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A new definition of geometric multiplicity of eigenvalues of tensors and some results based on it

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Abstract   We give a new definition of geometric multiplicity of eigenvalues of tensors, and based on this, we study the geometric and algebraic multiplicity of irreducible tensors' eigenvalues. We get the result that the eigenvalues with modulus $\rho(\mathscr{A})$ have the same geometric multiplicity. We also prove that two-dimensional nonnegative tensors' geometric multiplicity of eigenvalues is equal to algebraic multiplicity of eigenvalues.

Keywords   Irreducible tensor, Perron-Frobenius theorem, geometrically simple
MSC   15A18, 15A69, 74B99

1   Introduction

Eigenvalue problems of nonnegative tensors, especially, the largest eigenvalue of a nonnegative tensor, have attracted special attention in the recent years, see [4,5,7,9,12,14,15]. In particular, the Perron-Frobenius theorem for nonnegative tensors is related to measuring higher order connectivity in linked objects and hypergraphs [1,2]. From the matrix theory, we know that the largest eigenvalue of a nonnegative irreducible matrix is geometrically simple and the eigenvalues with modulus $\rho(A)$ are equally distributed on the spectral circle. But the geometric and algebraic multiplicity for a higher order tensor becomes complicated.

In the literature, the geometric simplicity of the largest eigenvalue (here we call it the spectral radius) of a nonnegative irreducible tensor was studied by Chang et al. [4], where some conditions were proposed to ensure the real geometric simplicity of the spectral radius. Pearson [13] defined essentially positive tensors and proved that the spectral radius of an essentially positive

tensor with order even is real geometrically simple and the modulus of any other eigenvalue of the essentially positive tensor with even order is smaller than the unique positive eigenvalue. Chang et al. [6] showed the modulus of any other eigenvalue of the primitive tensor is smaller than the unique positive eigenvalue. In [19], it was proved that the spectral radius is complex geometrically simple for any positive tensors. The real geometric simplicity of an even order non-negative irreducible tensor can be deduced from a conclusion of [20]. In [3], the real geometric simplicity of the spectral radius of an even order nonnegative irreducible tensor was proved in totally different way from that of [20].

This paper is organized as follows. In Section 2, we recall some definitions and well-known results. In Section 3, we give the new definition of geometric multiplicity of tensor's eigenvalues, and based on this, we study the geometric and algebraic multiplicity of irreducible tensors' eigenvalues. We get the result that the eigenvalues with modulus $\rho(\mathscr{A})$ have the same geometric multiplicity. The eigenvalues $\lambda e^{2\pi ij/k}$ $(j = 1, \ldots, k)$ of an order m dimension n nonnegative weakly irreducible tensor have the same algebraic multiplicity. Although this result can be deduced from [16], the idea of our analysis is totally different from that of [16]. In Section 4, we prove that for two-dimensional nonnegative tensors, the geometric multiplicity is equal to the algebraic multiplicity.

We first add a comment on the notation that is used in the sequel. Vectors are written as lowercase letters $(x, y, \ldots)$, matrices correspond to italic capitals $(A, B, \ldots)$, and tensors are written as calligraphic capitals $(\mathscr{A}, \mathscr{B}, \ldots)$. The entry with row index i and column index j in a matrix A, $(A)_{ij}$, is symbolized by a_{ij} (also, $(\mathscr{A})_{i_1 \cdots i_p, j_1 \cdots j_q} = a_{i_1 \cdots i_p, j_1 \cdots j_q}$). The symbol $|\cdot|$ used on a matrix A (resp. tensor $\mathscr{A}$) means that $(|A|)_{ij} = |a_{ij}|$ (resp. $(|\mathscr{A}|)_{i_1 \cdots i_p, j_1 \cdots j_q} = |a_{i_1 \cdots i_p, j_1 \cdots j_q}|$). $\mathbb{R}^n_+$ (resp. $\mathbb{R}^n_{++}$) denotes the cone $\{x \in \mathbb{R}^n \mid x_i \geqslant (\text{resp.} >) \, 0, \, i = 1, \ldots, n\}$. The symbol $A \geqslant (>, \leqslant, <) \, B$ means that $a_{ij} \geqslant (>, \leqslant, <) \, b_{ij}$ for every i, j and it is the same for rectangular tensors.

2 Preliminaries

First, we recall some known definitions and results, and then we give some new definitions and remarks.

A tensor is a multidimensional array, and a real order m dimension n tensor $\mathscr{A}$ consists of n^m real entries:

$$a_{i_1 \cdots i_m} \in \mathbb{R},$$

where $i_j = 1, \ldots, n$ for $j = 1, \ldots, m$. Like the square matrix, setting this class of tensors can be regarded as "square" tensors. If there are a complex number λ and a nonzero complex vector x that are solutions of the following homogeneous polynomial equations:

$$\mathscr{A} x^{m-1} = \lambda x^{[m-1]},$$

then λ is called the eigenvalue of $\mathscr{A}$ and x the eigenvector of $\mathscr{A}$ associated with

λ, where $\mathscr{A}x^{m-1}$ and $x^{[m-1]}$ are vectors, whose ith component are

$$(\mathscr{A}x^{m-1})_i = \sum_{i_2,\dots,i_m=1}^{n} a_{i,i_2\cdots i_m}x_{i_2}\cdots x_{i_m}, \quad (x^{[m-1]})_i = x_i^{m-1},$$

respectively. This definition was introduced by Qi [14] where he assumed that $\mathscr{A}$ is an order m dimension n symmetric tensor and m is even. If λ and x are restricted in the real field, then (λ, x) is called an H-eigenpair. Independently, Lim [11] gave such a definition but restricted x to be a real vector and λ to be a real number. Here, we follow the definition due to Chang et al. [4], where they gave the general definition as above.

In the following analysis, the notions below will be used.

Definition 1 [19, Definition 2.2]   The spectral radius of tensor $\mathscr{A}$ is defined as

$$\rho(\mathscr{A}) = \max\{|\lambda|\colon \lambda \text{ is an eigenvalue of } \mathscr{A}\}.$$

Definition 2 [11]   A tensor $\mathscr{C} = (c_{i_1}\cdots c_{i_m})$ of order m dimension n is called reducible, if there exists a nonempty proper index subset $I \subset \{1,\dots,n\}$ such that

$$c_{i_1\cdots i_m} = 0, \quad \forall i_1 \in I, \forall i_2,\dots,i_m \notin I.$$

If $\mathscr{C}$ is not reducible, then we call $\mathscr{C}$ irreducible.

Definition 3 [6, Definition 2.6]   An order m dimension n nonnegative irreducible tensor $\mathscr{A}$ is called primitive, if $T_{\mathscr{A}}$ does not have a nontrivial invariant set S on ∂P. ($\{0\}$ is the trivial invariant set.)

Definition 4 [8, Definition 2.1]   A tensor $\mathscr{C} = (c_{i_1}\cdots c_{i_m})$ of order m dimension n is called essentially positive, if $\mathscr{C}x^{m-1} > 0$ for any nonzero $x \geqslant 0$.

Definition 5 [10, Definition 2.1]   A nonnegative matrix $M(\mathscr{A})$ is called the majorization associated to nonnegative tensor $\mathscr{A}$, if the (i,j)-th element of $M(\mathscr{A})$ is defined to be $a_{ij\dots j}$ for any $i, j \in 1,\dots,n$. $\mathscr{A}$ is called weakly positive if $[M(\mathscr{A})]_{ij} > 0$ for all $i \neq j$.

Definition 6   A tensor $D(\mathscr{A}) = (b_{i_1\cdots i_m})$ is called induced to the tensor $\mathscr{A} = (a_{i_1\cdots i_m})$, if

$$b_{i_1\cdots i_m} = \begin{cases} a_{i_1\cdots i_m}, & i_2 = \cdots = i_m = i, \, i = 1,\dots,n, \\ 0, & \text{otherwise,} \end{cases}$$

for any $i_1 \in \{1,\dots,n\}$. $\mathscr{A}$ is called a degenerated tensor, if $\mathscr{A}=D(\mathscr{A})$.

In the next section, we will give some results about degenerated tensor.

Definition 7 [10, Definition 2.2]   Suppose that $\mathscr{A}$ is a nonnegative tensor of order m and dimension n.

(1) We call a nonnegative matrix $G(\mathscr{A})$ the representation associated to the nonnegative tensor $\mathscr{A}$, if the (i, j)-th element of $G(\mathscr{A})$ is defined to be the summation of $\mathscr{A}_{\{ii_2...i_m\}}$ with indices $\{i_2 \cdots i_m\} \ni j$.

(2) We call the tensor $\mathscr{A}$ weakly reducible, if its representation $G(\mathscr{A})$ is a reducible matrix, and weakly primitive, if $G(\mathscr{A})$ is a primitive matrix. If $\mathscr{A}$ is not weakly reducible, then it is called weakly irreducible.

Definition 8   An order m dimension n tensor $\mathscr{A}$ is called strongly irreducible (resp. primitive) tensor, if $D(\mathscr{A})$ is irreducible (resp. primitive) tensor.

Lemma 1 $D(\mathscr{A})$ *is irreducible* (resp. *primitive*) *tensor if and only if* $M(\mathscr{A})$ *is irreducible* (resp. *primitive*) *matrix.*

Proof   It is obvious by Definitions 5 and 6.

Definition 9 [13, Definition 3.4]   For any vector $x \in \mathbb{R}^n$, we define

$$(x^{[1/(m-1)]})_i = x_i^{1/(m-1)}.$$

Let $\mathscr{A}$ and $\mathscr{B}$ be two order m dimension n nonnegative tensors. Let

$$\omega := (\mathscr{A}x^{m-1})^{[1/(m-1)]} \in \mathbb{R}^n.$$

We define the composite of the tensors for $x \in \mathbb{R}^n$ to be the function (not necessarily a tensor) $(\mathscr{B} \circ \mathscr{A})x = \mathscr{B}\omega^{m-1}$.

