\mathbb{R}^n)$ . Let

$$\begin{aligned} \mathcal{D}(A) := \left\{ f \in \mathcal{D}(Q) : \exists g \in L^2(\mathbb{R}^n) \text{ such that } \forall \varphi \in \mathcal{D}(Q), \right. \\ \left. Q(f, \varphi) = \int_{\mathbb{R}^n} g(x) \overline{\varphi(x)} dx \right\} \end{aligned} \quad (1.1)$$

and let  $Af := g$  for all  $f \in \mathcal{D}(A)$  and  $g \in L^2(\mathbb{R}^n)$  as in (1.1). Then the magnetic Schrödinger operator  $A$  is a self-adjoint operator by the symmetry of  $Q$ ; see [33]. Formally, we write

$$Af = \sum_{k=1}^n L_k^* L_k f + Vf \tag{1.2}$$

or

$$A = -(\nabla - i\mathbf{a}) \cdot (\nabla - i\mathbf{a}) + V.$$

On the other hand, a function  $\Phi: [0, +\infty) \rightarrow [0, +\infty)$  is called an *Orlicz function* if it is nondecreasing,  $\Phi(0) = 0$ ,  $\Phi(t) > 0$  for  $t \in (0, +\infty)$ , and  $\lim_{t \rightarrow +\infty} \Phi(t) = +\infty$  (see, for example, [32,34]). Unlike the classical case, an Orlicz function in this article may not be convex. The function  $\Phi$  is said to be of *upper* (resp. *lower*) *type*  $p$  for some  $p \in [0, +\infty)$ , if there exists a positive constant  $C$  such that, for all  $s \in [1, +\infty)$  (resp.  $s \in [0, 1]$ ) and  $t \in [0, +\infty)$ ,

$$\Phi(st) \leq C s^p \Phi(t).$$

For a given function  $\varphi: \mathbb{R}^n \times [0, +\infty) \rightarrow [0, +\infty)$  such that, for any given  $x \in \mathbb{R}^n$ ,  $\varphi(x, \cdot)$  is an Orlicz function,  $\varphi$  is said to be of *uniformly upper* (resp. *lower*) *type*  $p$  for some  $p \in [0, +\infty)$  if there exists a positive constant  $C$  such that, for all  $x \in \mathbb{R}^n$ ,  $s \in [1, +\infty)$  (resp.  $s \in [0, 1]$ ) and  $t \in [0, +\infty)$ ,

$$\varphi(x, st) \leq C s^p \varphi(x, t).$$

Moreover, let  $I(\varphi)$  and  $i(\varphi)$  be, respectively, the *critical uniformly upper type index* and the *critical uniformly lower type index* defined, respectively, by

$$I(\varphi) := \inf\{p \in (0, +\infty) : \varphi \text{ is of uniformly upper type } p\} \tag{1.3}$$

and

$$i(\varphi) := \sup\{p \in (0, +\infty) : \varphi \text{ is of uniformly lower type } p\}. \tag{1.4}$$

Observe that  $i(\varphi)$  and  $I(\varphi)$  may not be attainable, namely,  $\varphi$  may not be of uniformly lower type  $i(\varphi)$  or of uniformly upper type  $I(\varphi)$  (see [23,31] for some examples).

**Definition 1.1** [29] A function  $\varphi: \mathbb{R}^n \times [0, +\infty) \rightarrow [0, +\infty)$  is said to satisfy the *uniformly Muckenhoupt condition* for some  $q \in [1, +\infty)$ , denoted by  $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$ , if, when  $q \in (1, +\infty)$ ,

$$\sup_{t \in (0, +\infty)} \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B \varphi(x, t) dx \left\{ \frac{1}{|B|} \int_B [\varphi(x, t)]^{-q'/q} dx \right\}^{q/q'} < +\infty,$$

or

$$\sup_{t \in (0, +\infty)} \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B \varphi(x, t) dx \left( \operatorname{ess\,sup}_{y \in B} [\varphi(y, t)]^{-1} \right) < +\infty.$$

Here, the first suprema are taken over all  $t \in (0, +\infty)$  and the second ones over all balls  $B \subset \mathbb{R}^n$ .

Observe that, in Definition 1.1, if  $\varphi$  is independent of  $t$ , then  $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$  for  $q \in [1, +\infty)$  just means  $\varphi \in A_q(\mathbb{R}^n)$ , the classical *class of Muckenhoupt weights* (see, for example, [19,39]).

**Definition 1.2** [29] A function  $\varphi: \mathbb{R}^n \times [0, +\infty) \rightarrow [0, +\infty)$  is called a *growth function* if the following hold true:

- (i)  $\varphi$  is a *Musielak-Orlicz function*, namely,
  - (a) the function  $\varphi(x, \cdot): [0, +\infty) \rightarrow [0, +\infty)$  is an Orlicz function for any given  $x \in \mathbb{R}^n$ ;
  - (b) the function  $\varphi(\cdot, t)$  is a measurable function for any given  $t \in [0, +\infty)$ ;
- (ii)

$$\varphi \in \mathbb{A}_\infty(\mathbb{R}^n) := \bigcup_{q \in [1, +\infty)} \mathbb{A}_q(\mathbb{R}^n);$$

- (iii) the function  $\varphi$  is of uniformly upper type 1 and of uniformly lower type  $p$  for some  $p \in (0, 1]$ .

Throughout the article, we always *assume that  $\varphi$  is a growth function* as in Definition 1.2. Clearly, the functions

$$\varphi(x, t) := t^p, \quad \forall x \in \mathbb{R}^n, t \in (0, +\infty), p \in (0, 1], \quad (1.5)$$

and

$$\varphi(x, t) := \omega(x)t^p, \quad \forall x \in \mathbb{R}^n, t \in (0, +\infty), p \in (0, 1], \quad (1.6)$$

are both growth functions in Definition 1.2, where

$$\omega \in A_\infty(\mathbb{R}^n) := \bigcup_{q \in [1, +\infty)} A_q(\mathbb{R}^n).$$

Another typical growth function is

$$\varphi(x, t) := \frac{t}{\log(e + |x|) + \log(e + t)}, \quad \forall x \in \mathbb{R}^n, t \in (0, +\infty). \quad (1.7)$$

If  $\varphi$  is as in (1.7), then it is easy to show that  $\varphi \in \mathbb{A}_1(\mathbb{R}^n)$ ,  $I(\varphi) = i(\varphi) = 1$ ,  $i(\varphi)$  is not attainable, but  $I(\varphi)$  is attainable (see [5,29]). For more examples of growth functions, see, for example, [5,23,29]. The *Musielak-Orlicz space*  $L^\varphi(\mathbb{R}^n)$  is then defined as the set of all measurable functions  $f$  such that

$$\int_{\mathbb{R}^n} \varphi(x, |f(x)|) dx < +\infty$$

with *Luxemburg norm*

$$\|f\|_{L^\varphi(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, +\infty) : \int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

We now recall the definition of  $H_{\varphi,A}(\mathbb{R}^n)$  in [5]; see also [1,20,26] for the definition of  $H_A^p(\mathbb{R}^n)$  with  $p \in (0, 1]$ , which corresponds to  $\varphi$  as in (1.5). For all functions  $f \in L^2(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ , define the *Lusin-area function*  $S_A f$  by

$$S_A f(x) := \left[ \iint_{\Gamma(x)} |t^2 A e^{-t^2 A} f(y)|^2 \frac{dy dt}{t^{n+1}} \right]^{1/2},$$

where, for all  $x \in \mathbb{R}^n$ ,

$$\Gamma(x) := \{(y, t) \in \mathbb{R}^n \times (0, +\infty) : |x - y| < t\}. \tag{1.8}$$

It is known that  $S_A$  is bounded on  $L^2(\mathbb{R}^n)$ ; see, for example, [13,20].

**Definition 1.3** Let  $\varphi$  and  $A$  be as in Definition 1.2 and (1.2), respectively. A function  $f \in L^2(\mathbb{R}^n)$  is said to be in  $\tilde{H}_{\varphi,A}(\mathbb{R}^n)$  if  $S_A f \in L^\varphi(\mathbb{R}^n)$ ; moreover, define

$$\|f\|_{H_{\varphi,A}(\mathbb{R}^n)} := \|S_A f\|_{L^\varphi(\mathbb{R}^n)}.$$

The *Musielaik-Orlicz-Hardy space*  $H_{\varphi,A}(\mathbb{R}^n)$  is then defined as the completion of  $\tilde{H}_{\varphi,A}(\mathbb{R}^n)$  in the quasi-norm  $\|\cdot\|_{H_{\varphi,A}(\mathbb{R}^n)}$ .

**Remark 1.4** The space  $H_{\varphi,L}(\mathcal{X})$  was introduced in [42] (see also [5]) when  $\mathcal{X}$  is a metric measure space,  $L$  is a nonnegative self-adjoint operator satisfying the Davies-Gaffney estimates, and  $\varphi$  is a growth function satisfying the *additional* assumption that  $\varphi \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$ , namely,  $\varphi(\cdot, t)$  satisfies the uniformly reverse Hölder inequality of order  $2/[2 - I(\varphi)]$  (see Definition 2.2 below).

We now recall the *Lusin-area function*  $S_P f$  and the *maximal functions*  $\mathcal{N}_h f$ ,  $\mathcal{N}_P f$ ,  $\mathcal{R}_h f$ , and  $\mathcal{R}_P f$  in [26], respectively, as follows. For all  $\beta \in (0, +\infty)$ ,  $f \in L^2(\mathbb{R}^n)$ , and  $x \in \mathbb{R}^n$ , let

$$\begin{aligned} S_P f(x) &:= \left[ \iint_{\Gamma(x)} |t\sqrt{A} e^{-t\sqrt{A}} f(y)|^2 \frac{dy dt}{t^{n+1}} \right]^{1/2}, \\ \mathcal{N}_h^\beta f(x) &:= \sup_{y \in B(x, \beta t), t > 0} |e^{-t^2 A} f(y)|, \quad \mathcal{N}_P^\beta f(x) := \sup_{y \in B(x, \beta t), t > 0} |e^{-t\sqrt{A}} f(y)|, \\ \mathcal{R}_h f(x) &:= \sup_{t > 0} |e^{-t^2 A} f(x)|, \quad \mathcal{R}_P f(x) := \sup_{t > 0} |e^{-t\sqrt{A}} f(x)|, \end{aligned}$$

where  $\Gamma(x)$  for  $x \in \mathbb{R}^n$  is as in (1.8). Denote  $\mathcal{N}_h^1 f$  and  $\mathcal{N}_P^1 f$  simply by  $\mathcal{N}_h f$  and  $\mathcal{N}_P f$ , respectively. It is known that all these operators are bounded on  $L^2(\mathbb{R}^n)$ ; see the proofs of [25, Theorem 5.2] and [26, Theorem 1.4].

