

Relative locations of subwords in free operated semigroups and Motzkin words

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Abstract Bracketed words are basic structures both in mathematics (such as Rota-Baxter algebras) and mathematical physics (such as rooted trees) where the locations of the substructures are important. In this paper, we give the classification of the relative locations of two bracketed subwords of a bracketed word in an operated semigroup into the separated, nested, and intersecting cases. We achieve this by establishing a correspondence between relative locations of bracketed words and those of words by applying the concept of Motzkin words which are the algebraic forms of Motzkin paths.

Keywords Bracketed word, relative location, operated semigroup, Motzkin word, Motzkin path, rooted tree

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1 Introduction

As a basic property of sets, there are three relative locations of any two subsets of a given set: separated (disjoint), nested (including), and intersecting (overlapping). See the proof of Theorem 2.11 below for example. Similarly, there are three relative locations of two subwords in a given word, a property that is essential in rewriting systems (critical pairs) and Gröbner bases [2,8,19]. Analogous classification of relative locations of combinatorial objects, such as Feynman graphs, plays an important role in combinatorics and physics, for example, in the renormalization of quantum field theory [3,10,16,17].

The classification of relative locations can be quite subtle in some structures, especially when a non-identity operator is present, such as in differential algebras, Rota-Baxter algebras [14] and, more generally, operated algebras [5,7,15,20]. Further by [13], free operated semigroups have natural

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combinatorial presentation as rooted trees which serve as the baby models for Feynman graphs [9,17]. Considering the importance of such classification in the study of these mathematics and physics structures, especially their Gröbner-Shirshov bases (compositions and diamond lemma) [4,6], it is crucial to establish such a classification. In this paper, we give an explicit correspondence between relative locations of two bracketed subwords and those of Motzkin words and subwords. As a direct consequence, we obtain the classification of the relative locations of bracketed subwords.

We take two steps in our treatment to deal with two subtle points of bracketed words. In Section 2, we deal with the first subtle point which is already present in studying relative locations of two subwords in comparison with two subsets, namely that one subword can appear at multiple locations in a given word. The concept of a \star -word [7] allows us to give a unique label to each appearance of a subword, called a placement. We then show that each placement corresponds uniquely to a substring of the string corresponding to the given word, converting the problem of studying subwords to that of subsets which can be solved easily as mentioned above. The second subtle point arise when we deal with bracketed words in Section 3 since the action of the bracket together with the product of the word gives the bracketed words a quite complicated structure. To resolve this difficulty, we make use of a bijection introduced in [13] between bracketed words and a class of words called Motzkin words on a larger set. We then show in Section 4 that this bijection preserves the relative locations of the (bracketed) word, reducing the study of relative locations of bracketed words to the one of words for which we can apply Section 2.

2 Relative locations of subwords

In this section, we consider the relative locations of two subwords of a fixed word. This serves as both the prototype and preparation for our study of the relative locations of two bracketed subwords in later sections.

2.1 Subwords

Definition 2.1 Let Z be a set. Let $M(Z)$ be the free monoid on Z consisting of words in the alphabet set Z . Thus, a word is either the *empty word* 1 or of the form

$$w = z_1 \cdots z_n, \quad z_i \in Z, \quad 1 \leq i \leq n.$$

A *subword* of w is defined to be a word $u \neq 1$ that is a part of w . Let $S(Z) := M(Z) \setminus \{1\}$ be the free semigroup on Z .

We emphasize that the empty word 1 is not taken to be a subword in this paper.

Note that a subword u of $w \in M(Z)$ may appear in w at multiple locations. For example, for $z \in Z$, $u := z$ appears in $w := zzz$ at three different locations.

It is often important to be precise about the location of a subword. This is the case, for example, in the study of rewriting systems and Gröbner-Shirshov bases [2,7,15,20]. For this purpose, we need additional information for the location of the subword u . Such information is provided by the concept of \star -words [7].

Definition 2.2 Let Z be a set.