Definition 10 [4, Definition 3.1]   Let λ be an eigenvalue of

$$\mathscr{A}x^{m-1} = \lambda x^{[m-1]}.$$

We say that λ has geometric multiplicity q, if the maximum number of linearly independent eigenvectors corresponding to λ equals q. If $q = 1$, then λ is called geometrically simple.

We follow [14] to define the characteristic polynomial $\psi(\lambda)$ of λ by

$$\psi(\lambda) = \mathrm{Res}((\mathscr{A}x^{m-1})_1 - \lambda x_1^{[m-1]}, \ldots, (\mathscr{A}x^{m-1})_1 - \lambda x_n^{[m-1]}),$$

where $\mathrm{Res}(P_1, \ldots, P_n)$ is the resultant of n homogeneous polynomials $P_1, \ldots, P_n$. For each λ, such $\psi(\lambda)$ is unique up to an extraneous factor.

Definition 11 [4, Definition 3.1]   Let λ be an eigenvalue of λ. We say that λ has algebraic multiplicity p, if λ is a root of $\psi(\lambda)$ of multiplicity p. And we call λ an algebraically simple eigenvalue, if $p = 1$.

Theorem 1 [18, Theorem 3.6]   *Let $\mathscr{A}$ and $\mathscr{B}$ be order m dimension n tensors satisfying $|\mathscr{B}| \leqslant \mathscr{A}$, and let $\mathscr{A}$ be weakly irreducible. Let β be an eigenvalue of $\mathscr{B}$. Then*

(1) $|\beta| \leqslant \rho(\mathscr{A})$;

(2) if $\beta = \rho(\mathscr{A})\mathrm{e}^{\mathrm{i}\varphi}$, then there exists a diagonal matrix D with modular-1 diagonal entries such that

$$\mathscr{A} = \mathrm{e}^{-\mathrm{i}\varphi}\mathscr{B} \cdot D^{-(m-1)} \cdot \overbrace{D \cdots D}^{m-1}.$$

Theorem 2 [18, Theorem 3.7] *Let $\mathscr{A}$ be an order m dimension n nonnegative weakly irreducible tensor. Suppose (λ, y) is an eigen-pair of $\mathscr{A}$ with modulus of λ equal to $\rho(\mathscr{A})$, i.e., $|\lambda| = \rho(\mathscr{A})$. Then $|y|$ is the unique positive eigenvector corresponding to $\rho(\mathscr{A})$.*

Theorem 3 [19, Theorem 3.1] *Let $\mathscr{A}$ be a nonnegative irreducible tensor and it has k different eigenvalues with modulus $\rho(\mathscr{A})$. Then these eigenvalues are $\rho(\mathscr{A})\mathrm{e}^{2\pi\mathrm{i}j/k}$, $j = 1, \ldots, k$.*

Theorem 4 [6, Theorem 4.5] *Let $\mathscr{A}$ be primitive. If λ is an eigenvalue with $|\lambda| = \rho(\mathscr{A})$, then $\lambda = \rho(\mathscr{A})$.*

Theorem 5 [13, Theorem 3.7] *If $\mathscr{A}$ is an order m dimension n nonnegative, essentially positive tensor with even m, then the modulus of any other eigenvalue of $\mathscr{A}$ is smaller than the unique positive eigenvalue; moreover, the unique positive eigenvalue is real geometrically simple.*

3   Geometric multiplicity of tensors' eigenvalue

As we have known, for a matrix, the set of eigenvectors corresponding to an eigenvalue λ is linear space, so we can call the maximum number of linearly independent eigenvectors the geometric multiplicity of λ. If a set is not linear space, the dimension is not meaning for this set. But if the set have property similar to linear space, we should give definitions. By Theorem 8 below, the set of eigenvectors corresponding to $\rho(\mathscr{A})$ for an irreducible tensor have property of

$$G = G_1 \cup \cdots \cup G_p, \quad p \leqslant (m-1)^{r(n-1)},$$

where

$$G_j = \{kx \neq 0 \mid x^{[(m-1)^r]} = |x|^{[(m-1)^r]}, k \in \mathscr{C}\}, \quad G_i \cap G_j = \emptyset, 1 \leqslant i, j \leqslant p.$$

Thus, we give a new definition of geometric multiplicity.

Before we proceed with the new definition of geometric multiplicity, we use three examples to preview some of the main ideas of the definition.

(i)

$$G = \{x \neq 0 \mid x = k(k_1x_1 + k_2x_2), k \in \mathscr{C}, k_1, k_2, k_1 + k_2 = 0 \text{ or } 1\}.$$

We think that the multiplicity of G is 2 and $B = \{x_1, x_2\} \subset G$ is the set of extreme directions.

(ii)
$$G = \{x \neq 0 \mid x = k[k_1 x_1 + k_2 x_2 + k_3(x_1 + 3x_2)],$$
$$k \in \mathscr{C},\ k_1, k_2, k_3,\ k_1 + k_2 + k_3 = 0 \text{ or } 1\}.$$

We think that the multiplicity of G is 3 and $B = \{x_1, x_2, x_1 + 3x_2\} \subset G$ is the set of extreme directions.

(iii)
$$G = \{x \neq 0 \mid x = k_1 x_1 + k_2(k_{21} x_2 + k_{22} x_3),$$
$$k_1, k_2 \in \mathscr{C},\ k_{21}, k_{22},\ k_{21} + k_{22} = 0 \text{ or } 1\}.$$

We think that the multiplicity of G is 3 and $B = \{x_{11}, x_{21}, x_{22}\} \subset G$ is the set of extreme directions.

In order to make the definition of geometric multiplicity easy to be understood, we give the following definition and corollary.

Definition 12 If

$$A \cap B = \emptyset, \quad C = \{k_1 x + k_2 y \neq 0 \mid \forall\, x \in A,\ \forall\, y \in B,\ k_1, k_2 \in \mathscr{C}\}.$$

Then we say that C is the general direct sum of A and B, and write it as $C = A \oplus B$.

Corollary 1 *We have*

$$\left\{ x \neq 0 \,\Big|\, x = \sum_{i=1}^{i_1} k_i \left(\sum_{j=1}^{j_1} k_{ij} x_{ij} \right),\ \sum_{j=1}^{j_1} k_{ij} = 1,\ k_{ij} = 0 \text{ or } 1,\ x_{ij} \in B,\ k_i \in \mathscr{C} \right\}$$

$$= \bigoplus_{i=1}^{i_1} \left\{ \bigcup_{j=1}^{j_i} A_{ij} \right\},$$

where

$$A_{ij} = \{y \neq 0 \mid y = k_i x_{ij} \neq 0,\ x_{ij} \in B,\ k_i \in \mathscr{C}\}.$$

Proof For any

$$x \in \left\{ k_i \left(\sum_{j=1}^{j_1} k_{ij} x_{ij} \right) \neq 0 \,\Big|\, \sum_{j=1}^{j_1} k_{ij} = 1,\ k_i \in \mathscr{C} \right\}, \tag{3.1}$$

we have $x = k_i x_{ij}$. Then $x \in \cup_{j=1}^{j_i} A_{ij}$. And for $x \in \cup_{j=1}^{j_i} A_{ij}$, we have $x = k_i x_{ij}$, and then we get (3.1). Hence,

$$\left\{ k_i \left(\sum_{j=1}^{j_1} k_{ij} x_{ij} \right) \neq 0 \,\Big|\, \sum_{j=1}^{j_1} k_{ij} = 1,\ k_i \in \mathscr{C} \right\} = \bigcup_{j=1}^{j_i} A_{ij}.$$

And by Definition 12, we have

$$\left\{ x \neq 0 \,\Big|\, x = \sum_{i=1}^{i_1} k_i x_i,\ x_i \in \left\{ \bigcup_{j=1}^{j_i} A_{ij} \right\},\ k_i \in \mathscr{C} \right\} = \bigoplus_{i=1}^{i_1} \left\{ \bigcup_{j=1}^{j_i} A_{ij} \right\},$$

which gives the desired result.   □

Definition 13 Let

$$G = \{x \neq 0 \mid \mathscr{A}x^{m-1} = \lambda x^{[m-1]}\}$$

be the set of eigenvectors corresponding to eigenvalue λ of the tensor $\mathscr{A}$. If

$$G = \bigoplus_{i=1}^{i_1} \left\{ \bigcup_{j=1}^{j_i} A_{ij} \right\}, \quad A_{ij} = \{y \mid y = k_{ij}x_{ij} \neq 0, \ x_{ij} \in B, \ k_{ij} \in \mathscr{C}\}$$

(or

$$G = \left\{ x \neq 0 \,\Big|\, x = \sum_{i=1}^{i_1} k_i \left(\sum_{j=1}^{j_1} k_{ij}x_{ij} \right), \ \sum_{j=1}^{j_1} k_{ij} = 1, \right.$$

$$\left. k_{ij} = 0 \text{ or } 1, \ x_{ij} \in B, \ k_i \in \mathscr{C} \right\}$$

by the Corollary 1). If q is the minimum of $\sum_{i=1}^{i_1} j_i$, then we say that λ has geometric multiplicity q and we call $B = \{x_1, \ldots, x_q\} \subset G$ the generalized base corresponding to λ; if $q = 1$, then λ is called geometrically simple.