**Definition 1.5** A function  $f \in L^2(\mathbb{R}^n)$  is said to be in  $\tilde{H}_{\varphi, \tilde{\mathcal{M}}}(\mathbb{R}^n)$  if  $\tilde{\mathcal{M}} f \in L^\varphi(\mathbb{R}^n)$ , where  $\tilde{\mathcal{M}} f$  is one of  $S_P f$ ,  $\mathcal{N}_h f$ ,  $\mathcal{N}_P f$ ,  $\mathcal{R}_h f$ , and  $\mathcal{R}_P f$  as above; moreover, let

$$\|f\|_{H_{\varphi, \tilde{\mathcal{M}}}(\mathbb{R}^n)} := \|\tilde{\mathcal{M}} f\|_{L^\varphi(\mathbb{R}^n)}.$$

The Musielak-Orlicz-Hardy space  $H_{\varphi, \tilde{\mathcal{M}}}(\mathbb{R}^n)$  is then defined as the completion of  $\tilde{H}_{\varphi, \tilde{\mathcal{M}}}(\mathbb{R}^n)$  in  $\|\cdot\|_{H_{\varphi, \tilde{\mathcal{M}}}(\mathbb{R}^n)}$ .

The main result of this article is as follows.

**Theorem 1.6** *Let  $\varphi$  and  $A$  be as in Definition 1.2 and (1.2), respectively. Then the spaces  $H_{\varphi, A}(\mathbb{R}^n)$ ,  $H_{\varphi, S_P}(\mathbb{R}^n)$ ,  $H_{\varphi, \mathcal{A}_h}(\mathbb{R}^n)$ ,  $H_{\varphi, \mathcal{A}_P}(\mathbb{R}^n)$ ,  $H_{\varphi, \mathcal{N}_h}(\mathbb{R}^n)$ , and  $H_{\varphi, \mathcal{N}_P}(\mathbb{R}^n)$  coincide with equivalent norms.*

**Remark 1.7** To the best of our knowledge, Theorem 1.6 is known only when  $\varphi$  is as in (1.5), and new for other cases. To be precise, when  $\varphi$  is as in (1.5) with  $p \in (0, 1]$  therein, we see that

$$H_{\varphi, A}(\mathbb{R}^n) = H_A^p(\mathbb{R}^n),$$

which was introduced in [25]. Theorem 1.6 in this case was established in [25, Theorem 5.2] and [26, Theorem 1.4]. Otherwise, Theorem 1.6 is new, even when  $\varphi$  is as in (1.6) or in (1.7).

We also obtain the following boundedness of the Riesz transforms  $L_k A^{-1/2}$ ,  $k \in \{1, 2, \dots, n\}$ , from the space  $H_{\varphi, A}(\mathbb{R}^n)$  to  $L^\varphi(\mathbb{R}^n)$ .

**Theorem 1.8** *Let  $\varphi$  be as in Definition 1.2, and let  $I(\varphi)$  and  $r(\varphi)$  be as in (1.3) and (2.3), respectively. Assume that  $r(\varphi) \in (2/[2 - I(\varphi)], +\infty]$ . Then the Riesz transforms  $L_k A^{-1/2}$ ,  $k \in \{1, 2, \dots, n\}$ , are bounded from  $H_{\varphi, A}(\mathbb{R}^n)$  to  $L^\varphi(\mathbb{R}^n)$ .*

**Remark 1.9** (i) To the best of our knowledge, Theorem 1.8 is known only when  $\varphi$  is as in (1.5), and new for other cases. To be precise, when  $\varphi$  is as in (1.5) with  $p \in (0, 1]$  therein, we see that, in this case,  $r(\varphi) = +\infty$  and  $I(\varphi) = p$ , and hence, the assumption  $r(\varphi) \in (2/[2 - I(\varphi)], +\infty]$  of Theorem 1.8 holds true automatically. In this case, Theorem 1.8 is [26, Theorem 1.5]. Otherwise, Theorem 1.8 is new, even when  $\varphi$  is as in (1.6) or in (1.7).

(ii) We mention that the range of  $r(\varphi) \in (2/[2 - I(\varphi)], +\infty]$  is determined by the atomic characterization of  $H_{\varphi, A}(\mathbb{R}^n)$  and the  $L^q(\mathbb{R}^n)$ -boundedness of the Riesz transforms  $L_k A^{-1/2}$ . To be precise, Bui et al. [5, Theorem 5.4] showed that  $H_{\varphi, A}(\mathbb{R}^n)$  and  $H_{\varphi, A, \text{at}}^{M, q}(\mathbb{R}^n)$  coincide for all  $M \in \mathbb{N}$  with  $M > \frac{n}{2} \frac{q(\varphi)}{i(\varphi)}$ , and  $q > r(\varphi)I(\varphi)/(r(\varphi) - 1)$  (this is equivalent to that  $r(\varphi) > q/(q - I(\varphi))$ ). On the other hand, Duong et al. [13] showed that the Riesz transforms  $L_k A^{-1/2}$ ,  $k \in \{1, 2, \dots, n\}$ , are bounded on  $L^q(\mathbb{R}^n)$  for all  $q \in (1, 2]$ . In the proof of Theorem 1.8, we need to use the aforementioned two facts at the same time, which induce that the best choice for  $r(\varphi)$  is  $r(\varphi) \in (2/[2 - I(\varphi)], +\infty]$ .

We also notice that, when  $A$  is the Schrödinger operator  $L := -\Delta + V$ , it was shown in [5, Theorem 8.5(i)] that, if the Riesz transforms  $\nabla L^{-1/2}$  are bounded on  $L^q(\mathbb{R}^n)$  for all  $q \in (1, p_0)$  with  $p_0 \in (2, +\infty)$ , then  $\nabla L^{-1/2}$  are also bounded from the Musielak-Orlicz-Hardy space  $H_{\varphi, L}(\mathbb{R}^n)$  to  $L^\varphi(\mathbb{R}^n)$  with  $r(\varphi) \in (p_0/(p_0 - I(\varphi)), +\infty]$ . Thus, the range of  $r(\varphi)$  in Theorem 1.8 coincides with this range.

The organization of the article is as follows.

Section 2 is devoted to some basic lemmas needed in Sections 3 and 4. We recall some known basic properties of Musielak-Orlicz functions established in [23,29] and a bounded criterion of linear operators from  $H_{\varphi,A}(\mathbb{R}^n)$  to  $L^\varphi(\mathbb{R}^n)$  in [5].

The proof of Theorem 1.6 is presented in Section 3. As in [20], we show Theorem 1.6 by proving the following inclusion link:

$$\begin{aligned} H_{\varphi,A}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) &\subset H_{\varphi,\mathcal{N}_h}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \\ &\subset H_{\varphi,\mathcal{R}_h}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \\ &\subset H_{\varphi,\mathcal{R}_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \\ &\subset H_{\varphi,\mathcal{N}_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \\ &\subset H_{\varphi,S_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \\ &\subset H_{\varphi,A}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n). \end{aligned}$$

We point out that important tools used in the proof of Theorem 1.6 include the properties of  $\varphi$  (see Lemmas 2.1 and 2.3 below), the Besicovitch covering lemma, the Whitney decomposition, the semigroup properties of  $\{e^{-tA}\}_{t>0}$ , the Caccioppoli inequality associated with  $A$  (this was established in [26]; see also Lemma 3.1 below), and the fact that the kernels of  $\{e^{-tA}\}_{t>0}$  satisfy the Gaussian upper bound (see (3.3) below). Besides these key tools, by the atomic characterization of  $H_{\varphi,A}(\mathbb{R}^n)$  and the bounded criterion, from  $H_{\varphi,A}(\mathbb{R}^n)$  to  $L^\varphi(\mathbb{R}^n)$ , of linear operators from [5], we show the inclusion

$$H_{\varphi,A}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset H_{\varphi,\mathcal{N}_h}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

On the other hand, as in [42], we show the inclusion

$$H_{\varphi,S_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset H_{\varphi,A}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$$

by establishing a pointwise estimate concerning a truncated Lusin-area function  $\tilde{S}_P^{\varepsilon,R,\tau} f$  and the non-tangential maximal function  $\mathcal{N}_P f$  (see (3.1) for the definition of  $\tilde{S}_P^{\varepsilon,R,\tau} f$ ), and a ‘good- $\lambda$  inequality’ between these two operators. Also, it is worth to point out that, in the proof of this inclusion, the special differential structure of the operator  $A$  itself plays an essential role.

In Section 4, by using the properties of  $\varphi$ , the semigroup properties of  $\{e^{-tA}\}_{t>0}$ , the Davies-Gaffney estimates of  $\{tL_k e^{-t^2 A}\}_{t>0}$  from [26], the atomic characterization of  $H_{\varphi,A}(\mathbb{R}^n)$ , and the bounded criterion, from  $H_{\varphi,A}(\mathbb{R}^n)$  to  $L^\varphi(\mathbb{R}^n)$ , of linear operators from [5], we show Theorem 1.8. We mention that this result is different from some known results, for example, [5, Theorem 8.5 (ii)], where the Riesz transform  $\nabla L^{-1/2}$ , associated with the Schrödinger operator  $L := -\Delta + V$ , is bounded from the Musielak-Orlicz-Hardy space  $H_{\varphi,L}(\mathbb{R}^n)$  to  $H_\varphi(\mathbb{R}^n)$ . The reason for this difference is that, because of the existence of  $\mathbf{a}$ , it is unclear whether

$$\int_{\mathbb{R}^n} L_k A^{-1/2} \alpha(x) dx = 0$$

for  $k \in \{1, 2, \dots, n\}$  and any molecule  $\alpha$  associated with  $A$ , is true or not.

We now make some conventions on notation. Throughout this article, we always use  $C$  to denote a positive constant that is independent of the main parameters involved, but it may differ from line to line. The symbol  $f \lesssim g$  means that  $f \leq Cg$  for some positive number  $C$  independent of  $f$  and  $g$ , and  $f \sim g$  means  $f \lesssim g \lesssim f$ . For any complex number  $z$ , its real part is denoted by  $\operatorname{Re} z$ . For any  $x \in \mathbb{R}^n$  and  $\lambda, r > 0$ , let  $Q := Q(x, r)$  be the *cube centered at  $x$  with side length  $r$*  and  $\lambda Q := Q(x, \lambda r)$ ; similarly,  $B := B(x, r)$  denotes the *ball centered at  $x$  with radius  $r$*  and  $\lambda B := B(x, \lambda r)$ . Moreover, for any ball  $B \subset \mathbb{R}^n$ , let

$$S_0(B) := B, \quad S_j(B) := (2^j B) \setminus (2^{j-1} B), \quad j \in \mathbb{N} := \{1, 2, \dots\}. \quad (1.9)$$

Let  $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ . Also, for any set  $E \subset \mathbb{R}^n$ ,  $\chi_E$  denotes its *characteristic function*. Moreover, for all sets  $E, F \subset \mathbb{R}^n$  and  $z \in \mathbb{R}^n$ ,

$$\operatorname{dist}(E, F) := \inf_{x \in E, y \in F} |x - y|, \quad \operatorname{dist}(z, E) := \inf_{x \in E} |x - z|.$$

Finally, for any growth function  $\varphi$ , measurable subset  $E$  of  $\mathbb{R}^n$ , and  $t \in [0, +\infty)$ , let

$$\varphi(E, t) := \int_E \varphi(x, t) dx.$$

## 2 Basic lemmas

In this section, we recall some basic lemmas used in Sections 3 and 4. We begin with the following lemma on the estimates of  $\varphi$ , which was first established by Ky in [29].