(a) Let \star be an element not in Z . A \star -word on Z is a word in $M(Z \cup \{\star\})$ in which \star appears exactly once. The set of \star -words on Z is denoted by $M^\star(Z)$.

(b) More generally, for \star_1, \dots, \star_k not in Z , denote $\vec{\star} = (\star_1, \dots, \star_k)$. A $\vec{\star}$ -word on Z is a word in $M(Z \cup \{\star_1, \dots, \star_k\})$ in which \star_i appears exactly once for each $i = 1, \dots, k$. The set of $\vec{\star}$ -words is denoted by $M^{\vec{\star}}(Z)$. When $k = 2$, it is denoted by $M^{\star_1, \star_2}(Z)$.

(c) For $p \in M^{\vec{\star}}(Z)$ and $u_1, \dots, u_k \in M(Z)$, let

$$p|_{u_1, \dots, u_k} = p|_{\star_1 \mapsto u_1, \dots, \star_k \mapsto u_k}$$

denote the word in $M(Z)$ when the \star_i in p is replaced by u_i , where $i = 1, \dots, k$.

Now, we can be more precise on a particular appearance of a subword.

Definition 2.3 Let $u, w \in M(Z)$ be words on Z . A *placement* of u in w (by p) is a pair (u, p) , where p is in $M^\star(Z)$ such that $p|_u = w$.

Thus, $(u_1, p_1) = (u_2, p_2)$ means $u_1 = u_2$ and $p_1 = p_2$.

Of course, u is a subword of w if there is $p \in M^\star(Z)$ such that (u, p) is a placement of u in w . However, the usefulness of the placement notion is its role in distinguishing different appearances of u in w . For example, the three appearances of $u = z$ in $w = zzz$ are identified by the three placements (u, p_1) , (u, p_2) , and (u, p_3) , where

$$p_1 = \star zz, \quad p_2 = z \star z, \quad p_3 = zz \star.$$

The concept of a placement is also essential in determining the relative locations of two subwords of a given word.

Example 2.4 Let $Z = \{x, y\}$, and let $w = xyxyxy$. Then $u := xyx$ appears at two locations in w and $v := xy$ appears at three locations, as shown in the following equation:

$$w = \underbrace{xyx}_{(u,p)} xy = \overbrace{xy}^{(v,q_1)} \overbrace{xy}^{(v,q_2)} \overbrace{xy}^{(v,q_3)}.$$

Take (u, p) to be the first appearance (from the left) of u in w . Thus, $p = \star xyx$. Then the three placements of $v = xy$ in w , given by (v, q_i) , $i = 1, 2, 3$, with

$$q_1 = \star xyxy, \quad q_2 = xy \star xy, \quad q_3 = xyxy \star.$$

These three placements of v are in three different kinds of relative locations with respect to u : the left v (in (v, q_1)) is a subword of u , the middle v (in

(v, q_2)) is not a subword of u but has a nonempty intersection with u , and the right v (in (v, q_3)) is disjoint with u .

This situation can again be made precise by \star -words.

Definition 2.5 Let w be a word in $M(Z)$. Two placements (u_1, p_1) and (u_2, p_2) are called

(a) *separated* if there exists an element p in $M^{\star 1, \star 2}(Z)$ such that

$$w = p|_{u_1, u_2}, \quad p_1|_{\star 1} = p|_{\star 1, u_2}, \quad p_2|_{\star 2} = p|_{u_1, \star 2};$$

(b) *nested* if there exists an element p in $M^\star(Z)$ such that $p_1 = p_2|_p$ or $p_2 = p_1|_p$;

(c) *intersecting* if there exist an element p in $M^\star(Z)$ and elements a, b , and c in $S(Z)$ such that

(i) either $w = p|_{abc}$, $p_1 = p|_{\star c}$, $p_2 = p|_{a\star}$;

(ii) or $w = p|_{abc}$, $p_1 = p|_{a\star}$, $p_2 = p|_{\star c}$.