Remark Definition 13 is given based on the geometric multiplicity of nonnegative irreducible tensors and two-dimensional nonnegative tensors. Although we cannot find a counter-example for the unreasonable of the definition for nonnegative tensors, we should use the definition for the nonnegative irreducible tensors and two-dimensional nonnegative tensors only. From Theorems 7 and 8 below, we can get that geometric multiplicity of λ with modulus $\rho(\mathscr{A})$ in Definition 13 $\leqslant (m-1)^{r(n-1)}$. From Theorem 16 below, for an order m dimension two nonnegative weakly irreducible tensor, the geometric multiplicity of λ with modulus $\rho(\mathscr{A})$ is equal to the algebraic multiplicity of it. The difference from the definition before is that the extreme direction corresponding to λ can be linearly dependent. For example,

$$G = G_1 \cup \cdots \cup G_q, \quad q \leqslant (m-1)^{r(n-1)},$$

where

$$G_j = \{kx \neq 0 \mid x^{[(m-1)^r]} = |x|^{[(m-1)^r]}, \ k \in C\}, \quad G_i \cap G_j = \emptyset, \quad 1 \leqslant i, j \leqslant p,$$

and thus,

$$G = \left\{ x \neq 0 \,\Big|\, x = k\sum_{i=1}^{q} k_i x_i, \ \sum_{i=1}^{q} k_i = 1, \ k_i = 0 \text{ or } 1, \ x_i \in B, \ k \in C \right\}.$$

Then the extreme direction of G is $B = \{x_1, \ldots, x_q\}$. If $x_1, \ldots, x_q$ are linearly dependent and

$$(x_1, \ldots, x_q) = (x_1, \ldots, x_r)(\alpha_1^{\mathrm{T}}, \alpha_2^{\mathrm{T}} \cdots \alpha_q^{\mathrm{T}}),$$

then $x_1, \ldots, x_r$ cannot be worked out G without $\alpha_{r+1}^{\mathrm{T}}, \alpha_{r+1}^{\mathrm{T}} \cdots \alpha_k^{\mathrm{T}}$. And the geometric multiplicity in Definition 11 is r, but the geometric multiplicity in Definition 13 is q. Obviously, $r \leqslant q$. Then we will give a general result about these two definitions.

Corollary 2 *Geometric multiplicity in Definition* 11 $\leqslant$ *complex geometric multiplicity in Definition* 13.

Proof Suppose that

$$B = \{x_{ij} \mid 1 \leqslant i \leqslant i_1, \ 1 \leqslant j \leqslant j_i\} \subseteq G$$

is the generalized base corresponding to λ and $B_0 = \{x_1, \ldots, x_q\}$ is the maximum linearly independent group of B. Then $q \leqslant \sum_{i=1}^{i_1} j_i$ and

$$
\begin{aligned}
B_0 &\subset B \\
&\subset G \\
&= \left\{ x \neq 0 \,\Big|\, x = \sum_{i=1}^{i_1} k_i \left(\sum_{j=1}^{j_1} k_{ij} x_{ij} \right), \ \sum_{j=1}^{j_1} k_{ij} = 1, \right. \\
&\qquad \left. k_{ij} = 0 \text{ or } 1, \ x_{ij} \in B, \ k_i \in \mathscr{C} \right\} \\
&\subset \left\{ x \neq 0 \,\Big|\, x = \sum_{j=1}^{q} k_i x_i, \ k_i \in \mathscr{C} \right\}.
\end{aligned}
$$

So q is geometric multiplicity in Definition 11. And $\sum_{i=1}^{i_1} j_i$ is geometric multiplicity in Definition 13. Hence, geometric multiplicity in Definition 11 $\leqslant$ complex geometric multiplicity in Definition 13. □

Remark If

$$B_0 = \{x_1, \ldots, x_q\} \subset G_1 \subset G_2 \subset \left\{ x \neq 0 \,\Big|\, x = \sum_{j=1}^{q} k_i x_i, \ k_i \in \mathscr{C} \right\},$$

then G_1 and G_2 have the same geometric multiplicity in Definition 11. But this is different in Definition 13. For example,

$$
\begin{aligned}
B_0 &= \{x_1, x_2\} \\
&\subset G_1 \\
&= \{x \neq 0 \mid x = k[k_1 x_1 + k_2 x_2], \ k \in \mathscr{C}, \ k_1, k_2, k_1 + k_2 = 0 \text{ or } 1\} \\
&\subset G_2 \\
&= \{x \neq 0 \mid x = k[k_1 x_1 + k_2 x_2 + k_3(x_1 + 3x_2)], \ k \in \mathscr{C}, \\
&\qquad k_1, k_2, k_3, k_1 + k_2 + k_3 = 0 \text{ or } 1\} \\
&\subset \{x \neq 0 \mid x = k_1 x_1 + k_2 x_2, \ k_1, k_2 \in \mathscr{C}\},
\end{aligned}
$$

G_1 and G_2 both have the geometric multiplicity 2 in Definition 11, but G_1 has the geometric multiplicity 2 in Definition 13 and G_1 has the geometric multiplicity 3 in Definition 13.

Theorem 6 *Let $\mathscr{A}$ be an order m dimension n nonnegative strongly irreducible tensor, and let G be the set of eigenvectors corresponding to $\rho(\mathscr{A})$. If $x \in G$, then*

$$x^{[m-1]} = \mathrm{e}^{\mathrm{i}\theta}|x|^{[m-1]}.$$

Moreover, $\rho(\mathscr{A})$ is real geometrically simple when m is even.

Proof Because $\mathscr{A}$ is a nonnegative strongly irreducible tensor, $M(\mathscr{A})$ is a nonnegative irreducible matrix. Suppose that

$$\exists\, x \in A, \quad x^{[m-1]} \neq \mathrm{e}^{\mathrm{i}\theta}|x|^{[m-1]}, \quad y = \left(|x_1|, \frac{|x_1|}{x_1}x_2, \ldots, \frac{|x_1|}{x_1}x_n\right).$$

Then $y^{[m-1]} \neq |y|^{[m-1]}$, and

$$\mathscr{A}y^{m-1} = \rho(\mathscr{A})y^{[m-1]}, \quad \mathscr{A}|y|^{m-1} = \rho(\mathscr{A})|y|^{[m-1]}.$$

Suppose

$$(|y|^{[m-1]} - y^{[m-1]})_j \begin{cases} = 0, & 1 \leqslant j \leqslant i \\ \neq 0, & i+1 \leqslant j \leqslant n. \end{cases}$$

Then

$$\mathrm{Re}(|y_{i_2}|\cdots|y_{i_m}| - y_{i_2}\cdots y_{i_m}) > 0, \quad i+1 \leqslant i_2, \ldots, i_m \leqslant n,$$
$$\mathrm{Re}(|y_j|^{[m-1]} - y_j^{[m-1]}) > 0, \quad i+1 \leqslant j \leqslant n.$$

So

$$\begin{aligned} \mathrm{Re}[\rho(\mathscr{A})(|y|^{[m-1]} - y^{[m-1]})] &= \mathrm{Re}(\mathscr{A}|y|^{m-1} - \mathscr{A}y^{m-1}) \\ &\geqslant \mathrm{Re}[M(\mathscr{A})(|y|^{[m-1]} - y^{[m-1]})] \\ &= M(\mathscr{A})\mathrm{Re}(|y|^{[m-1]} - y^{[m-1]}). \end{aligned}$$

And we also have

$$(\mathscr{A}|y|^{m-1} - \mathscr{A}y^{m-1})_k = 0, \quad 1 \leqslant k \leqslant i.$$

Thus, we can get

$$a_{k,j\cdots j} = 0, \quad 1 \leqslant k \leqslant i,\, i+1 \leqslant j \leqslant n.$$

Hence, $D(\mathscr{A})$ is a nonnegative reducible tensor, which contradicts that $D(\mathscr{A})$ is a nonnegative reducible tensors. Then

$$x^{[m-1]} = \mathrm{e}^{\mathrm{i}\theta}|x|^{[m-1]}.$$

Clearly, $\rho(\mathscr{A})$ is real geometrically simple when m is even. $\qquad\square$

Theorem 7   *Let $\mathscr{A}$ be an order m dimension n nonnegative irreducible tensor, and let G be the set of eigenvectors corresponding to $\rho(\mathscr{A})$. If $x \in G$, then*

$$x^{[(m-1)^r]} = e^{i\theta}|x|^{[(m-1)^r]},$$

where θ relies on x, and r is the number of $P^{\mathrm{T}}M(A)P$ irreducible blocks. Moreover, $\rho(\mathscr{A})$ is real geometrically simple with m even.

Proof   Suppose

$$P^{\mathrm{T}}M(A)P = (D_{ij})_{1 \leqslant i,j \leqslant r},$$

where P is an $n \times n$ permutation matrix, D_{ii}, $i = 1, \ldots, r$, are irreducible matrices. Let

$$z^{[m-1]} = P^{\mathrm{T}}x^{[m-1]} = (z_1^{[m-1]}, \ldots, z_r^{[m-1]})^{\mathrm{T}}.$$

Since

$$\rho(\mathscr{A})(|x|^{[m-1]} - x^{m-1}) = \mathscr{A}|x|^{m-1} - \mathscr{A}x^{m-1},$$

we have

$$\mathrm{Re}[\rho(\mathscr{A})(|x|^{[m-1]} - x^{[m-1]})] \geqslant \mathrm{Re}[M(\mathscr{A})(|x|^{[m-1]} - x^{[m-1]})].$$

Thus,

$$\mathrm{Re}[\rho(\mathscr{A})(P^{\mathrm{T}}|x|^{[m-1]} - P^{\mathrm{T}}x^{[m-1]})] \geqslant \mathrm{Re}[P^{\mathrm{T}}M(\mathscr{A})P(P^{\mathrm{T}}|x|^{[m-1]} - P^{\mathrm{T}}x^{[m-1]})],$$
$$\mathrm{Re}[\rho(\mathscr{A})(|z|^{[m-1]} - z^{[m-1]})] \geqslant \mathrm{Re}[P^{\mathrm{T}}M(\mathscr{A})P(|z|^{[m-1]} - z^{[m-1]})].$$