**Lemma 2.1** *Let  $\varphi$  be as in Definition 1.2. Then the following statements hold true.*

(i) *There exists a positive constant  $C$  such that, for all  $(x, t_j) \in \mathbb{R}^n \times [0, +\infty)$  with  $j \in \mathbb{N}$ ,*

$$\varphi\left(x, \sum_{j=1}^{+\infty} t_j\right) \leq C \sum_{j=1}^{+\infty} \varphi(x, t_j).$$

(ii) *For all  $(x, t) \in \mathbb{R}^n \times [0, +\infty)$ , let*

$$\tilde{\varphi}(x, t) := \int_0^t \frac{\varphi(x, s)}{s} ds.$$

*Then  $\tilde{\varphi}$  is a growth function equivalent to  $\varphi$ ; moreover,  $\tilde{\varphi}$  is continuous and strictly increasing.*

(iii) *For all  $f \in L^\varphi(\mathbb{R}^n) \setminus \{0\}$ ,*

$$\int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{\|f\|_{L^\varphi(\mathbb{R}^n)}}\right) dx = 1.$$

Similar to the class  $A_\infty(\mathbb{R}^n)$  of Muckenhoupt weights, it turns out that uniformly weights in  $A_\infty(\mathbb{R}^n)$  also have some very useful properties, including the following uniformly reverse Hölder condition introduced in [23].

**Definition 2.2** A function  $\varphi: \mathbb{R}^n \times [0, +\infty) \rightarrow [0, +\infty)$  is said to satisfy the *uniformly reverse Hölder condition* for some  $q \in (1, +\infty]$ , denoted by  $\varphi \in \mathbb{RH}_q(\mathbb{R}^n)$ , if, when  $q \in (1, +\infty)$ ,

$$\sup_{t \in (0, +\infty)} \sup_{B \subset \mathbb{R}^n} \left\{ \frac{1}{|B|} \int_B [\varphi(x, t)]^q dx \right\}^{1/q} \left\{ \frac{1}{|B|} \int_B \varphi(x, t) dx \right\}^{-1} < +\infty,$$

or

$$\sup_{t \in (0, +\infty)} \sup_{B \subset \mathbb{R}^n} \left\{ \operatorname{ess\,sup}_{y \in B} \varphi(y, t) \right\} \left\{ \frac{1}{|B|} \int_B \varphi(x, t) dx \right\}^{-1} < +\infty,$$

where the first suprema are taken over all  $t \in (0, +\infty)$  and the second ones over all balls  $B \subset \mathbb{R}^n$ .

The following lemma was obtained in [23]; see also [29,31].

**Lemma 2.3** *The following statements hold true:*

- (i)  $A_1(\mathbb{R}^n) \subset A_p(\mathbb{R}^n) \subset A_q(\mathbb{R}^n)$  for  $1 \leq p \leq q < +\infty$ ;
- (ii)  $\mathbb{RH}_\infty(\mathbb{R}^n) \subset \mathbb{RH}_p(\mathbb{R}^n) \subset \mathbb{RH}_q(\mathbb{R}^n)$  for  $1 < q \leq p \leq +\infty$ ;
- (iii) if  $\varphi \in A_p(\mathbb{R}^n)$  with  $p \in (1, +\infty)$ , then there exists  $q \in (1, p)$  such that  $\varphi \in A_q(\mathbb{R}^n)$ ;
- (iv) if  $\varphi \in \mathbb{RH}_q(\mathbb{R}^n)$  with  $q \in [1, +\infty)$ , then there exists  $p \in (q, +\infty)$  such that  $\varphi \in \mathbb{RH}_p(\mathbb{R}^n)$ ;
- (v)  $A_\infty(\mathbb{R}^n) = \cup_{p \in [1, +\infty)} A_p(\mathbb{R}^n) = \cup_{q \in (1, +\infty]} \mathbb{RH}_q(\mathbb{R}^n)$ ;
- (vi) if  $p \in (1, +\infty)$  and  $\varphi \in A_p(\mathbb{R}^n)$ , then there exists a positive constant  $C$  such that, for all measurable functions  $f$  on  $\mathbb{R}^n$  and  $t \in [0, +\infty)$ ,

$$\int_{\mathbb{R}^n} [\mathcal{M}(f)(x)]^p \varphi(x, t) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \varphi(x, t) dx,$$

where  $\mathcal{M}$  denotes the Hardy-Littlewood maximal function on  $\mathbb{R}^n$ , defined by setting, for all  $x \in \mathbb{R}^n$ ,

$$\mathcal{M}f(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy \tag{2.1}$$

and the supremum is taken over all balls  $B$  containing  $x$ ;

- (vii) if  $\varphi \in A_p(\mathbb{R}^n)$  with  $p \in [1, +\infty)$ , then there exists a positive constant  $C$  such that, for all balls  $B_1, B_2 \subset \mathbb{R}^n$  with  $B_1 \subset B_2$  and  $t \in (0, +\infty)$ ,

$$\frac{\varphi(B_2, t)}{\varphi(B_1, t)} \leq C \left[ \frac{|B_2|}{|B_1|} \right]^p;$$

(viii) if  $\varphi \in \mathbb{RH}_q(\mathbb{R}^n)$  with  $q \in [1, +\infty)$ , then there exists a positive constant  $C$  such that, for all balls  $B_1, B_2 \subset \mathbb{R}^n$  with  $B_1 \subset B_2$  and  $t \in (0, +\infty)$ ,

$$\frac{\varphi(B_2, t)}{\varphi(B_1, t)} \geq C \left[ \frac{|B_2|}{|B_1|} \right]^{1-\frac{1}{q}}.$$

For  $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$ , its *critical indices* of  $\varphi$ ,  $q(\varphi)$  and  $r(\varphi)$ , are defined, respectively, as follows:

$$q(\varphi) := \inf\{q \in [1, +\infty) : \varphi \in \mathbb{A}_q(\mathbb{R}^n)\} \quad (2.2)$$

$$r(\varphi) := \sup\{q \in [1, +\infty) : \varphi \in \mathbb{RH}_q(\mathbb{R}^n)\}. \quad (2.3)$$

Recall that, if  $q(\varphi) \in (1, +\infty)$ , then, by Lemma 2.3 (iii), we see that  $\varphi \notin \mathbb{A}_{q(\varphi)}(\mathbb{R}^n)$ , and there exists  $\varphi \notin \mathbb{A}_1(\mathbb{R}^n)$  such that  $q(\varphi) = 1$  (see, for example, [27]). Similarly, if  $r(\varphi) \in (1, +\infty)$ , then, by Lemma 2.3 (iv), we find that  $\varphi \notin \mathbb{RH}_{r(\varphi)}(\mathbb{R}^n)$ , and there exists  $\varphi \notin \mathbb{RH}_\infty(\mathbb{R}^n)$  such that  $r(\varphi) = +\infty$  (see, for example, [9]).

We now recall the atomic characterization of  $H_{\varphi, A}(\mathbb{R}^n)$  from [5]. First, we recall the following notion of  $(\varphi, q, M)_A$ -atoms.

**Definition 2.4** Let  $\varphi$  be a growth function as in Definition 1.2,  $M \in \mathbb{N}$  and  $q \in (1, +\infty)$ . A function  $\alpha \in L^q(\mathbb{R}^n)$  is called a  $(\varphi, q, M)_A$ -atom associated to  $A$  if there exists a function  $b \in \mathcal{D}(A^M)$  and a ball  $B := B(x_B, r_B)$  for  $x_B \in \mathbb{R}^n$  and  $r_B \in (0, +\infty)$  such that

$$(A)_i \quad \alpha = A^M b;$$

$$(A)_{ii} \quad \text{supp}(A^k b) \subset B, \quad k \in \{0, 1, \dots, M\};$$

$$(A)_{iii} \quad \|(r_B^2 A)^k b\|_{L^q(\mathbb{R}^n)} \leq r_B^{2M} |B|^{1/q} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}, \quad k \in \{0, 1, \dots, M\}.$$

A function  $f \in L^2(\mathbb{R}^n)$  is said to have an *atomic*  $(\varphi, q, M)_A$ -representation,  $f = \sum_j \lambda_j \alpha_j$ , if, for each  $j$ ,  $\alpha_j$  is a  $(\varphi, q, M)_A$ -atom associated to a ball  $B_j \subset \mathbb{R}^n$ , the summation converges in  $L^2(\mathbb{R}^n)$  and  $\{\lambda_j\}_j \subset \mathbb{C}$  satisfies

$$\sum_j \varphi(B_j, |\lambda_j| \|\chi_{B_j}\|_{L^\varphi(\mathbb{R}^n)}^{-1}) < +\infty.$$

Let

$$\tilde{H}_{\varphi, A, \text{at}}^{M, q}(\mathbb{R}^n) := \{f : f \text{ has an atomic } (\varphi, q, M)_A\text{-representation}\}$$

with the quasi-norm  $\|\cdot\|_{\tilde{H}_{\varphi, A, \text{at}}^{M, q}(\mathbb{R}^n)}$  given by setting, for all  $f \in \tilde{H}_{\varphi, A, \text{at}}^{M, q}(\mathbb{R}^n)$ ,

$$\|f\|_{\tilde{H}_{\varphi, A, \text{at}}^{M, q}(\mathbb{R}^n)} := \inf \left\{ \Lambda(\{\lambda_j \alpha_j\}_j) : f = \sum_j \lambda_j \alpha_j \text{ is an atomic } (\varphi, q, M)_A\text{-representation} \right\},$$

where the infimum is taken over all the atomic  $(\varphi, q, M)_A$ -representation of  $f$  as above and

$$\Lambda(\{\lambda_j \alpha_j\}_j) := \inf \left\{ \lambda \in (0, +\infty) : \sum_j \varphi \left( B_j, \frac{|\lambda_j|}{\lambda \|\chi_{B_j}\|_{L^\varphi(\mathbb{R}^n)}} \right) \leq 1 \right\}.$$

The *atomic Musielak-Orlicz-Hardy space*  $H_{\varphi, A, \text{at}}^{M, q}(\mathbb{R}^n)$  is then defined as the completion of  $\tilde{H}_{\varphi, A, \text{at}}^{M, q}(\mathbb{R}^n)$  with respect to the quasi-norm  $\|\cdot\|_{H_{\varphi, A, \text{at}}^{M, q}(\mathbb{R}^n)}$ .

The following atomic characterization of  $H_{\varphi, A}(\mathbb{R}^n)$  was established in [5, Theorem 5.4].

**Lemma 2.5** *Let  $\varphi$  be as in Definition 1.2,  $A$  as in (1.2),*

$$q \in \left( \frac{r(\varphi)}{r(\varphi) - 1} I(\varphi), +\infty \right),$$

and  $M \in \mathbb{N}$  with  $M > \frac{n}{2} \frac{q(\varphi)}{i(\varphi)}$ , where  $i(\varphi)$  and  $q(\varphi)$  are as in (1.4) and (2.2), respectively. Then the spaces  $H_{\varphi, A}(\mathbb{R}^n)$  and  $H_{\varphi, A, \text{at}}^{M, q}(\mathbb{R}^n)$  coincide with equivalent quasi-norms.

From Lemma 2.5 and the boundedness criterion for nonnegative (sub)linear operators in [5, Lemma 5.7], we deduce the following conclusion. Recall that a sublinear operator  $T$  is said to be *nonnegative* if, for any function  $f$  in its domain,  $Tf \geq 0$ .