Example 2.6 With the w, u and the three appearances of v in Example 2.4, we have the corresponding placements (u, p) with $p = \star yxy$, and (v, q_1) , (v, q_2) , and (v, q_3) . Then (u, p) and (v, q_1) (resp. (v, q_2) , (v, q_3)) are nested (resp. intersecting, separated).

2.2 Substrings

We now give another description of a placement of a subword that makes it easier to classify the relative locations of two subwords. Denote

$$[n] := \{1, \dots, n\}, \quad [i, k] := \{i, \dots, k\}, \quad n, i \geq 1, \quad k \geq i.$$

Let Z be a set. Let

$$w = z_1 \cdots z_n, \quad z_i \in Z, \quad 1 \leq i \leq n,$$

be a word in $M(Z)$. Let $\mathbf{p} := (u, p)$ be a placement of u in w . Then p is of the form

$$p = x_1 \cdots x_j \star x_{j+1} \cdots x_m, \quad x_i \in Z, \quad 1 \leq i \leq m,$$

and

$$u = y_1 \cdots y_t, \quad y_j \in Z, \quad 1 \leq j \leq t.$$

Comparing

$$x_1 \cdots x_j y_1 \cdots y_t x_{j+1} \cdots x_m = p|_u = w = z_1 \cdots z_n$$

in the free monoid $M(Z)$, we obtain

$$p = z_1 \cdots z_{j-1} \star z_{k+1} \cdots z_n$$

for unique $j = j_{\mathbf{p}}$ and $k = k_{\mathbf{p}}$. Thus,

$$u = z_j \cdots z_k.$$

Definition 2.7 With notations as above, the set $I_{\mathbf{p}} := [j_{\mathbf{p}}, k_{\mathbf{p}}]$ is called the *location* of the placement $\mathbf{p} = (u, p)$ in w . Denote

$$\mathcal{LO}(w) := \{[j, k] \mid 1 \leq j \leq k \leq n\}. \tag{1}$$

Also denote

$$\mathcal{PL}(w) := \{(u, p) \mid (u, p) \text{ is a placement in } w\}. \tag{2}$$

Proposition 2.8 Let $1 \neq w \in M(Z)$. The map

$$\begin{aligned} \eta: \mathcal{PL}(w) &\rightarrow \mathcal{LO}(w), \\ \mathbf{p} = (u, p) &\mapsto [j_{\mathbf{p}}, k_{\mathbf{p}}], \end{aligned}$$

is a bijection.

Proof Let $w = z_1 \cdots z_n$. Given a placement (u, p) in w . Then

$$p = z_1 \cdots z_{j-1} \star z_{k+1} \cdots z_n \in M^*(Z), \quad u = z_j \cdots z_k, \quad 1 \leq j \leq k \leq n.$$

Thus, if $\mathbf{p}_1 = (u_1, p_1)$ and $\mathbf{p}_2 = (u_2, p_2)$ are in $\mathcal{PL}(w)$ and $\mathbf{p}_1 \neq \mathbf{p}_2$, then we have $p_1 \neq p_2$. Hence, either $j_{\mathbf{p}_1} \neq j_{\mathbf{p}_2}$ or $k_{\mathbf{p}_1} \neq k_{\mathbf{p}_2}$. Then

$$\eta(u_1, p_1) \neq \eta(u_2, p_2).$$

Hence, η is injective.

Furthermore, for any $[j, k] \in \mathcal{LO}(w)$, define

$$u = z_j \cdots z_k, \quad p = z_1 \cdots z_{j-1} \star z_{k+1} \cdots z_n.$$

Then we have $p|_u = w$ and $\eta(u, p) = [j, k]$. Hence, η is surjective. □

Definition 2.9 Two nonempty subsets I and J of $[n]$ are called

- (a) *separated*, if $I \cap J = \emptyset$;
- (b) *nested*, if $I \subseteq J$ or $J \subseteq I$;
- (c) *intersecting*, if $I \cap J \neq \emptyset$, $I \not\subseteq J$ and $J \not\subseteq I$.