Hence,

$$\mathrm{Re}[\rho(\mathscr{A})(|z|_i^{[m-1]} - z_i^{[m-1]})] \geqslant \mathrm{Re}[D_{ii}(|z|_i^{[m-1]} - z_i^{[m-1]})], \quad 1 \leqslant i \leqslant r.$$

Then, by the proof of Theorem 6, we can get

$$z_i^{[m-1]} = e^{i\theta_i}|z|_i^{[m-1]}, \quad 1 \leqslant i \leqslant r.$$

Suppose that

$$z = P^{\mathrm{T}}x = (z_1, \ldots, z_r), \quad z_i^{[m-1]} = e^{i\theta_i}|z|_i^{[m-1]}, \quad 1 \leqslant i \leqslant r,$$

and z_1 contains p elements. Next, we will prove that if $x \in G$, then

$$x^{[(m-1)^r]} = e^{i\theta}|x|^{[(m-1)^r]}$$

by inductive method.
Case 1   When $r = 2$, let

$$y = z = (z_1, z_2), \quad \xi = P^{\mathrm{T}}(1, \ldots, n)^{\mathrm{T}}.$$

Then $y_i = x_{\xi_i}$. Since $\mathscr{A}$ is a nonnegative irreducible tensor, $\exists\, a_{i_1 \cdots i_m} > 0$, $\widetilde{i}_2, \ldots,$ $\widetilde{i}_m \in \{\xi_1, \ldots, \xi_p\}$, p is the side of z_1, and $\widetilde{i}_1 \in \{\xi_{p+1}, \ldots, \xi_n\}$. Then we have

$$\sum_{j_2, \ldots, j_m = 1}^{n} a_{\widetilde{i}_1 j_2 \cdots j_m} x_{j_2} \cdots x_{j_m} = e^{i\theta_2} \rho(\mathscr{A}) |x_{\widetilde{i}_1}|^{m-1},$$

$$\sum_{j_2, \ldots, j_m = 1}^{n} a_{\widetilde{i}_1 j_2 \cdots j_m} |x_{j_2}| \cdots |x_{j_m}| = \rho(\mathscr{A}) |x_{\widetilde{i}_1}|^{m-1},$$

$$\sum_{j_2, \ldots, j_m = 1}^{n} a_{\widetilde{i}_1 j_2 \cdots j_m} |x_{j_2}| \cdots |x_{j_m}| = \left| \sum_{j_2, \ldots, j_m = 1}^{n} a_{\widetilde{i}_1 j_2 \cdots j_m} x_{j_2} \cdots x_{j_m} \right|,$$

$$a_{\widetilde{i}_1 \cdots \widetilde{i}_m} x_{\widetilde{i}_2} \cdots x_{\widetilde{i}_m} = e^{i\theta_2} a_{\widetilde{i}_1 \cdots \widetilde{i}_m} |x_{\widetilde{i}_2}| \cdots |x_{\widetilde{i}_m}|,$$

$$e^{i(m-1)\theta_1} a_{\widetilde{i}_1 \cdots \widetilde{i}_m}^{m-1} |x_{\widetilde{i}_2}|^{m-1} \cdots |x_{\widetilde{i}_m}|^{m-1} = e^{i(m-1)\theta_2} a_{\widetilde{i}_1 \cdots \widetilde{i}_m}^{m-1} |x_{\widetilde{i}_2}|^{m-1} \cdots |x_{\widetilde{i}_m}|^{m-1}.$$

Since $a_{\widetilde{i}_1 \cdots \widetilde{i}_m} > 0$, we have

$$e^{i(m-1)\theta_1} = e^{i(m-1)\theta_2}.$$

Thus,

$$z^{[(m-1)^2]} = e^{i\theta} |z|^{[(m-1)^2]}.$$

Then

$$x^{[(m-1)^2]} = e^{i\theta} |x|^{[(m-1)^2]}.$$

Case 2 When

$$r = k, \quad z = P^{\mathrm{T}} x = (z_1, \ldots, z_k), \quad z_i^{[m-1]} = e^{i\theta_i} |z|_i^{[m-1]}, \quad 1 \leqslant i \leqslant k,$$

we have

$$x^{(m-1)^k} = e^{i\theta} |x|^{(m-1)^k}.$$

Case 3 When $r = k + 1$, let

$$y = z = (z_1, \ldots, z_{k+1}), \quad z_i^{[m-1]} = e^{i\theta_i} |z|_i^{[m-1]}, \quad 1 \leqslant i \leqslant k + 1,$$
$$\xi = P^{\mathrm{T}} (1, \ldots, n)^{\mathrm{T}}.$$

Then $y_i = x_{\xi_i}$. Since $\mathscr{A}$ is a nonnegative irreducible tensor, $\exists\, a_{i_1 \cdots i_m} > 0$, $\widetilde{i}_2, \ldots,$ $\widetilde{i}_m \in \{\xi_1, \ldots, \xi_p\}$, p is the side of z_1, $\widetilde{i}_1 \in \{\xi_{p+1}, \ldots, \xi_n\}$, and $x_{\widetilde{i}_1}$ is the component of z_j. Then we have

$$\sum_{j_2, \ldots, j_m = 1}^{n} a_{\widetilde{i}_1 j_2 \cdots j_m} x_{j_2} \cdots x_{j_m} = e^{i\theta_j} \rho(\mathscr{A}) |x_{\widetilde{i}_1}|^{m-1},$$

$$\sum_{j_2, \ldots, j_m = 1}^{n} a_{\widetilde{i}_1 j_2 \cdots j_m} |x_{j_2}| \cdots |x_{j_m}| = \rho(\mathscr{A}) |x_{\widetilde{i}_1}|^{m-1},$$

$$\sum_{j_2,\ldots,j_m=1}^{n} a_{\widetilde{i}_1 j_2 \cdots j_m} |x_{j_2}| \cdots |x_{j_m}| = \left| \sum_{j_2,\ldots,j_m=1}^{n} a_{\widetilde{i}_1 j_2 \cdots j_m} x_{j_2} \cdots x_{j_m} \right|,$$

$$a_{\widetilde{i}_1 \cdots \widetilde{i}_m} x_{\widetilde{i}_2} \cdots x_{\widetilde{i}_m} = \mathrm{e}^{\mathrm{i}\theta_j} a_{\widetilde{i}_1 \cdots \widetilde{i}_m} |x_{\widetilde{i}_2}| \cdots |x_{\widetilde{i}_m}|,$$

$$\mathrm{e}^{\mathrm{i}(m-1)\theta_1} a_{\widetilde{i}_1 \cdots \widetilde{i}_m}^{m-1} |x_{\widetilde{i}_2}|^{m-1} \cdots |x_{\widetilde{i}_m}|^{m-1} = \mathrm{e}^{\mathrm{i}(m-1)\theta_j} a_{\widetilde{i}_1 \cdots \widetilde{i}_m}^{m-1} |x_{\widetilde{i}_2}|^{m-1} \cdots |x_{\widetilde{i}_m}|^{m-1}.$$

Since $a_{i_1 \cdots i_m} > 0$, we have

$$\mathrm{e}^{\mathrm{i}(m-1)\theta_1} = \mathrm{e}^{\mathrm{i}(m-1)\theta_j}.$$

Thus,

$$z_1^{[(m-1)^2]} = \mathrm{e}^{\mathrm{i}\theta} |z_1|^{[(m-1)^2]}, \quad z_j^{[(m-1)^2]} = \mathrm{e}^{\mathrm{i}\theta} |z_j|^{[(m-1)^2]}.$$

Let

$$w = (w_1, \ldots, w_k), \quad w_i = \begin{cases} (z_1, z_j), & i = 1, \\ z_i, & 2 \leqslant i \leqslant j-1, \\ z_{i+1}, & j+1 \leqslant i \leqslant k. \end{cases}$$

Then

$$r = k, \quad w = P^{\mathrm{T}} x = (w_1, \ldots, w_k), \quad w_i^{[m-1]^2} = \mathrm{e}^{\mathrm{i}\theta_i} |z|_i^{[m-1]^2}, \quad 1 \leqslant i \leqslant k.$$

Thus, we have

$$x^{(m-1)^{k+1}} = \mathrm{e}^{\mathrm{i}\theta} |x|^{(m-1)^{k+1}}.$$

Moreover, $\rho(\mathscr{A})$ is real geometrically simple when m is even. □

Corollary 3 *If*

$$G_0 = \{x \mid x^{[(m-1)^r]} = |x|^{[(m-1)^r]}\}, \quad G_\theta = \{x \mid x^{[(m-1)^r]} = \mathrm{e}^{\mathrm{i}\theta} |x|^{[(m-1)^r]}\},$$

then

$$G_\theta = \{\mathrm{e}^{\mathrm{i}\theta/(m-1)^r} x \mid x \in G_0\}.$$

Moreover, if $\mathscr{A}$ is an order m dimension n nonnegative irreducible tensor and G is the set of eigenvectors corresponding to $\rho(\mathscr{A})$, then the side of G is not more than $(m-1)^r$ up to a multiplicative constant.