**Lemma 2.6** *Let  $\varphi$  be as in Definition 1.2,  $A$  as in (1.2),*

$$q \in \left( \frac{r(\varphi)}{r(\varphi) - 1} I(\varphi), +\infty \right),$$

and  $M \in \mathbb{N}$  with  $M > \frac{n}{2} \frac{q(\varphi)}{i(\varphi)}$ , where  $i(\varphi)$  and  $q(\varphi)$  are as in (1.4) and (2.2), respectively. Assume that  $T$  is a linear (resp. nonnegative sublinear) operator which maps  $L^2(\mathbb{R}^n)$  continuously into  $L^{2, \infty}(\mathbb{R}^n)$ . If there exists a positive constant  $C$  such that, for any  $\lambda \in \mathbb{C}$  and  $(\varphi, q, M)$ -atom  $\alpha$  associated with the ball  $B$ ,

$$\int_{\mathbb{R}^n} \varphi(x, |T(\lambda\alpha)(x)|) dx \leq C \varphi \left( B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \right),$$

then  $T$  extends to a bounded linear (resp. sublinear) operator from  $H_{\varphi, A}(\mathbb{R}^n)$  to  $L^\varphi(\mathbb{R}^n)$ .

### 3 Proof of Theorem 1.6

In this section, we show Theorem 1.6. To this end, we first recall some notation.

For the moment, we denote by  $L_{n+1}$  the closure in  $L^2(\mathbb{R}^{n+1})$  of  $\frac{\partial}{\partial t}$  and write  $\frac{\partial}{\partial t}$  as  $\frac{\partial}{\partial x_{n+1}}$ . To prove the inclusion

$$H_{\varphi, \mathcal{N}_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset H_{\varphi, S_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$$

in Theorem 1.6, we need to establish a pointwise estimate for the following truncated operator  $\tilde{S}_P^{\varepsilon, R, \tau} f$ , defined by setting, for all  $f \in L^2(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ ,

$$\tilde{S}_P^{\varepsilon, R, \tau} f(x) := \left[ \iint_{\Gamma_{\tau}^{\varepsilon, R}(x)} \sum_{k=1}^{n+1} |tL_k e^{-t\sqrt{A}} f(y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2}, \quad (3.1)$$

where  $\tau \in (0, +\infty)$ ,  $\varepsilon, R \in (0, +\infty)$  with  $\varepsilon < R$  and

$$\Gamma_{\tau}^{\varepsilon, R}(x) := \{(y, t) \in \mathbb{R}^n \times (\varepsilon, R) : |x - y| < \tau t\}.$$

To this end, we recall a Caccioppoli inequality for weak solutions of the equation

$$-\frac{\partial^2 u}{\partial t^2} + Au = 0 \quad (3.2)$$

in an open ball  $\tilde{B}$  of  $\mathbb{R}^{n+1}$  in [26]. Define

$$W_{\mathbf{a}, V}^{1,2}(\tilde{B}) := \{u \in L^2(\tilde{B}) : L_k u \in L^2(\tilde{B}), k \in \{1, 2, \dots, n+1\}, \sqrt{V} u \in L^2(\tilde{B})\},$$

and let  $W_{\mathbf{a}, V, 0}^{1,2}(\tilde{B})$  be the subspace of  $W_{\mathbf{a}, V}^{1,2}(\tilde{B})$  with trace 0 on  $\partial\tilde{B}$ . Here and hereafter, for  $k \in \{1, 2, \dots, n\}$  and all  $u \in L^2(\tilde{B})$ ,

$$L_k u(x, t) := L_k(u(\cdot, t))(x), \quad \forall (x, t) \in \tilde{B}.$$

The function  $u \in W_{\mathbf{a}, V}^{1,2}(\tilde{B})$  is called a weak solution of (3.2) in  $\tilde{B}$  if

$$\sum_{k=1}^{n+1} \iint_{\tilde{B}} L_k u \overline{L_k \varphi} dydt + \iint_{\tilde{B}} u V \overline{\varphi} dydt = 0, \quad \forall \varphi \in W_{\mathbf{a}, V, 0}^{1,2}(\tilde{B}).$$

The following is the Caccioppoli inequality.

**Lemma 3.1** [26, Lemma 2.1] *Let  $(x_0, t_0) \in \mathbb{R}^{n+1}$ , let  $R \in (0, +\infty)$ , and let  $u$  be a weak solution of (3.2) in the ball  $B((x_0, t_0), 2R) \subset \mathbb{R}^{n+1}$ . Then there exists a positive constant  $C$ , independent of  $(x_0, t_0) \in \mathbb{R}^{n+1}$ ,  $R$ , and  $u$ , such that*

$$\sum_{k=1}^{n+1} \iint_{B((x_0, t_0), R)} |L_k u(y, t)|^2 dydt \leq \frac{C}{R^2} \iint_{B((x_0, t_0), 2R)} |u(y, t)|^2 dydt.$$

Using Lemma 3.1, we now prove the following conclusion.

**Lemma 3.2** *Let  $\alpha \in (0, 1)$ , and let  $\varepsilon, R \in (0, +\infty)$  such that  $\varepsilon < R$ . Then there exists a positive constant  $C$  such that, for all  $f \in L^2(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ ,*

$$\tilde{S}_P^{\varepsilon, R, \alpha} f(x) \leq C \left(1 + \log \frac{R}{\varepsilon}\right)^{1/2} \mathcal{N}_P f(x).$$

*Proof* Let

$$u(y, t) := e^{-t\sqrt{A}} f(y), \quad \forall (y, t) \in \mathbb{R}^n \times (0, +\infty).$$

Observe that the kernel  $p_t(y, z)$  of  $e^{-tA}$  satisfies the *Gaussian upper bound* that, for all  $t \in (0, +\infty)$  and almost all  $y, z \in \mathbb{R}^n$ ,

$$|p_t(y, z)| \leq (4\pi t)^{-n/2} \exp\left(-\frac{|y-z|^2}{4t}\right); \tag{3.3}$$

see [16]. This fact, together with the well-known *subordination formula* that, for all  $f \in L^2(\mathbb{R}^n)$ ,

$$e^{-t\sqrt{A}} f = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \frac{e^{-u}}{\sqrt{u}} e^{-\frac{t^2}{4u}A} f du, \tag{3.4}$$

implies that  $e^{-t\sqrt{A}}$  is bounded on  $L^2(\mathbb{R}^n)$  for each  $t \in (0, +\infty)$ . Let  $\alpha \in (0, 1)$ ,  $\varepsilon, R \in (0, +\infty)$  with  $\varepsilon < R$ , and  $x \in \mathbb{R}^n$ . Moreover, for any  $(z, \tau) \in \Gamma_\alpha^{\varepsilon, R}(x)$ , let

$$\tilde{B}(z, \tau) := B((z, \tau), r)$$

with  $r := \delta\tau$ , where  $\delta \in (0, 1)$  is small enough. By the Besicovitch covering lemma, we know that there exists a subsequence

$$\{\tilde{B}_j\}_j := \{B((z_j, \tau_j), r_j)\}_j$$

of balls covering  $\Gamma_\alpha^{\varepsilon, R}(x)$  with bounded overlap. Observe that, for any  $(y, t) \in \tilde{B}_j$ ,  $t \sim d_j$ , where  $d_j$  denotes the distance between  $\tilde{B}_j$  and the bottom boundary  $\mathbb{R}^n \times \{0\}$ . Also, we see that, if  $(y, t) \in 2\tilde{B}_j$ , then  $(y, t) \in \Gamma(x)$  for  $\delta$  small enough. Hence,

$$|e^{-t\sqrt{A}} f(y)| \leq \mathcal{N}_P f(x).$$

On the other hand, by the semigroup property, we find that, for fixed  $t \in (0, +\infty)$ ,

$$Au(\cdot, t) - \frac{\partial^2}{\partial t^2} u(\cdot, t) = 0$$

in  $L^2(\mathbb{R}^n)$ , which implies that  $u$  is a weak solution of (3.2) for each  $2\tilde{B}_j$ . By the bounded overlap of  $\{\tilde{B}_j\}_j$  and Lemma 3.1, we conclude that, for all  $x \in \mathbb{R}^n$ ,

$$[\tilde{S}_P^{\varepsilon, R, \alpha} f(x)]^2 \leq \sum_j \iint_{\tilde{B}_j} \sum_{k=1}^{n+1} |tL_k e^{-t\sqrt{A}} f(y)|^2 \frac{dy dt}{t^{n+1}}$$

$$\begin{aligned}
&\lesssim \sum_j r_j^{-(n-1)} \iint_{\tilde{B}_j} \sum_{k=1}^{n+1} |L_k e^{-t\sqrt{A}} f(y)|^2 dy dt \\
&\lesssim \sum_j r_j^{-(n+1)} \iint_{2\tilde{B}_j} \sum_{k=1}^{n+1} |e^{-t\sqrt{A}} f(y)|^2 dy dt \\
&\lesssim [\mathcal{N}_P f(x)]^2 \sum_j r_j^{-(n+1)} |\tilde{B}_j| \\
&\sim [\mathcal{N}_P f(x)]^2 \sum_j \iint_{\tilde{B}_j} \frac{dy dt}{t^{n+1}} \\
&\sim [\mathcal{N}_P f(x)]^2 \iint_{\Gamma_{\alpha}^{\varepsilon, R}(x)} \frac{dy dt}{t^{n+1}} \\
&\lesssim \left(1 + \log \frac{R}{\varepsilon}\right) [\mathcal{N}_P f(x)]^2,
\end{aligned}$$

which implies the desired conclusion. This finishes the proof.  $\square$

*Proof of Theorem 1.6 Step 1 Show*

$$H_{\varphi, A}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset H_{\varphi, \mathcal{N}_h}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

To this end, by Lemma 2.6, it suffices to show that, for any  $\lambda \in \mathbb{C}$  and  $(\varphi, q, M)_A$ -atom  $\alpha$  associated with a ball  $B := B(x_B, r_B)$  for some  $x_B \in \mathbb{R}^n$  and  $r_B \in (0, +\infty)$ ,

$$\int_{\mathbb{R}^n} \varphi(x, \mathcal{N}_h(\lambda\alpha)(x)) dx \lesssim \varphi\left(B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\right), \quad (3.5)$$

where  $M \in \mathbb{N}$  with  $M > \frac{n}{2} \frac{q(\varphi)}{i(\varphi)}$ . Indeed, we first observe that  $\mathcal{N}_h$  is bounded on  $L^q(\mathbb{R}^n)$  for all  $q \in (1, +\infty)$  (see, for example, the proof of [26, Theorem 1.4]). Since  $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$ , from (1.3), (1.4), (2.2), and (iii)–(v) of Lemma 2.3, it follows that there exist  $q_0 \in (q(\varphi), +\infty)$ ,  $p_2 \in (0, i(\varphi)]$ ,  $p_1 \in [I(\varphi), 1]$ , and  $q \in (I(\varphi)/[r(\varphi)]', +\infty)$  such that  $\varphi$  is of uniformly upper type  $p_1$  and of uniformly lower type  $p_2$ ,  $\varphi \in \mathbb{A}_{q_0}(\mathbb{R}^n)$ , and  $\varphi \in \mathbb{RH}_{(q/p_1)' }(\mathbb{R}^n)$ . We now write

$$\int_{\mathbb{R}^n} \varphi(x, \mathcal{N}_h(\lambda\alpha)(x)) dx = \sum_{j=0}^{+\infty} \int_{S_j(B)} \varphi(x, \mathcal{N}_h(\lambda\alpha)(x)) dx =: \sum_{j=0}^{+\infty} I_j,$$

where  $\{S_j(B)\}_{j \in \mathbb{Z}_+}$  are as in (1.9).