Proof For any $x \in G_\theta$, we have

$$x^{[(m-1)^r]} = \mathrm{e}^{\mathrm{i}\theta} |x|^{[(m-1)^r]}.$$

Then

$$x = (\mathrm{e}^{\mathrm{i}\frac{\theta+2j_1\pi}{(m-1)^r}} |x_1|, \ldots, \mathrm{e}^{\mathrm{i}\frac{\theta+2j_n\pi}{(m-1)^r}} |x_n|)$$
$$= \mathrm{e}^{\mathrm{i}\frac{\theta}{(m-1)^r}} (\mathrm{e}^{\mathrm{i}\frac{2j_1\pi}{(m-1)^r}} |x_1|, \ldots, \mathrm{e}^{\mathrm{i}\frac{2j_n\pi}{(m-1)^r}} |x_n|).$$

Since

$$(\mathrm{e}^{\mathrm{i}\frac{2j_1\pi}{(m-1)^r}} |x_1|, \ldots, \mathrm{e}^{\mathrm{i}\frac{2j_n\pi}{(m-1)^r}} |x_n|) \in G_0,$$

we have

$$G_\theta = \{\mathrm{e}^{\mathrm{i}\frac{\theta}{(m-1)^r}} x \mid x \in G_0\}.$$

The side of G_0 is not more than $(m-1)^r$, so the side of G is not more than $(m-1)^r$ up to a multiplicative constant by Theorem 7. □

Lemma 2 *Let $\mathscr{A}$ be an order m dimension n nonnegative irreducible tensor, and let G be the set of eigenvectors corresponding to $\rho(\mathscr{A})$. If $x, y, k_1 x + k_2 y \in G$, $k_1 k_2 \neq 0$, then*

$$\frac{x_j}{y_j} = \frac{x_1}{y_1} \quad or \quad \frac{x_j}{y_j} = \frac{x_1}{y_1} \mathrm{e}^{\mathrm{i}\theta},$$

and

$$\mathrm{e}^{\mathrm{i}\theta} \frac{k_1}{k_2} = \frac{\overline{k_1}}{\overline{k_2}}.$$

Proof Since $x, y \in G$, $|y| = k|x|$ by Theorem 2. Since $k_1 x + k_2 y \in G$, we have

$$|k_1 x + k_2 y| = p|x|.$$

Thus,

$$p|x_j| = |k_1 x_j + k_2 y_j|.$$

Hence,

$$\frac{p}{|k_1|} = \left|1 + \frac{k_2}{k_1} \frac{y_j}{x_j}\right|, \quad 1 \leqslant j \leqslant n,$$

and the desired result follows. □

Theorem 8 *Let $\mathscr{A}$ be an order m dimension n nonnegative irreducible tensor, and let G be the set of eigenvectors corresponding to $\rho(\mathscr{A})$. Then*

$$G = G_1 \cup \cdots \cup G_p, \quad p \leqslant (m-1)^{r(n-1)}, \tag{3.2}$$

where

$$G_j = \{kx \neq 0 \mid x^{[(m-1)^r]} = |x|^{[(m-1)^r]}, 0 \neq k \in C\}, \quad G_i \cap G_j = \emptyset, \quad 1 \leqslant i, j \leqslant p.$$

Hence, $\{x_1, \ldots, x_p\} \subset G$ is the generalized base of $\rho(\mathscr{A})$ and $\rho(\mathscr{A})$ has geometric multiplicity p.

Proof We get (3.2) by Theorem 7 and Corollary 3. From (3.2), we can get the result that $\forall\, x \in G$, $x = p_0 \sum_{i=1}^{k} p_i x_i$, $x_i \in G_i$, $1 \leqslant i \leqslant p$, $p_0 \in C$, $\{p_i, 1 \leqslant i \leqslant p\}$, does not have more than one nonzero element. Suppose that the geometric multiplicity of $\rho(\mathscr{A})$ is $k_0 < p$. Then we have

$$G = B_1 \cup \cdots \cup B_{k_0},$$

where

$$B_j = \{kx \neq 0 \mid x^{[(m-1)^r]} = |x|^{[(m-1)^r]}, 0 \neq k \in C\}, \quad B_i \cap B_j = \emptyset, \quad 1 \leqslant i, j \leqslant k_0,$$

by Lemma 2. Then we get that there exist i_0 and j_0 such that $1 \leqslant i_0, j_0 \leqslant p$, $G_{i_0} = G_{j_0}$, which contradicts that $G_i \cap G_j = \emptyset$, $1 \leqslant i, j \leqslant p$. So we get the conclusion that for $0 \neq x_j \in G_j$, $1 \leqslant j \leqslant p$, $\{x_1, \ldots, x_k\} \subset G$ is the generalized base of $\rho(\mathscr{A})$ and $\rho(\mathscr{A})$ has geometric multiplicity p by Definition 13. $\square$

Lemma 3   *Suppose that G_1 is the set of eigenvectors corresponding to λ_1 and G_2 is the set of eigenvectors corresponding to λ_2. If G_1 is isomorphic to G_2 and the geometric multiplicity of λ_1 is finite, then λ_1 and λ_2 have the same geometric multiplicity.*

Proof   Because G_1 is isomorphic to G_2 and the geometric multiplicity of λ_1 is finite, we can suppose that $\{x_1, \ldots, x_k\} \subset G_1$ is the base of G_1 and there exists isomorphism φ, which satisfy $\{\varphi(x_1), \ldots, \varphi(x_p)\} \subset G_2$ and G_2 can not only be linear represented by $\{\varphi(x_1), \ldots, \varphi(x_p)\}$, but also can be worked out by $\{\varphi(x_1), \ldots, \varphi(x_p)\}$. Thus, the geometric multiplicity of λ_2 is finite and the geometric multiplicity of λ_2 is not more than the geometric multiplicity of λ_1. Thus, the geometric multiplicity of λ_1 is not more than the geometric multiplicity of λ_2 by the same proof. Therefore, λ_1 and λ_2 have the same geometric multiplicity. $\square$

Theorem 9   *Let $\mathscr{A}$ be an order m dimension n nonnegative irreducible tensor. Suppose that $\rho(\mathscr{A})$ and $\rho(\mathscr{A})\mathrm{e}^{\mathrm{i}\theta}$ are eigenvalues of $\mathscr{A}$. Then these eigenvalues have the same geometric multiplicity.*

Proof   Suppose

$$\mathscr{A}y^{m-1} = \rho(\mathscr{A})\mathrm{e}^{\mathrm{i}\theta}y^{[m-1]}, \quad \mathscr{A}x^{m-1} = \rho(\mathscr{A})x^{[m-1]},$$

$$D = \mathrm{diag}\Big(\frac{y_1}{|y_1|}, \ldots, \frac{y_n}{|y_n|}\Big).$$

Then

$$\mathscr{A} = \mathrm{e}^{-\mathrm{i}\theta}\mathscr{A} \cdot D^{-(m-1)} \cdot \overbrace{D \cdots D}^{m-1}.$$

Thus,

$$\sum_{i_2, \ldots, i_m=1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}$$

$$= \mathrm{e}^{-\mathrm{i}\theta} \sum_{i_2, \ldots, i_m=1}^{n} a_{i,i_2 \cdots i_m} \Big(\frac{y_i}{|y_i|}\Big)^{-(m-1)} \frac{y_{i_2}}{|y_{i_2}|} \cdots \frac{y_{i_m}}{|y_{i_m}|} \cdot x_{i_2} \cdots x_{i_m}$$

$$= \mathrm{e}^{-\mathrm{i}\theta} \Big(\frac{y_i}{|y_i|}\Big)^{-(m-1)} \sum_{i_2, \ldots, i_m=1}^{n} a_{i,i_2 \cdots i_m} \frac{y_{i_2}}{|y_{i_2}|} \cdots \frac{y_{i_m}}{|y_{i_m}|} \cdot x_{i_2} \cdots x_{i_m}$$

$$= \rho(\mathscr{A})x_i^{m-1},$$

and then

$$\sum_{i_2, \ldots, i_m=1}^{n} a_{i,i_2 \cdots i_m} \frac{y_{i_2}}{|y_{i_2}|} \cdots \frac{y_{i_m}}{|y_{i_m}|} \cdot x_{i_2} \cdots x_{i_m} = \rho(\mathscr{A})\mathrm{e}^{\mathrm{i}\theta}\Big(\frac{y_i}{|y_i|} x_i\Big)^{m-1}.$$

Letting

$$z = \left(\frac{y_1}{|y_1|} x_1, \ldots, \frac{y_n}{|y_n|} x_n \right),$$

we have

$$\mathscr{A} z^{m-1} = \rho(\mathscr{A}) e^{i\theta} z^{[m-1]}.$$

For

$$x = (x_1, \ldots, x_n) \in G_0, \quad y = (y_1, \ldots, y_n) \in G_1,$$

where G_0 is the set of eigenvectors corresponding to $\rho(\mathscr{A})$ and G_1 is the set of eigenvectors corresponding to $\rho(\mathscr{A}) e^{i\theta}$, define project φ as follows:

$$\varphi(x) = z = \left(\frac{y_1}{|y_1|} x_1, \ldots, \frac{y_n}{|y_n|} x_n \right).$$

Then, $\forall\, x, x' \in A_k$, $x \neq x'$, we have $\varphi(x) \neq \varphi(x')$. So φ is an injection from G_0 to G_1. In addition, for $y \in G_1$, since

$$\mathscr{A} y^{m-1} = \rho(\mathscr{A}) e^{i\theta} y^{[m-1]},$$

we have

$$\mathscr{A} \overline{y}^{m-1} = \rho(\mathscr{A}) e^{-i\theta} \overline{y}^{[m-1]},$$

where $\overline{y}$ denotes the dual of y. Let

$$D_1 = \operatorname{diag}\left(\frac{\overline{y}_1}{|\overline{y}_1|}, \ldots, \frac{\overline{y}_n}{|\overline{y}_n|} \right).$$

Then

$$\mathscr{A} = e^{i\theta} \mathscr{A} \cdot D_1^{-(m-1)} \cdot \overbrace{D_1 \cdots D_1}^{m-1}.$$