For  $j \in \{0, 1, 2, 3, 4\}$ , by the fact that  $\varphi$  is of uniformly upper type  $p_1$  and of uniformly lower type  $p_2$ , the Hölder inequality, the  $L^q(\mathbb{R}^n)$ -boundedness of  $\mathcal{N}_h$ , Definition 2.4 (A)<sub>iii</sub>,  $\varphi \in \mathbb{A}_{q_0}(\mathbb{R}^n)$  and  $\varphi \in \mathbb{RH}_{(q/p_1)' }(\mathbb{R}^n)$ ,  $p_2 \leq p_1$ , (vi) and (vii) of Lemma 2.3, we have

$$\begin{aligned}
 I_j &\lesssim \sum_{i=1}^2 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_i} \int_{S_j(B)} \varphi(x, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}) [\mathcal{N}_h(\alpha)(x)]^{p_i} dx \\
 &\lesssim \sum_{i=1}^2 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_i} \left\{ \int_{S_j(B)} [\mathcal{N}_h(\alpha)(x)]^q dx \right\}^{p_i/q} \\
 &\quad \times \left\{ \int_{S_j(B)} [\varphi(x, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1})]^{(q/p_i)'} dx \right\}^{1/(q/p_i)'} \\
 &\lesssim \sum_{i=1}^2 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_i} \|\alpha\|_{L^q(\mathbb{R}^n)}^{p_i} |2^j B|^{-p_i/q} \varphi(2^j B, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}) \\
 &\lesssim \sum_{i=1}^2 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_i} |B|^{p_i/q} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-p_i} |2^j B|^{-p_i/q} 2^{jnq_0} \\
 &\quad \times \varphi(B, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}) \\
 &\lesssim 2^{-jn p_2 (\frac{1}{q} - \frac{q_0}{p_2})} \varphi(B, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}). \tag{3.6}
 \end{aligned}$$

Now, we turn to the case when  $j \geq 5$ . From the fact that  $\varphi$  is of uniformly upper type  $p_1$  and of uniformly lower type  $p_2$ , we deduce that

$$I_j \lesssim \sum_{i=1}^2 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_i} \int_{S_j(B)} \varphi(x, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}) [\mathcal{N}_h(\alpha)(x)]^{p_i} dx.$$

By the Hölder inequality, the facts that  $\varphi \in \mathbb{RH}_{(q/p_1)'(\mathbb{R}^n)} \cap \mathbb{A}_{q_0}(\mathbb{R}^n)$ , and (vii) and (viii) of Lemma 2.3, we further conclude that, for all  $j \in \mathbb{N} \cap [5, +\infty)$ ,

$$\begin{aligned}
 I_j &\leq \sum_{i=1}^2 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_i} \|\mathcal{N}_h(\alpha)\chi_{S_j(B)}\|_{L^q(\mathbb{R}^n)}^{p_i} \left\| \varphi\left(\cdot, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\right) \chi_{S_j(B)} \right\|_{L^{(q/p_i)'(\mathbb{R}^n)}} \\
 &\lesssim \sum_{i=1}^2 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_i} \|\mathcal{N}_h(\alpha)\chi_{S_j(B)}\|_{L^q(\mathbb{R}^n)}^{p_i} |2^j B|^{-p_i/q} \varphi\left(2^j B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\right) \\
 &\lesssim \sum_{i=1}^2 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_i} \|\mathcal{N}_h(\alpha)\chi_{S_j(B)}\|_{L^q(\mathbb{R}^n)}^{p_i} |2^j B|^{-p_i/q} 2^{jnq_0} \varphi\left(B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\right).
 \end{aligned}$$

Let  $a \in (0, 1)$  such that  $ap_2(2M + n) > n$ . We see that, for all  $j \in \mathbb{N} \cap [5, +\infty)$  and  $x \in S_j(B)$ ,

$$\mathcal{N}_h(\alpha)(x) \leq \sup_{y \in B(x,t), t \leq 2^{aj-2}r_B} |e^{-t^2 A} \alpha(y)| + \sup_{y \in B(x,t), t > 2^{aj-2}r_B} \dots =: I_{j,1} + I_{j,2}.$$

On one hand, by (3.3), the definition of  $\alpha$ , and the Hölder inequality, we

know that, for all  $j \in \mathbb{N} \cap [5, +\infty)$ ,

$$\begin{aligned} I_{j,1} &\lesssim \sup_{y \in B(x,t), t \leq 2^{aj-2}r_B} t^{-n} \int_B e^{-|y-z|^2/(4t^2)} |\alpha(z)| dz \\ &\lesssim \sup_{t \leq 2^{aj-2}r_B} t^{-n} \left( \frac{t}{2^j r_B} \right)^{N+n} \|\alpha\|_{L^1(\mathbb{R}^n)} \\ &\lesssim (2^{aj-2}r_B)^N (2^j r_B)^{-(N+n)} |B|^{1/q'} |B|^{1/q} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} \\ &\lesssim 2^{-j[n+(1-a)N]} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}, \end{aligned}$$

where  $N$  satisfies  $p_2[n + (1-a)N] > q_0 n$ . On the other hand, recall that, for all  $k \in \mathbb{N}$ , there exist positive constants  $C_{(k)}$  and  $\tilde{C}_{(k)}$ , depending on  $k$ , such that, for almost all  $y, z \in \mathbb{R}^n$ ,

$$\left| \frac{\partial^k}{\partial t^k} p_t(y, z) \right| \leq \frac{C_{(k)}}{t^{k+\frac{n}{2}}} \exp\left(-\frac{|y-z|^2}{\tilde{C}_{(k)}t}\right);$$

see [33, Theorem 6.16]. From this, the semigroup property, the definition of  $\alpha$ , and the Hölder inequality, we deduce that, for all  $j \in \mathbb{N} \cap [5, +\infty)$ ,

$$\begin{aligned} I_{j,2} &= \sup_{y \in B(x,t), t > 2^{aj-2}r_B} |A^M e^{-t^2 A} b(y)| \\ &= \sup_{y \in B(x,t), t > 2^{aj-2}r_B} \left| \left( \frac{\partial}{\partial s} \right)^M \Big|_{s=t^2} e^{-sA} b(y) \right| \\ &\lesssim \sup_{y \in B(x,t), t > 2^{aj-2}r_B} t^{-(2M+n)} \int_{\mathbb{R}^n} \exp\left(-\frac{|y-z|^2}{\tilde{C}_M t^2}\right) |b(z)| dz \\ &\lesssim (2^{aj} r_B)^{-(2M+n)} \|b\|_{L^1(\mathbb{R}^n)} \\ &\lesssim (2^{aj} r_B)^{-(2M+n)} r_B^{2M} |B|^{1/q'} |B|^{1/q} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} \\ &\lesssim 2^{-aj(2M+n)} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}. \end{aligned}$$

Combining these two inequalities, we find that, for all  $j \in \mathbb{N} \cap [5, +\infty)$  and  $x \in \mathbb{R}^n$ ,

$$\mathcal{N}_h(\alpha)(x) \lesssim \{2^{-j[n+(1-a)N]} + 2^{-aj(2M+n)}\} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}.$$

By this, we conclude that, for all  $j \in \mathbb{N} \cap [5, +\infty)$ ,

$$\begin{aligned} I_j &\lesssim \sum_{i=1}^2 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_i} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-p_i} |2^j B|^{p_i/q} |2^j B|^{-p_i/q} 2^{jq_0 n} \varphi\left(B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\right) \\ &\quad \times \{2^{-p_i j[n+(1-a)N]} + 2^{-aj p_i (2M+n)}\} \\ &\lesssim \sum_{i=1}^2 \{2^{j(q_0 n - p_i[n+(1-a)N])} + 2^{j[q_0 n - a p_i (2M+n)]}\} \varphi\left(B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\right), \end{aligned}$$

which, together with the facts that

$$p_2 \leq p_1, \quad p_2[n + (1 - a)N] > q_0n, \quad ap_2(2M + n) > q_0n,$$

implies that

$$\sum_{j=5}^{+\infty} I_j \lesssim \varphi\left(B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\right).$$

From this and (3.6), we deduce (3.5), which further implies the desired inclusion relation that

$$H_{\varphi, A}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset H_{\varphi, \mathcal{N}_h}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

**Step 2** Prove

$$H_{\varphi, \mathcal{N}_h}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset H_{\varphi, \mathcal{R}_h}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

Observe that, for all  $f \in L^2(\mathbb{R}^n)$ ,

$$\mathcal{R}_h f \leq \mathcal{N}_h f.$$

By this fact, we conclude that, for all  $f \in H_{\varphi, \mathcal{N}_h}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ ,

$$f \in H_{\varphi, \mathcal{R}_h}(\mathbb{R}^n), \quad \|f\|_{H_{\varphi, \mathcal{R}_h}(\mathbb{R}^n)} \leq \|f\|_{H_{\varphi, \mathcal{N}_h}(\mathbb{R}^n)},$$

which further implies the desired conclusion.

**Step 3** Show

$$H_{\varphi, \mathcal{R}_h}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset H_{\varphi, \mathcal{R}_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

From the subordination formula (3.4), it follows that, for all  $f \in L^2(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ ,

$$\mathcal{R}_P f(x) \lesssim \sup_{t>0} \int_0^{+\infty} \frac{e^{-u}}{\sqrt{u}} |e^{-t^2 A/(4u)} f(x)| du \lesssim \mathcal{R}_h f(x) \int_0^{+\infty} \frac{e^{-u}}{\sqrt{u}} du \lesssim \mathcal{R}_h f(x),$$

which further implies that, for all  $f \in H_{\varphi, \mathcal{R}_h}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ ,

$$f \in H_{\varphi, \mathcal{R}_P}(\mathbb{R}^n), \quad \|f\|_{H_{\varphi, \mathcal{R}_P}(\mathbb{R}^n)} \lesssim \|f\|_{H_{\varphi, \mathcal{R}_h}(\mathbb{R}^n)}.$$

From this, we deduce the desired inclusion relation.

**Step 4** Prove

$$H_{\varphi, \mathcal{R}_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset H_{\varphi, \mathcal{N}_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

By [26, (2.12)], we know that, for any  $q \in (0, 1)$  and  $x \in \mathbb{R}^n$ ,

$$\mathcal{N}_P^{1/4} f(x) \lesssim [\mathcal{M}([\mathcal{R}_P f]^q)(x)]^{1/q}, \quad (3.7)$$

where  $\mathcal{M}f$  denotes the Hardy-Littlewood maximal function of  $f$  as in (2.1). Using this fact, Lemma 2.1 (ii), Lemma 2.3 (vi), Definition 1.2, and arguing as the proof of [42, (7.17)], we conclude that, for all  $f \in L^2(\mathbb{R}^n)$ ,

$$\mathcal{N}_P^{1/4}f \in L^\varphi(\mathbb{R}^n), \quad \mathcal{N}_P f \in L^\varphi(\mathbb{R}^n), \quad \|\mathcal{N}_P f\|_{L^\varphi(\mathbb{R}^n)} \sim \|\mathcal{N}_P^{1/4}f\|_{L^\varphi(\mathbb{R}^n)}.$$

This, together with (3.7), Definition 1.2, Lemma 2.1 (ii), Lemma 2.3 (vi), and an argument similar to [42, (7.16)], further implies that, for all  $f \in H_{\varphi, \mathcal{R}_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ ,

$$\|\mathcal{N}_P f\|_{L^\varphi(\mathbb{R}^n)} \sim \|\mathcal{N}_P^{1/4}f\|_{L^\varphi(\mathbb{R}^n)} \lesssim \|\mathcal{R}_P f\|_{L^\varphi(\mathbb{R}^n)}.$$

By this, we obtain the desired inclusion relation.

**Step 5** Show

$$H_{\varphi, \mathcal{N}_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset H_{\varphi, S_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

By an argument similar to that used in the proof of [42, Proposition 7.6], we see that, to prove the desired inclusion relation, it suffices to show that there exist positive constants  $C$  and  $\varepsilon_0 \in (0, 1)$  such that, for all  $\gamma \in (0, 1]$ ,  $\lambda, \varepsilon, R \in (0, +\infty)$  with  $\varepsilon < R$ ,  $f \in H_{\varphi, \mathcal{N}_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ , and  $t \in (0, +\infty)$ ,

$$\begin{aligned} & \int_{\{x \in \mathbb{R}^n : \tilde{S}_P^{\varepsilon, R, 1/20} f(x) > 2\lambda, \mathcal{N}_P f(x) \leq \gamma\lambda\}} \varphi(x, t) dx \\ & \leq C\gamma^{\varepsilon_0} \int_{\{x \in \mathbb{R}^n : \tilde{S}_P^{\varepsilon, R, 1/2} f(x) > \lambda\}} \varphi(x, t) dx. \end{aligned} \quad (3.8)$$

To this end, fix  $0 < \varepsilon < R < +\infty$ ,  $\gamma \in (0, 1]$ , and  $\lambda \in (0, +\infty)$ . Let

$$f \in H_{\varphi, \mathcal{N}_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \quad O := \{x \in \mathbb{R}^n : \tilde{S}_P^{\varepsilon, R, 1/2} f(x) > \lambda\}.$$

Then  $O$  is an open subset of  $\mathbb{R}^n$ . Let  $O = \cup_k Q_k$  be a Whitney decomposition of  $O$  such that  $\{Q_k\}_k$  has disjoint interiors,

$$2Q_k \subset O, \quad 4Q_k \cap (\mathbb{R}^n \setminus O) \neq \emptyset.$$

By  $O = \cup_k Q_k$  and  $\{Q_k\}_k$  is disjoint mutually, to show (3.8), it suffices to prove that, for each  $k$ ,

$$\int_{\{x \in Q_k : \tilde{S}_P^{\varepsilon, R, 1/20} f(x) > 2\lambda, \mathcal{N}_P f(x) \leq \gamma\lambda\}} \varphi(x, t) dx \lesssim \gamma^{\varepsilon_0} \int_{Q_k} \varphi(x, t) dx. \quad (3.9)$$

Denote by  $\ell_k$  the *side length* of  $Q_k$ . Observe that, if  $x \in Q_k$ , then

$$\tilde{S}_P^{\max\{10\ell_k, \varepsilon\}, R, 1/20} f(x) \leq \lambda; \quad (3.10)$$

see [42, (7.8)]. It follows, from (3.10), that, if  $\varepsilon \geq 10\ell_k$ , then

$$\{x \in Q_k : \tilde{S}_P^{\varepsilon, R, 1/20} f(x) > 2\lambda, \mathcal{N}_P f(x) \leq \gamma\lambda\} = \emptyset,$$

and hence, (3.9) holds true. When  $\varepsilon < 10\ell_k$ , by (3.10) and the fact that

$$\tilde{S}_P^{\varepsilon, R, 1/20} f \leq \tilde{S}_P^{\varepsilon, 10\ell_k, 1/20} f + \tilde{S}_P^{10\ell_k, R, 1/20} f,$$

it remains to show that there exists  $\varepsilon_0 \in (0, 1)$  such that, for all  $t \in (0, +\infty)$ ,

$$\int_{\{x \in Q_k : \tilde{S}_P^{\varepsilon, 10\ell_k, 1/20} f(x) > \lambda, \mathcal{N}_P f(x) \leq \gamma\lambda\}} \varphi(x, t) dx \lesssim \gamma^{\varepsilon_0} \int_{Q_k} \varphi(x, t) dx. \tag{3.11}$$

Let

$$F := \{x \in \mathbb{R}^n : \mathcal{N}_P f(x) \leq \gamma\lambda\}.$$

Then we claim that (3.11) can be deduced from the following inequality that

$$\int_{Q_k \cap F} [\tilde{S}_P^{\varepsilon, 10\ell_k, 1/20} f(x)]^2 dx \lesssim (\gamma\lambda)^2 |Q_k|. \tag{3.12}$$

Indeed, if (3.12) holds true, we first deduce, from the Tchebychev inequality, that

$$|\{x \in Q_k \cap F : \tilde{S}_P^{\varepsilon, 10\ell_k, 1/20} f(x) > \lambda\}| \lesssim \gamma^2 |Q_k|. \tag{3.13}$$

On the other hand, by the fact that  $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$  and Lemma 2.3 (v), we conclude that there exists  $r \in (1, +\infty)$  such that  $\varphi \in \mathbb{RH}_r(\mathbb{R}^n)$ , which, together with (3.13) and Lemma 2.3 (vii), implies that, for all  $t \in (0, +\infty)$ ,

$$\begin{aligned} & \frac{1}{\varphi(Q_k, t)} \int_{\{x \in Q_k \cap F : \tilde{S}_P^{\varepsilon, 10\ell_k, 1/20} f(x) > \lambda\}} \varphi(x, t) dx \\ & \lesssim \left[ \frac{|\{x \in Q_k \cap F : \tilde{S}_P^{\varepsilon, 10\ell_k, 1/20} f(x) > \lambda\}|}{|Q_k|} \right]^{(r-1)/r} \\ & \lesssim \gamma^{2(r-1)/r}. \end{aligned}$$

Let

$$\varepsilon_0 := \frac{2(r-1)}{r}.$$

Then we have

$$\int_{\{x \in Q_k \cap F : \tilde{S}_P^{\varepsilon, 10\ell_k, 1/20} f(x) > \lambda\}} \varphi(x, t) dx \lesssim \gamma^{\varepsilon_0} \varphi(Q_k, t),$$

which implies (3.11). Thus, the claim holds true.

Now, we show (3.12). If  $\varepsilon \geq 5\ell_k$ , then, by the definitions of  $\tilde{S}_P^{\varepsilon, 10\ell_k, 1/20} f$  and  $F$ , together with Lemma 3.2, we conclude that

$$\int_{Q_k \cap F} [\tilde{S}_P^{\varepsilon, 10\ell_k, 1/20} f(x)]^2 dx \lesssim \int_{Q_k \cap F} [\mathcal{N}_P f(x)]^2 dx \lesssim (\gamma\lambda)^2 |Q_k|.$$

Assume that  $\varepsilon < 5\ell_k$ . Let

$$G_k := \left\{ (y, t) \in \mathbb{R}^n \times (\varepsilon, 10\ell_k) : \Psi_k(y) := \text{dist}(y, Q_k \cap F) < \frac{t}{20} \right\}.$$

By the definition of  $\tilde{S}_P^{\varepsilon, 10\ell_k, 1/20} f$ , we see that

$$\begin{aligned} \int_{Q_k \cap F} [\tilde{S}_P^{\varepsilon, 10\ell_k, 1/20} f(x)]^2 dx &\lesssim \iint_{G_k} \sum_{k=1}^{n+1} |tL_k e^{-t\sqrt{A}} f(y)|^2 \frac{dy dt}{t} \\ &\sim \iint_{G_k} \sum_{k=1}^{n+1} t |L_k e^{-t\sqrt{A}} f(y)|^2 dy dt. \end{aligned}$$

Let

$$E_k := \left\{ y \in \mathbb{R}^n : \text{there exists } t \in (\varepsilon, 10\ell_k) \text{ such that } \Psi_k(y) < \frac{t}{20} \right\}.$$

Then we claim that  $E_k \subset 2Q_k$ . Indeed, if  $y \in E_k$ , then there exists  $t \in (\varepsilon, 10\ell_k)$  such that  $(y, t) \in G_k$ . Furthermore, we see that there exists  $x \in Q_k \cap F$  such that  $|x - y| < t/20$ . By  $t < 10\ell_k$ , we know that  $|x - y| < \ell_k/2$ , which implies that  $E_k \subset 2Q_k$ , and hence, the claim holds true.

Let

$$\tilde{G}_k := \left\{ (y, t) \in \mathbb{R}^n \times \left(\frac{\varepsilon}{5}, 40\ell_k\right) : \Psi_k(y) < \frac{t}{10} \right\}.$$

Then, for any  $(y, t) \in \tilde{G}_k$ ,

$$|u(y, t)| := |e^{-t\sqrt{A}} f(y)| \leq \gamma\lambda.$$

Indeed, for any  $(y, t) \in \tilde{G}_k$ , there exists  $x \in Q_k \cap F$  such that  $|x - y| < t$  and  $t \in (\varepsilon/5, 40\ell_k)$ . This implies that  $(y, t) \in \Gamma(x)$ , where  $\Gamma(x)$  is as in (1.8). Thus, from the definitions of  $F$  and  $\mathcal{N}_P(f)$ , it follows that, for all  $(y, t) \in \Gamma(x)$ ,

$$|e^{-t\sqrt{A}} f(y)| \leq \mathcal{N}_P(f)(x) \leq \gamma\lambda.$$

Let

$$G_{k,1} := \left\{ (y, t) \in \mathbb{R}^n \times \left(\frac{\varepsilon}{2}, 20\ell_k\right) : \Psi_k(y) < \frac{t}{10} \right\}.$$

Then, by [41, Lemma 3.6], we see that there exists a function

$$\xi \in C^\infty\left(\mathbb{R}^n \times \left(\frac{\varepsilon}{2}, 20\ell_k\right)\right) \cap C\left(\mathbb{R}^n \times \left[\frac{\varepsilon}{2}, 20\ell_k\right]\right)$$

such that  $\text{supp}(\xi) \subset G_{k,1}$ ,  $0 \leq \xi \leq 1$ ,  $\xi \equiv 1$  on  $G_k$ , and  $|\tilde{\nabla}\xi(y, t)| \lesssim t^{-1}$  for any