Thus, for any $z \in G_1$,

$$\sum_{j_2,\ldots,j_m=1}^{n} a_{ij_2\cdots j_m} z_{j_2} \cdots z_{j_m}$$

$$= e^{i\theta} \sum_{j_2,\ldots,i_m=1}^{n} a_{ij_2\cdots j_m} \left(\frac{\overline{y}_i}{|\overline{y}_i|} \right)^{-(m-1)} \frac{\overline{y}_{j_2}}{|\overline{y}_{j_2}|} \cdots \frac{\overline{y}_{j_m}}{|\overline{y}_{j_m}|} \cdot z_{j_2} \cdots z_{j_m}$$

$$= e^{i\theta} \left(\frac{\overline{y}_i}{|\overline{y}_i|} \right)^{-(m-1)} \sum_{j_2,\ldots,j_m=1}^{n} a_{i,j_2\cdots i_m} \frac{\overline{y}_{j_2}}{|\overline{y}_{j_2}|} \cdots \frac{\overline{y}_{j_m}}{|\overline{y}_{j_m}|} \cdot z_{j_2} \cdots z_{j_m}$$

$$= \rho(\mathscr{A}) e^{i\theta} z_i^{m-1},$$

and then

$$\sum_{j_2,\ldots,j_m=1}^{n} a_{i,j_2\cdots i_m} \frac{\overline{y}_{j_2}}{|\overline{y}_{j_2}|} \cdots \frac{\overline{y}_{j_m}}{|\overline{y}_{j_m}|} \cdot z_{j_2} \cdots z_{j_m} = \rho(\mathscr{A}) \left(\frac{\overline{y}_i}{|\overline{y}_i|} z_i \right)^{m-1}.$$

Letting

$$x = \left(\left(\frac{\overline{y}_1}{|\overline{y}_1|} \right) z_1, \ldots, \left(\frac{\overline{y}_n}{|\overline{y}_n|} \right) z_n \right),$$

we have

$$\mathscr{A} x^{m-1} = \rho(\mathscr{A}) x^{[m-1]}.$$

So for any $z \in G_1$, there exists

$$x = \left(\left(\frac{\overline{y}_1}{|\overline{y}_1|} \right) z_1, \ldots, \left(\frac{\overline{y}_n}{|\overline{y}_n|} \right) z_n \right) \in G_0$$

such that $\varphi(x) = z$, and then φ is a surjection. And we also have

$$\varphi(k_1 x + k_2 x') = \left(\frac{y_1}{|y_1|} (k_1 x + k_2 x')_1, \ldots, \frac{y_n}{|y_n|} (k_1 x + k_2 x')_n \right) = k_1 \varphi(x) + k_2 \varphi(x').$$

So G_0 is isometry to G_1. Then $\rho(\mathscr{A})$ and $\rho(\mathscr{A}) \mathrm{e}^{\mathrm{i}\theta}$ have the same geometric multiplicity by Theorem 8 and Lemma 3.   □

Theorem 10 *Let $\mathscr{A}$ be an order m dimension n nonnegative irreducible tensor, which has k different eigenvalues with modulus $\rho(\mathscr{A})$. Suppose that G_j is the set of eigenvectors corresponding to $\rho(\mathscr{A}) \mathrm{e}^{2\pi \mathrm{i} j/k}$, $j = 1, \ldots, k$. Then these eigenvalues have the same geometric multiplicity.*

Proof Take $\theta = 2\pi j/k$, $j = 1, \ldots, k$. Then the conclusion follows immediately from Theorem 9.   □

Theorem 11 *Let $\mathscr{A}$ be an order m dimension n nonnegative irreducible tensor, which has k different eigenvalues with modulus $\rho(\mathscr{A})$. Suppose that G_j is the set of eigenvectors corresponding to $\lambda \mathrm{e}^{2\pi \mathrm{i} j/k}$, $j = 1, \ldots, k$, and y_1 is an eigenvector corresponding to $\rho(\mathscr{A}) \mathrm{e}^{2\pi \mathrm{i}/k}$. Then G_j, $j = 1, \ldots, k$, are isometry each other and*

$$G_p = \left\{ z \in \mathbb{C}^n \,\Big|\, x \in G_k, \, z = \left(\left(\frac{y_{11}}{|y_{11}|} \right)^p x_1, \ldots, \left(\frac{y_{1n}}{|y_{1n}|} \right)^p x_n \right) \right\}, \quad p = 1, \ldots, k-1.$$

Proof The conclusion is easily obtained from Theorem 9.   □

Theorem 12 *Let $\mathscr{A}$ be an order m dimension n nonnegative irreducible tensor, which has k different eigenvalues with modulus $\rho(\mathscr{A})$, and let A_p be the set of eigenvectors corresponding to $\rho(\mathscr{A}) \mathrm{e}^{2\pi \mathrm{i} p/k}$, $p = 1, \ldots, k$. If $x \in G_p$, $y \in A_k$, then*

$$z_j = \left(\left(\frac{y_1}{|y_1|} \right)^j x_1, \ldots, \left(\frac{y_n}{|y_n|} \right)^j x_n \right) \in A_p, \quad j \in \mathbb{Z}, \, p = 1, \ldots, k.$$

Proof Suppose that

$$\mathscr{A} x^{m-1} = \rho(\mathscr{A}) \mathrm{e}^{2\pi \mathrm{i} p/k} x^{[m-1]}, \quad \mathscr{A} y^{m-1} = \rho(\mathscr{A}) y^{[m-1]},$$

$$D = \mathrm{diag}\left(\frac{y_1}{|y_1|}, \ldots, \frac{y_n}{|y_n|} \right), \quad D_1 = \mathrm{diag}\left(\frac{\overline{y}_{j_0 1}}{|\overline{y}_{j_0 1}|}, \ldots, \frac{\overline{y}_{j_0 n}}{|\overline{y}_{j_0 n}|} \right).$$

Then

$$\mathscr{A} = \mathscr{A} \cdot D^{-(m-1)} \cdot \overbrace{D \cdots D}^{m-1} = \mathscr{A} \cdot D_1^{-(m-1)} \cdot \overbrace{D_1 \cdots D_1}^{m-1}.$$

Thus,

$$\sum_{i_2,\ldots,i_m=1}^{n} a_{i,i_2\cdots i_m} x_{i_2} \cdots x_{i_m}$$

$$= \sum_{i_2,\ldots,i_m=1}^{n} a_{i,i_2\cdots i_m} \left(\frac{y_i}{|y_i|}\right)^{-(m-1)} \frac{y_{i_2}}{|y_{i_2}|} \cdots \frac{y_{i_m}}{|y_{i_m}|} \cdot x_{i_2} \cdots x_{i_m}$$

$$= \left(\frac{y_i}{|y_i|}\right)^{-(m-1)} \sum_{i_2,\ldots,i_m=1}^{n} a_{i,i_2\cdots i_m} \frac{y_{i_2}}{|y_{i_2}|} \cdots \frac{y_{i_m}}{|y_{i_m}|} \cdot x_{i_2} \cdots x_{i_m}$$

$$= \rho(\mathscr{A}) e^{2\pi i p/k} x_i^{m-1},$$

and then

$$\sum_{i_2,\ldots,i_m=1}^{n} a_{i,i_2\cdots i_m} \frac{y_{i_2}}{|y_{i_2}|} \cdots \frac{y_{i_m}}{|y_{i_m}|} \cdot x_{i_2} \cdots x_{i_m} = \rho(\mathscr{A}) e^{2\pi i p/k} \left(\frac{y_i}{|y_i|} x_i\right)^{m-1}.$$

Letting

$$z_1 = \left(\frac{y_1}{|y_1|} x_1, \ldots, \frac{y_n}{|y_n|} x_n\right),$$

we have

$$\mathscr{A} z_1^{m-1} = \rho(\mathscr{A}) e^{2\pi i p/k} z_1^{[m-1]}.$$

Repeating the procedure above, we have

$$z_j = \left(\left(\frac{y_1}{|y_1|}\right)^j x_1, \ldots, \left(\frac{y_n}{|y_n|}\right)^j x_n\right),$$

which satisfies

$$\mathscr{A} z_j^{m-1} = \rho(\mathscr{A}) e^{2\pi i p/k} z_j^{[m-1]}, \quad j \in \mathbb{N}.$$

With the same progress, we can get

$$\mathscr{A} z_j^{m-1} = \rho(\mathscr{A}) e^{2\pi i p/k} z_j^{[m-1]}, \quad j \in \mathbb{N}.$$

Thus, $z_j \in G_p$, $j \in \mathbb{Z}$, $p = 1, \ldots, k$. □

Corollary 4 *Let $\mathscr{A}$ be an order m dimension n nonnegative irreducible degenerated tensor, and let G be the set of eigenvectors corresponding to $\rho(\mathscr{A})$. Then*

complex geometric multiplicity of $\rho(\mathscr{A})$ = algebraic multiplicity of $\rho(\mathscr{A})$

$$= (m-1)^{n-1}.$$

Proof Suppose that G is the set of eigenvectors of $\rho(\mathscr{A})$. Because $\mathscr{A}$ is a nonnegative irreducible degenerated tensor, $M(\mathscr{A})$ is an irreducible matrix. Then

$$\rho(\mathscr{A})x^{[m-1]} = M(\mathscr{A})x^{[m-1]} = \rho(\mathscr{A})x^{[m-1]}.$$

Thus,

$$x^{[m-1]} = |x|^{[m-1]}, \quad G = G_1 \cup G_2 \cup \cdots \cup G_{(m-1)^{n-1}},$$

where

$$G_j = \{kx \neq 0 \mid x^{[(m-1)^r]} = |x|^{[(m-1)^r]}, \, k \in C\},$$
$$G_i \cap G_j = \emptyset, \quad 1 \leqslant i, j \leqslant (m-1)^{n-1}.$$

So we establish the complex geometric multiplicity corresponding to $\rho(\mathscr{A}) = (m-1)^{n-1}$ by Theorem 8. And when λ is not the root of $\mathrm{Res}(M(\mathscr{A}) - \lambda I) = 0$,

$$
\begin{aligned}
\mathrm{Res}(\mathscr{A}&x^{m-1} - \lambda x^{[m-1]}) \\
&= \mathrm{Res}(M(\mathscr{A})x^{[m-1]} - \lambda x^{[m-1]}) \\
&= \det(M(\mathscr{A}) - \lambda I)^{(m-1)^{n-1}} \mathrm{Res}(x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1}) \\
&= \det(M(\mathscr{A}) - \lambda I)^{(m-1)^{n-1}}.
\end{aligned}
$$

Because the side of $\{\lambda \mid \det(M(\mathscr{A}) - \lambda I) = 0\}$ is finite, for any λ, we have

$$\mathrm{Res}(\mathscr{A}x^{m-1} - \lambda x^{[m-1]}) = \det(M(\mathscr{A}) - \lambda I)^{(m-1)^{n-1}}.$$

Then we can get the algebraic multiplicity of $\rho(\mathscr{A}) = (m-1)^{n-1}$, too. Thus, we get the desired result. $\qquad\square$

In [4], Chang et al. pointed out that "By definition, we see complex geometric multiplicity $\leqslant$ algebraic multiplicity, but not equal in general". However, by Definition 13, we can get

$$\text{geometric multiplicity of } \rho(\mathscr{A}) = \text{algebraic multiplicity of } \rho(\mathscr{A})$$

for some kinds of nonnegative tensors.