$(y, t) \in \mathbb{R}^n \times (\varepsilon/2, 20\ell_k)$ . By  $0 \leq \xi \leq 1$  and  $\xi \equiv 1$  on  $G_k$ , we conclude that

$$\begin{aligned} & \iint_{G_k} t \sum_{k=1}^{n+1} |L_k e^{-t\sqrt{A}} f(y)|^2 dy dt \\ & \leq \iint_{\mathbb{R}^n \times (0, +\infty)} t \sum_{k=1}^{n+1} |L_k e^{-t\sqrt{A}} f(y)|^2 \xi(y, t) dy dt \\ & = \sum_{k=1}^{n+1} \iint_{G_{k,1}} t L_k u(y, t) \overline{L_k u(y, t)} \xi(y, t) dy dt \\ & \sim \operatorname{Re} \left\{ \sum_{k=1}^{n+1} \iint_{G_{k,1}} t L_k u(y, t) \left[ \overline{L_k(u\xi)(y, t)} - \overline{u(y, t)} \frac{\partial \xi(y, t)}{\partial x_k} \right] dy dt \right\}. \end{aligned}$$

From integration by parts, the fact that, for fixed  $t$ ,

$$Au(\cdot, t) - \frac{\partial^2}{\partial t^2} u(\cdot, t) = 0$$

in  $L^2(\mathbb{R}^n)$  and the definition of  $A$ , it follows that

$$\begin{aligned} & \operatorname{Re} \left\{ \iint_{G_{k,1}} t L_{n+1} u(y, t) \overline{L_{n+1}(u\xi)(y, t)} dy dt \right\} \\ & = - \operatorname{Re} \left\{ \iint_{G_{k,1}} \left[ t \frac{\partial^2}{\partial t^2} u(y, t) \overline{u(y, t)} \xi(y, t) + \frac{\partial u(y, t)}{\partial t} \overline{u(y, t)} \xi(y, t) \right] dy dt \right\} \\ & = - \operatorname{Re} \left\{ \iint_{G_{k,1}} \left[ t Au(y, t) \overline{u(y, t)} \xi(y, t) + \frac{\partial u(y, t)}{\partial t} \overline{u(y, t)} \xi(y, t) \right] dy dt \right\} \\ & = - \operatorname{Re} \left\{ \sum_{k=1}^n \iint_{G_{k,1}} t L_k u(y, t) \overline{L_k(u\xi)(y, t)} dy dt \right\} \\ & \quad - \iint_{G_{k,1}} \left[ t |u(y, t)|^2 V(y) \xi(y, t) + \frac{1}{2} \frac{\partial |u(y, t)|^2}{\partial t} \xi(y, t) \right] dy dt. \end{aligned}$$

Since  $\xi, V \geq 0$ , from this fact, integration by parts, the choice of  $\xi$ , and the Cauchy-Schwarz inequality, we further deduce that

$$\begin{aligned} & \iint_{G_k} t \sum_{k=1}^{n+1} |L_k e^{-t\sqrt{A}} f(y)|^2 dy dt \\ & \lesssim \iint_{G_{k,1}} \left\{ - \operatorname{Re} \left[ \sum_{k=1}^{n+1} t L_k u(y, t) \overline{u(y, t)} \frac{\partial \xi(y, t)}{\partial x_k} \right] + \frac{1}{2} |u(y, t)|^2 \frac{\partial \xi(y, t)}{\partial t} \right\} dy dt \\ & \lesssim \iint_{G_{k,1} \setminus G_k} \left[ \sum_{k=1}^{n+1} |L_k u(y, t) u(y, t)| + t^{-1} |u(y, t)|^2 \right] dy dt \end{aligned}$$

$$\begin{aligned} &\lesssim \iint_{G_{k,1} \setminus G_k} \sum_{k=1}^{n+1} t |L_k u(y, t)|^2 dy dt + \iint_{G_{k,1} \setminus G_k} t^{-1} |u(y, t)|^2 dy dt \\ &=: J_1 + J_2. \end{aligned}$$

We first estimate  $J_2$ . By the fact that  $G_{k,1} \subset \tilde{G}_k$ , we conclude that

$$|u(y, t)| \leq \gamma \lambda, \quad \forall (y, t) \in G_{k,1} \setminus G_k.$$

Moreover, we write

$$\begin{aligned} G_{k,1} \setminus G_k &\subset \left\{ (y, t) \in \mathbb{R}^n \times \left( \frac{\varepsilon}{2}, 20\ell_k \right) : \frac{t}{20} \leq \Psi_k(y) < \frac{t}{10} \right\} \\ &\cup \left\{ (y, t) \in \mathbb{R}^n \times \left( \frac{\varepsilon}{2}, 20\ell_k \right) : \Psi_k(y) < \frac{t}{10}, \frac{\varepsilon}{2} < t \leq \varepsilon \right\} \\ &\cup \left\{ (y, t) \in \mathbb{R}^n \times \left( \frac{\varepsilon}{2}, 20\ell_k \right) : \Psi_k(y) < \frac{t}{10}, 10\ell_k \leq t < 20\ell_k \right\}. \end{aligned}$$

From these facts, we deduce that

$$\begin{aligned} J_2 &\lesssim \iint_{G_{k,1} \setminus G_k} (\gamma \lambda)^2 \frac{dy dt}{t} \\ &\lesssim (\gamma \lambda)^2 \int_{H_{k,1}} \left\{ \int_{\varepsilon/2}^{\varepsilon} \frac{dt}{t} + \int_{10\ell_k}^{20\ell_k} \frac{dt}{t} + \int_{10\Psi_k(y)}^{20\Psi_k(y)} \frac{dt}{t} \right\} dy \\ &\lesssim (\gamma \lambda)^2 |H_{k,1}|, \end{aligned}$$

where

$$H_{k,1} := \left\{ y \in \mathbb{R}^n : \text{there exists } t \in \left( \frac{\varepsilon}{2}, 20\ell_k \right) \text{ such that } (y, t) \in G_{k,1} \right\}.$$

Moreover, we claim that  $H_{k,1} \subset 5Q_k$ . Indeed, for any  $y \in H_{k,1}$ , there exists  $t \in (\varepsilon/2, 20\ell_k)$  such that  $(y, t) \in G_{k,1}$ . From this and the definition of  $G_{k,1}$ , it follows that there exists  $x \in Q_k \cap F$  such that  $|x - y| < t/10$  and  $t \in (\varepsilon/2, 20\ell_k)$ . This implies that  $|x - y| < 2\ell_k$ , and hence,  $y \in 5Q_k$ . Thus, the claim holds true, from which it follows that

$$|J_2| \lesssim (\gamma \lambda)^2 |Q_k|.$$

To estimate  $J_1$ , for any  $(y, t) \in (G_{k,1} \setminus G_k)$  and  $\delta \in (0, 1)$ , let

$$E_{(y,t)} := B((y, t), r)$$

with  $r := \delta t$ , and  $\tilde{E}_{(y,t)} := B((y, t), 2r)$ . Take  $\delta$  small enough such that, for any  $(y, t) \in (G_{k,1} \setminus G_k)$ ,

$$\begin{aligned} \tilde{E}_{(y,t)} &\subset \left\{ (z, s) \in \mathbb{R}^n \times \left( \frac{\varepsilon}{5}, 30\ell_k \right) : \frac{s}{40} < \Psi_k(z) < \frac{s}{10} \right\} \\ &\cup \left\{ (z, s) \in \mathbb{R}^n \times \left( \frac{\varepsilon}{5}, 30\ell_k \right) : \Psi_k(z) < \frac{s}{2}, \frac{\varepsilon}{5} \leq s < \frac{2}{\varepsilon} \right\} \\ &\cup \left\{ (z, s) \in \mathbb{R}^n \times \left( \frac{\varepsilon}{5}, 30\ell_k \right) : \Psi_k(z) < \frac{s}{5}, 5\ell_k \leq s \leq 30\ell_k \right\} \\ &=: G_{k,2}. \end{aligned}$$

By the Besicovitch covering lemma, we know that there exists a subsequence  $\{E_{(y_j, t_j)}\}_j$  of balls such that

$$(G_{k,1} \setminus G_k) \subset \bigcup_j E_{(y_j, t_j)}$$

and  $\{E_{(y_j, t_j)}\}_j$  has bounded overlap. Observe that, for any  $j$  and  $(y, t) \in E_j$ ,

$$t \sim t_j \sim r_j.$$

From this, the fact that  $G_{k,2} \subset \tilde{G}_k$ , and Lemma 3.1, we deduce that

$$\begin{aligned} J_1 &\lesssim \sum_j \iint_{E_j} \sum_{k=1}^{n+1} t |L_k u(y, t)|^2 dy dt \\ &\lesssim \sum_j \frac{1}{r_j^2} \iint_{\tilde{E}_j} t |u(y, t)|^2 dy dt \\ &\lesssim \sum_j \frac{1}{r_j} (\gamma\lambda)^2 \iint_{\tilde{E}_j} dy dt \\ &\lesssim \sum_j \frac{1}{r_j} (\gamma\lambda)^2 |E_j| \\ &\lesssim \sum_j (\gamma\lambda)^2 \iint_{E_j} \frac{dy dt}{t} \\ &\lesssim (\gamma\lambda)^2 \iint_{G_{k,2}} \frac{dy dt}{t} \\ &\lesssim (\gamma\lambda)^2 \int_{H_{k,2}} \left\{ \int_{\varepsilon/5}^{2\varepsilon} \frac{dt}{t} + \int_{5\ell_k}^{30\ell_k} \frac{dt}{t} + \int_{10\Psi_k(y)}^{40\Psi_k(y)} \frac{dt}{t} \right\} dy \\ &\lesssim (\gamma\lambda)^2 |H_{k,2}|, \end{aligned}$$

where

$$H_{k,2} := \left\{ y \in \mathbb{R}^n : \text{there exists } t \in \left( \frac{\varepsilon}{5}, 30\ell_k \right) \text{ such that } (y, t) \in G_{k,2} \right\}.$$

By an argument similar to that used in the estimate of  $H_{k,1}$ , we find that

$$|H_{k,2}| \lesssim |Q_k|,$$

which implies that

$$J_1 \lesssim (\gamma\lambda)^2 |Q_k|.$$

This shows (3.12) when  $\varepsilon < 5\ell_k$ .

**Step 6** Prove

$$H_{\varphi, S_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset H_{\varphi, A}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

To this end, for all  $f \in L^2(\mathbb{R}_+^{n+1})$  with compact support and  $x \in \mathbb{R}^n$ , define

$$\Pi_{\Psi,A}(f)(x) := C_{(M)} \int_0^{+\infty} (t^2 A)^{M+1} e^{-t^2 A} (f(\cdot, t))(x) \frac{dt}{t},$$

where  $C_{(M)}$  is a positive constant, depending on  $M$ , such that

$$C_{(M)} \int_0^{+\infty} t^{2(M+2)} e^{-2t^2} \frac{dt}{t} = 1.$$

It was shown that  $\Pi_{\Psi,A}$  is bounded from  $T_2^2(\mathbb{R}_+^{n+1})$  to  $L^2(\mathbb{R}^n)$  in [25, Proposition 4.2 (i)] and from  $T_\varphi(\mathbb{R}_+^{n+1})$  to  $H_{\varphi,A}(\mathbb{R}^n)$  in [5, Proposition 4.5 (ii)], where, for any measurable function  $g$  on  $\mathbb{R}_+^{n+1}$  and  $x \in \mathbb{R}^n$ ,

$$\begin{aligned} \mathcal{A}(g)(x) &:= \left\{ \iint_{\Gamma(x)} |g(y, t)|^2 \frac{dy dt}{t^{n+1}} \right\}^{1/2}, \\ T_\varphi(\mathbb{R}_+^{n+1}) &:= \{g : \|g\|_{T_\varphi(\mathbb{R}_+^{n+1})} := \|\mathcal{A}(g)\|_{L^\varphi(\mathbb{R}^n)} < +\infty\}, \\ T_2^2(\mathbb{R}_+^{n+1}) &:= \{g : \|g\|_{T_2^2(\mathbb{R}_+^{n+1})} := \|\mathcal{A}(g)\|_{L^2(\mathbb{R}^n)} < +\infty\}. \end{aligned}$$

Let  $f \in H_{\varphi,S_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ . Then we have

$$S_P f \in L^\varphi(\mathbb{R}^n) \cap L^2(\mathbb{R}^n),$$

which implies that

$$t\sqrt{A} e^{-t\sqrt{A}} f \in T_\varphi(\mathbb{R}_+^{n+1}) \cap T_2^2(\mathbb{R}_+^{n+1}).$$

On one hand, by the  $H_\infty$ -functional calculus, we see that

$$f = C \Pi_{\Psi,A}(t\sqrt{A} e^{-t\sqrt{A}} f)$$

in  $L^2(\mathbb{R}^n)$ . This, together with the boundedness of  $\Pi_{\Psi,A}$  from  $T_\varphi(\mathbb{R}_+^{n+1})$  to  $H_{\varphi,A}(\mathbb{R}^n)$ , further implies that  $f \in H_{\varphi,A}(\mathbb{R}^n)$ , which, combined with the inclusion relations in Steps 1–5, completes the proof of Theorem 1.6.  $\square$

#### 4 Proof of Theorem 1.8

This section is devoted to the proof of Theorem 1.8. To this end, we first recall that the semigroup  $\{tL_k e^{-t^2 A}\}_{t>0}$  for  $k \in \{1, 2, \dots, n\}$  satisfies the following Davies-Gaffney estimates, which was established in [26, Lemma 3.1].

**Lemma 4.1** *There exist positive constants  $C$  and  $\tilde{C}$  such that, for all  $t \in (0, +\infty)$ , disjoint closed sets  $E, F \subset \mathbb{R}^n$ , and  $f \in L^2(\mathbb{R}^n)$  with  $\text{supp}(f) \subset E$ ,*

$$\sum_{k=1}^n \|\chi_F t L_k e^{-t^2 A} f\|_{L^2(\mathbb{R}^n)} \leq \tilde{C} \exp\left(-\frac{[\text{dist}(E, F)]^2}{Ct^2}\right) \|f \chi_E\|_{L^2(\mathbb{R}^n)}.$$

*Proof of Theorem 1.8* Since  $r(\varphi) > 2/[2 - I(\varphi)]$ , we see that  $[r(\varphi)]'I(\varphi) < 2$  and, by Lemma 2.5,

$$H_{\varphi,A}(\mathbb{R}^n) = H_{\varphi,A,\text{at}}^{M,2}(\mathbb{R}^n).$$

By Lemma 2.6, to prove Theorem 1.8, it suffices to show that, for any  $k \in \{1, 2, \dots, n\}$ ,  $\lambda \in \mathbb{C}$ , and  $(\varphi, 2, M)_A$ -atom  $\alpha$  associated with a ball

$$B := B(x_B, r_B)$$

for some  $x_B \in \mathbb{R}^n$  and  $r_B \in (0, +\infty)$ ,

$$\int_{\mathbb{R}^n} \varphi(x, |L_k A^{-1/2}(\lambda\alpha)(x)|) dx \lesssim \varphi\left(B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\right). \tag{4.1}$$

We first write

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi(x, |L_k A^{-1/2}(\lambda\alpha)(x)|) dx &= \sum_{j=0}^{+\infty} \int_{S_j(B)} \varphi(x, |L_k A^{-1/2}(\lambda\alpha)(x)|) dx \\ &=: \sum_{j=0}^{+\infty} L_j. \end{aligned}$$

Recall that  $L_k A^{-1/2}$  is bounded on  $L^p(\mathbb{R}^n)$  for any  $p \in (1, 2]$ ; see [13]. Using this fact and an argument similar to (3.6), we find that, for  $j \in \{0, 1, 2, 3, 4\}$ ,

$$L_j \lesssim \varphi\left(B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\right).$$

Now, we turn to the case when  $j \geq 5$ . As in the proof of Theorem 1.6, since  $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$ , from (1.3), (1.4), (2.2), and Lemma 2.3 (iii)–(v), it follows that there exist  $q_0 \in (q(\varphi), +\infty)$ ,  $p_2 \in (0, i(\varphi)]$ , and  $p_1 \in [I(\varphi), 1]$  such that  $\varphi$  is of uniformly upper type  $p_1$  and of uniformly lower type  $p_2$ ,  $\varphi \in \mathbb{A}_{q_0}(\mathbb{R}^n)$ , and  $\varphi \in \mathbb{RH}_{(2/p_1)'}(\mathbb{R}^n)$ . Recall that

$$A^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} e^{-tA} \frac{dt}{\sqrt{t}};$$

see [16,33]. From this, together with the fact that  $\varphi$  is of uniformly upper type  $p_1$  and of uniformly lower type  $p_2$ , the Hölder inequality, and the fact that  $\varphi \in \mathbb{RH}_{(2/p_1)'}(\mathbb{R}^n)$  and  $\varphi \in \mathbb{A}_{q_0}(\mathbb{R}^n)$ , we deduce that, for all  $j \in \mathbb{N} \cap [5, +\infty)$ ,

$$\begin{aligned} L_j &\lesssim \sum_{i=1}^2 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_i} \int_{S_j(B)} \varphi(x, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}) |L_k A^{-1/2}(\alpha)(x)|^{p_i} dx \\ &\leq \sum_{i=1}^2 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_i} \left\{ \int_{S_j(B)} |L_k A^{-1/2}(\alpha)(x)|^2 dx \right\}^{p_i/2} \end{aligned}$$

$$\begin{aligned}
& \times \left\| \varphi \left( \cdot, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \right) \right\|_{L^{(2/p_i)'}(S_j(B))} \\
\lesssim & \sum_{i=1}^2 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_i} \left\{ \int_{S_j(B)} \left| L_k \int_0^{+\infty} e^{-t^2 A}(\alpha)(x) dt \right|^2 dx \right\}^{p_i/2} \\
& \times |2^j B|^{-p_i/2} \int_{S_j(B)} \varphi(x, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}) dx \\
\lesssim & \sum_{i=1}^2 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_i} \left\{ \int_0^{+\infty} \left[ \int_{S_j(B)} |L_k e^{-t^2 A}(\alpha)(x)|^2 dx \right]^{1/2} dt \right\}^{p_i} \\
& \times |2^j B|^{-p_i/2} 2^{jq_0 n} \varphi \left( B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \right) \\
= & \sum_{i=1}^2 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_i} \left\{ \int_0^{r_B} \left[ \int_{S_j(B)} |L_k e^{-t^2 A}(\alpha)(x)|^2 dx \right]^{1/2} dt \right\}^{p_i} \\
& \times |2^j B|^{-p_i/2} 2^{jq_0 n} \varphi \left( B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \right) \\
& + \sum_{i=1}^2 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_i} \left\{ \int_{r_B}^{+\infty} \left[ \int_{S_j(B)} |L_k e^{-t^2 A}(\alpha)(x)|^2 dx \right]^{1/2} dt \right\}^{p_i} \\
& \times |2^j B|^{-p_i/2} 2^{jq_0 n} \varphi \left( B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \right) \\
= & \sum_{i=1}^2 \mathbf{H}_{j,i}.
\end{aligned}$$

For  $\mathbf{H}_{j,1}$ , by Lemma 4.1, the definition of  $\alpha$ , and  $p_2 \leq p_1$ , we see that

$$\begin{aligned}
\mathbf{H}_{j,1} & \lesssim \sum_{i=1}^2 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_i} \left\{ \int_0^{r_B} \exp \left( -\frac{(2^j r_B)^2}{ct^2} \right) \|\alpha\|_{L^2(\mathbb{R}^n)} \frac{dt}{t} \right\}^{p_i} \\
& \times |2^j B|^{-p_i/2} 2^{jq_0 n} \varphi \left( B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \right) \\
& \lesssim \sum_{i=1}^2 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_i} |B|^{p_i/2} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-p_i} \left\{ \int_0^{r_B} \left( -\frac{t}{2^j r_B} \right)^{2M} \frac{dt}{t} \right\}^{p_i} \\
& \times |2^j B|^{-p_i/2} 2^{jq_0 n} \varphi \left( B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \right) \\
& \lesssim 2^{-j(p_2(2M + \frac{n}{2}) - q_0 n)} \varphi \left( B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \right),
\end{aligned}$$

where  $M$  is large enough such that  $p_2(2M + \frac{n}{2}) > q_0 n$ .

To estimate  $\mathbf{H}_{j,2}$ , we write

$$tL_k(t^2 A)^M e^{-t^2 A} = tL_k e^{-t^2 A/2} (t^2 A)^M e^{-t^2 A/2}.$$

Recall that the semigroup  $\{(t^2 A)^M e^{-t^2 A}\}_{t>0}$  satisfies the Davies-Gaffney estimates (see [20, Proposition 3.1]). Then, using [21, Lemma 2.3] and Lemma 4.1, we find that  $\{tL_k(t^2 A)^M e^{-t^2 A}\}_{t>0}$  also satisfies the Davies-Gaffney estimates. By this fact, the Hölder inequality, and the Minkowski inequality, we obtain

$$\begin{aligned} H_{j,2} &\sim \sum_{i=1}^2 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_i} \left\{ \int_{r_B}^{+\infty} \left[ \int_{S_j(B)} |L_k e^{-t^2 A} A^M b(x)|^2 dx \right]^{1/2} dt \right\}^{p_i} \\ &\quad \times |2^j B|^{-p_i/2} 2^{jq_0 n} \varphi\left(B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\right) \\ &\lesssim \sum_{i=1}^2 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_i} \left\{ \int_{r_B}^{+\infty} \exp\left(-\frac{(2^j r_B)^2}{ct^2}\right) \|b\|_{L^2(\mathbb{R}^n)} \frac{dt}{t^{2M+1}} \right\}^{p_i} \\ &\quad \times |2^j B|^{-p_i/2} 2^{jq_0 n} \varphi\left(B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\right) \\ &\lesssim \sum_{i=1}^2 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_i} \left\{ \int_{r_B}^{+\infty} \left(\frac{t}{2^j r_B}\right)^{2M-1} \frac{dt}{t^{2M+1}} \right\}^{p_i} \\ &\quad \times (r_B^{2M} |B|^{1/2} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1})^{p_i} |2^j B|^{-p_i/2} 2^{jq_0 n} \varphi\left(B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\right) \\ &\lesssim 2^{-j[p_2(2M-1+\frac{n}{2})-q_0 n]} \varphi\left(B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\right), \end{aligned}$$

where  $M$  is large enough such that  $p_2(2M-1+\frac{n}{2}) > q_0 n$ . Thus, if we choose  $M$  satisfying  $p_2(2M-1+\frac{n}{2}) > q_0 n$ , then we have (4.1). This finishes the proof of Theorem 1.8.  $\square$

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