Corollary 5 *Let $\mathscr{A}$ be an order m dimension n nonnegative irreducible tensor, which has k different eigenvalues with modulus $\rho(\mathscr{A})$. Suppose that G_j is the set of eigenvectors corresponding to $\lambda \mathrm{e}^{2\pi \mathrm{i} j / k}$, $j = 1, \ldots, k$. Then G_j, $j = 1, \ldots, k$, are isometry each other.*

Proof Suppose

$$\mathscr{A}y^{m-1} = \rho(\mathscr{A})\mathrm{e}^{\mathrm{i}\theta}y^{[m-1]}, \quad D = \mathrm{diag}\Big(\frac{y_1}{|y_1|}, \ldots, \frac{y_n}{|y_n|}\Big).$$

Then

$$\mathscr{A} = \mathrm{e}^{-\mathrm{i}\theta}\mathscr{A} \cdot D^{-(m-1)} \cdot \overbrace{D \cdots D}^{m-1}.$$

Thus,

$$\mathrm{Res}(\mathscr{A}x^{m-1} - \lambda x^{[m-1]})$$

$$= \mathrm{Res}(\mathrm{e}^{-\mathrm{i}\theta}\mathscr{A} \cdot D^{-(m-1)} \cdot \overbrace{D \cdots D}^{m-1} x^{m-1} - \lambda x^{[m-1]})$$
$$= (\mathrm{e}^{-\mathrm{i}\theta}\det(D))^{-(m-1)^n}\mathrm{Res}(\mathscr{A}(Dx)^{m-1} - \mathrm{e}^{\mathrm{i}\theta}\lambda(Dx)^{[m-1]})$$
$$= (\mathrm{e}^{-\mathrm{i}\theta}\det(D))^{-(m-1)^n}\mathrm{Res}(\mathscr{A}z^{m-1} - \mathrm{e}^{\mathrm{i}\theta}\lambda z^{[m-1]}),$$

where $z = Dx$. Suppose

$$\varphi(\lambda) = \mathrm{Res}(\mathscr{A}x^{m-1} - \lambda x^{[m-1]}),$$
$$\psi(\mathrm{e}^{\mathrm{i}\theta}\lambda) = (\mathrm{e}^{-\mathrm{i}\theta}\det(D))^{-(m-1)^n}\mathrm{Res}(\mathscr{A}z^{m-1} - \mathrm{e}^{\mathrm{i}\theta}\lambda z^{[m-1]}).$$

Then

$$\varphi(\lambda)^{(n)} = 0, \quad \varphi(\lambda)^{(n+1)} \neq 0$$

if and only if

$$\psi(\mathrm{e}^{\mathrm{i}\theta}\lambda)^{(n)} = 0, \quad \psi(\mathrm{e}^{\mathrm{i}\theta}\lambda)^{(n+1)} \neq 0.$$

Take $\theta = 2\pi j/k$, $j = 1, \ldots, k-1$. Then G_j, $j = 1, \ldots, k$, are isometry each other. $\square$

We should mention that, Corollary 5 can be deduced from [16, Theorem 2.4]. However, in view of the analysis, our idea is totally different from that of [16].

4 Algebraic multiplicity for two-dimensional nonnegative tensors

In this section, we will establish the algebraic multiplicity for order m dimension two tensors by the property of its resultant.

Suppose that $\mathscr{A}$ is a tensor, and

$$x = (x_1, x_2), \quad y = (x_1^{m-1}, x_1^{m-2}x_2, \ldots, x_2^{m-1}), \quad B = (b_{ij})_{2 \times m},$$

which satisfy

$$By = \mathscr{A}x^{m-1}.$$

Define

$$C = (c_{ij})_{(2m-2) \times (2m-2)},$$

where

$$c_{ij} = \begin{cases} b_{1k}, & 1 \leqslant k \leqslant m,\ 1 \leqslant i \leqslant m-1,\ j = k+i-1, \\ b_{2k}, & 1 \leqslant k \leqslant m,\ m \leqslant i \leqslant 2m-2,\ j = k+i-m, \\ 0, & \text{otherwise}. \end{cases}$$

Then we can get that the resultant of $\mathscr{A}$ is $\det(C - \lambda E)$. Thus, the algebraic multiplicity of $\mathscr{A}$ can be obtained by the property of C.

Lemma 4 *Assume that $\mathscr{A}$ is an order m dimension two nonnegative irreducible tensor and $b_{1j_{i_1}} \neq 0$, $1 \leqslant i_1 \leqslant r_1$, $1 \leqslant j_{i_1} < m$ (resp. $b_{2k_{i_2}} \neq 0$, $1 \leqslant i_2 \leqslant r_2$, $1 < k_{i_2} \leqslant m$). Let (a,b) be the greatest common divisor of a and b, $q = (m-1, j_1 - 1, \ldots, j_{r_1} - 1)$ (resp. $q = (m-1, k_1 - 1, \ldots, k_{r_2} - 1)$). Then C can be divided into q same irreducible blocks.*

Proof Since $\mathscr{A}$ is an order m dimension two nonnegative irreducible tensor, C is a nonnegative matrix and $b_{1m} \neq 0$, $b_{21} \neq 0$.

Case 1 $r = 2r_1$, where $r = 2m - 2$, $r_1 = m - 1$.

For $b_{21} \neq 0$,

$$\mathrm{patha}_{11} : 1 \to r_1 + 1 \to 1,$$
$$\mathrm{patha}_{12} : 2 \to r_1 + 2 \to 2,$$
$$\cdots,$$
$$\mathrm{patha}_{1r_1} : r_1 \to 2r_1 \to r_1.$$

Case 2 For $b_{1j_1} \neq 0$,

(1) $r_1 = k_1 r_2 + r_3$, where $r_2 = j_1 - 1$,

$$\mathrm{patha}_{21} : 1 \to r_2 + 1 \to \cdots \to k_1 r_2 + 1$$
$$\to r_1 + r_2 - r_3 + 1 \overrightarrow{\mathrm{patha}_{1r_1 - r_3 + 1}} r_2 - r_3 + 1,$$
$$\cdots,$$
$$\mathrm{patha}_{2r_3} : r_3 \to r_2 + r_3 \to \cdots \to k_1 r_2 + r_3 \to r_1 + r_2 \overrightarrow{\mathrm{patha}_{1r_2}} r_2,$$
$$\mathrm{patha}_{2r_3 + 1} : r_3 + 1 \to r_2 + r_3 + 1 \to \cdots \to k_1 r_2 + r_3 + 1 \overrightarrow{\mathrm{patha}_{11}} 1,$$
$$\cdots,$$
$$\mathrm{patha}_{2r_2} : r_2 \to 2r_2 \to \cdots \to k_1 r_2 + r_2 \overrightarrow{\mathrm{patha}_{1r_2 - r_3 + 1}} r_2 - r_3;$$

(2) $r_2 = k_2 r_3 + r_4$,

$$\mathrm{patha}_{31} : 1 \overrightarrow{\mathrm{patha}_{21}} r_2 - r_3 + 1 \overrightarrow{\mathrm{patha}_{r_2 - r_3 + 1}} \cdots \overrightarrow{\mathrm{patha}_{2r_2 - k_2 r_3 + 1}} r_4 + 1,$$
$$\cdots,$$
$$\mathrm{patha}_{3r_3 - r_4} : r_3 - r_4 \overrightarrow{\mathrm{patha}_{2r_3 - r_4}} r_2 - r_4 \overrightarrow{\mathrm{patha}_{r_2 - r_4}} \cdots \overrightarrow{\mathrm{patha}_{2r_2 - (k_2 - 2)r_3 - r_4}} r_3,$$
$$\mathrm{patha}_{3r_3 - r_4 + 1} : r_3 - r_4 + 1 \overrightarrow{\mathrm{patha}_{2r_3 - r_4 + 1}} r_2 - r_4$$
$$+ 1 \overrightarrow{\mathrm{patha}_{r_2 - r_4 + 1}} \cdots \overrightarrow{\mathrm{patha}_{2r_3 + 1)}} 1,$$
$$\cdots,$$
$$\mathrm{patha}_{3r_3} : r_3 \overrightarrow{\mathrm{patha}_{2r_3}} r_2 \overrightarrow{\mathrm{patha}_{2r_2}} r_2 - r_3 \overrightarrow{\mathrm{patha}_{2r_2 - r_3}} \cdots \overrightarrow{\mathrm{patha}_{2r_2 - (k_2 - 1)r_3}} r_4.$$

Continuing the process:

(p) $r_p = k_p r_{p+1}$,

$$\text{patha}_{p+11}: 1\overrightarrow{\text{patha}_{p1}}r_p - r_{p+1} + 1 \cdots \overrightarrow{\text{patha}_{pr_p - k_p r_{p+1} + 1}}1,$$

$$\cdots,$$

$$\text{patha}_{p+1r_{p+1}}: r_{p+1}\overrightarrow{\text{patha}_{2r_{p+1}}}r_p \cdots \overrightarrow{\text{patha}_{2r_p - (k_p - 1)r_{p+1}}}r_{p+1},$$

or

$$\text{patha}_{p+11}: 1\overrightarrow{\text{patha}_{p1}}r_{p+1} + 1 \cdots \overrightarrow{\text{patha}_{p(k_p-1)r_{p+1}+1}}r_p + 1\overrightarrow{\text{patha}_{pr_p+1}}1,$$

$$\cdots,$$

$$\text{patha}_{p+1r_{p+1}}: r_{p+1}\overrightarrow{\text{patha}_{pr_{p+1}}}r_p\overrightarrow{\text{patha}_{pr_p}}r_p - r_{p+1} \cdots \overrightarrow{\text{patha}_{pr_p(k_p-2)r_{p+1}}}r_{p+1}.$$

In (1),

$$\text{patha}_{2k(m-1,j_1-1)+i}, \quad 1 \leqslant i \leqslant (m-1, j_1-1), \ k(m-1, j_1-1) = j_1 - 1,$$

have the same number and

$$c_{i_1,j} = c_{i_2,j}, \quad 1 \leqslant i_1, i_2 \leqslant (m-1, j_1-1),$$

where $c_{i,j}$ is the j-th line segment of the $\text{patha}_{2k(m-1,j_1-1)+i}$. The same result can be got for $0 \leqslant k \leqslant r_2/(m-1, j_1-1)$. So let $r_{p+1} = (m-1, j_1-1)$, $b_{1j_1} \neq 0$, $b_{21} \neq 0$ make C have $r_{p+1} = (m-1, j_1-1)$ same irreducible blocks.

Case 3 For $b_{1j_2} \neq 0$, let $r_k = (m-1, j_2-1)$, $b_{1j_2} \neq 0$, $b_{21} \neq 0$ in order to C has $r_k = (m-1, j_2-1)$ same irreducible blocks.

In Case 2,

$$\text{patha}_1: 1 \to \cdots \to r_{p+1} + 1 \to \cdots \to 1,$$

$$\text{patha}_{r_{p+1}}: r_{p+1} \to \cdots \to 2r_{p+1} \to \cdots \to r_{p+1}.$$

In Case 3,

$$\text{patha}_1: 1 \to \cdots \to r_k + 1 \to \cdots \to 1,$$

$$\text{patha}_{r_k}: r_k \to \cdots \to 2r_k \to \cdots \to r_k.$$

Then let $r = (r_{p+1}, r_k) = (m-1, j_1-1, j_2-1)$, $b_{1j_1} \neq 0$, $b_{1j_2} \neq 0$, $b_{21} \neq 0$ make C have r same irreducible blocks.

Repeating the same procedure, let $q = (m-1, j_1-1, \ldots, j_{r_1} - 1)$. Then C can be divided into q same irreducible blocks. $\square$

Lemma 5 *Assume that $\mathscr{A}$ is an order m dimension two nonnegative irreducible tensor and $b_{1j_{i_1}} \neq 0, 1 \leqslant i_1 \leqslant r_1, 1 \leqslant j_{i_1} < m$, $b_{2k_{i_2}} \neq 0$, $1 \leqslant i_2 \leqslant r_2, 1 < k_{i_2} \leqslant m$. Let (a,b) be the greatest common divisor of a and b, $q = (m-1, j_1-1, \ldots, j_{r_1} - 1, k_1 - 1, \ldots, k_{r_2} - 1)$. Then C can be divided into q same irreducible blocks.*

Proof The conclusion can be reached by the same proof of Lemma 4. $\square$

Theorem 13 *Assume that $\mathscr{A}$ is an order m dimension two nonnegative irreducible tensor and $b_{1j_{i_1}} \neq 0, 1 \leqslant i_1 \leqslant r_1 < m, b_{2k_{i_2}} \neq 0, 2 \leqslant i_2 \leqslant r_2$. Let (a, b) be the greatest common divisor of a and b, $q = (m-1, j_1-1, \ldots, j_{r_1} - 1, k_1 - 1, \ldots, k_{r_2} - 1)$. Then its characteristic polynomial $\phi(t) := \det(tI - \mathscr{A})$ is of the form*

$$\phi(t) = \{t^m[t^k - \rho^k(\mathscr{A})][t^k - \delta_1\rho^k(\mathscr{A})]\cdots[t^k - \delta_r\rho^k(\mathscr{A})]\}^q,$$

where $|\delta| < 1$ for $1 < i \leqslant r$ if $r > 1$, and $m + rk = (2m - 2)/q$.

Proof C can be divided into q same irreducible blocks by Lemma 5. And every block have order $(2m - 2)/q$. So every block have characteristic polynomial

$$\phi(t) = t^m[t^k - \rho^k(\mathscr{A})][t^k - \delta_1\rho^k(\mathscr{A})]\cdots[t^k - \delta_r\rho^k(\mathscr{A})],$$

where $|\delta_i| < 1$ for $1 < i \leqslant r$ if $r > 1$, and $m + rk = n$. This can be deduced from [17, Corollary 2.12]. Thus, $\mathscr{A}$ have characteristic polynomial $\phi(t) := \det(tI - \mathscr{A})$ with the desired form. □

Theorem 14 *Assume that $\mathscr{A}$ is an order m dimension two nonnegative irreducible tensor and $b_{1j_{i_1}} \neq 0, 1 \leqslant i_1 \leqslant r_1, b_{2k_{i_2}} \neq 0, 1 \leqslant i_2 \leqslant r_2$. Let (a, b) be the greatest common divisor of a and b, $q = (j_1 - 1, \ldots, j_{r_1} - 1, m - k_1, k_2 - k_1, \ldots, k_{r_2} - k_1)$. Then its characteristic polynomial $\phi(t) := \det(tI - \mathscr{A})$ is of the form*

$$\phi(t) = t^{m-1-j_{r_1}+k_1}\{t^m[t^k - \rho^k(A)][t^k - \delta_1\rho^k(A)]\cdots[t^k - \delta_r\rho^k(A)]\}^q,$$

where $|\delta| < 1$ for $1 < i \leqslant r$ if $r > 1$, and $m + rk = (m - 1 + j_{r_1} - k_1)/q$.

Proof   The conclusion can be reached by the same proof for Theorem 13. □

Theorem 15 *If $\mathscr{A}$ is an order m dimension two nonnegative irreducible tensor for every eigenvalue with modulus $\rho(\mathscr{A})$, then its complex geometric multiplicity $=$ algebraic multiplicity.*

Proof   Since $\mathscr{A}$ is a nonnegative irreducible tensor with dimension two, we can suppose $b_{1j_{i_1}} \neq 0, 1 \leqslant i_1 \leqslant r_1, b_{2k_{i_2}} \neq 0, 1 \leqslant i_2 \leqslant r_2, q = (j_1 - 1, \ldots, j_{r_1} - 1, m - k_1, \ldots, m - k_{r_2})$. Assume that G is the set of eigenvectors corresponding to $\rho(\mathscr{A})$. Let $x = (x_1, x_2) \in A$. Then

$$\mathscr{A}x^{m-1} = \rho(\mathscr{A})x^{[m-1]}.$$

Thus,

$$\sum_{j_2,\ldots,j_m=1}^{n} a_{ij_2\cdots j_m}x_{j_2}\cdots x_{j_m} = x_i^{m-1}, \quad i = 1, 2.$$

$$\sum_{j_2,\ldots,j_m=1}^{n} a_{ij_2\cdots j_m}|x_{j_2}|\cdots|x_{j_m}| = |x_i|^{m-1}, \quad i = 1, 2.$$

$$\sum_{j_2,\ldots,j_m=1}^{n} a_{ij_2\cdots j_m}|x_{j_2}|\cdots|x_{j_m}| = \left| \sum_{j_2,\ldots,j_m=1}^{n} a_{ij_2\cdots j_m}x_{j_2}\cdots x_{j_m}\right|, \quad i=1,2.$$

Hence, we can get

$$b_{1j_{i_1}}x_1^{m-j_{i_1}}x_2^{j_{i_1}-1} = \left(\frac{x_1}{|x_1|}\right)^{m-1}|x_1|^{m-j_{i_1}}|x_2|^{j_{i_1}-1}, \quad 1\leqslant i_1\leqslant r_1;$$

$$b_{2k_{i_2}}x_1^{m-k_{i_2}}x_2^{k_{i_2}-1} = \left(\frac{x_2}{|x_2|}\right)^{m-1}|x_1|^{m-k_{i_2}}|x_2|^{k_{i_2}-1}, \quad 1\leqslant i_2\leqslant r_2.$$

So we have

$$\left(\frac{x_1}{|x_1|}\right)^{j_{i_1}-1} = \left(\frac{x_2}{|x_2|}\right)^{j_{i_1}-1}, \quad 1\leqslant i_1\leqslant r_1,$$

$$\left(\frac{x_1}{|x_1|}\right)^{m-k_{i_2}} = \left(\frac{x_2}{|x_2|}\right)^{m-k_{i_2}}, \quad 1\leqslant i_2\leqslant r_2.$$

Then

$$x^{[q]} = \mathrm{e}^{\mathrm{i}\theta}|x|^{[q]}.$$

And if

$$x^{[q]} = \mathrm{e}^{\mathrm{i}\theta}|x|^{[q]}, \quad |x|\in A,$$

then $x\in A$. Since

$$q = (j_1-1,\ldots,j_{r_1}-1,m-k_1,\ldots,m-k_{r_2})$$
$$= (j_1-1,\ldots,j_{r_1}-1,m-k_1,k_2-k_1,\ldots,k_{r_2}-k_1),$$

its complex geometric multiplicity = algebraic multiplicity. $\square$

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