Consider the w , u , and v in Example 2.4 and their corresponding placements (u, p) , (v, q_1) , (v, q_2) , and (v, q_3) in Example 2.6. The locations of the placements are

$$\eta(u, p) = [1, 3], \quad \eta(v, q_1) = [1, 2], \quad \eta(v, q_2) = [3, 4], \quad \eta(v, q_3) = [5, 6].$$

Then we see that, as subsets of $[6]$, $[1, 3]$ and $[1, 2]$ (resp. $[3, 4]$, $[5, 6]$) are also nested (resp. intersecting, separated). The next theorem shows that this equivalence holds in general.

Theorem 2.10 Let Z be a set. Let $1 \neq w = z_1 \cdots z_n$ be in $M(Z)$, where $z_i \in Z$, $1 \leq i \leq n$. Then placements $\mathbf{p}_1 = (u_1, p_1)$ and $\mathbf{p}_2 = (u_2, p_2)$ in w are

separated (resp. nested, intersecting) if and only if the corresponding subsets I_{p_1} and I_{p_2} of $[n]$ are separated (resp. nested, intersecting).

Proof First, suppose that the placements (u_1, p_1) and (u_2, p_2) in w are separated. Then there exists an element p in $M^{\star_1, \star_2}(Z)$ such that

$$w = p|_{u_1, u_2}, \quad p_1|_{\star_1} = p|_{\star_1, u_2}, \quad p_2|_{\star_2} = p|_{u_1, \star_2}.$$

From $p|_{u_1, u_2} = z_1 \cdots z_n$, there are $1 \leq j_1 \leq k_1 < j_2 \leq k_2 \leq n$ such that

$$p = z_1 \cdots z_{j_1-1} \star_1 z_{k_1+1} \cdots z_{j_2-1} \star_2 z_{k_2+1} \cdots z_n,$$

or

$$p = z_1 \cdots z_{j_1-1} \star_2 z_{k_1+1} \cdots z_{j_2-1} \star_1 z_{k_2+1} \cdots z_n.$$

Thus, we have

$$\{I_{p_1}, I_{p_2}\} = \{[j_1, k_1], [j_2, k_2]\},$$

and hence,

$$I_{p_1} \cap I_{p_2} = \emptyset.$$

Conversely, suppose $I_{p_1} \cap I_{p_2} = \emptyset$. Let

$$I_{p_1} = [j_1, k_1], \quad I_{p_2} = [j_2, k_2].$$

Then we have $k_1 < j_2$ or $k_2 < j_1$. Without loss of generality, assume $k_1 < j_2$. Then take

$$u_1 = z_{j_1} \cdots z_{k_1}, \quad u_2 = z_{j_2} \cdots z_{k_2}.$$

Define

$$p = z_1 \cdots z_{j_1-1} \star_1 z_{k_1+1} \cdots z_{j_2-1} \star_2 z_{k_2+1} \cdots z_n.$$

Then we have

$$w = p|_{u_1, u_2}, \quad p_1|_{\star_1} = p|_{\star_1, u_2}, \quad p_2|_{\star_2} = p|_{u_1, \star_2},$$

as needed.

Next, suppose that (u_1, p_1) and (u_2, p_2) are nested. Then there exists an element p in $M^\star(Z)$ such that $p_1 = p_2|_p$ or $p_2 = p_1|_p$. Without loss of generality, we can assume $p_1 = p_2|_p$. From $w = p_1|_{u_1}$ and $w = p_2|_{u_2}$, we obtain

$$z_1 \cdots z_{j_1-1} \star z_{k_1+1} \cdots z_n = p_1 = p_2|_p = z_1 \cdots z_{j_2-1} \star|_p z_{k_2+1} \cdots z_n.$$

Thus, $j_2 \leq j_1 \leq k_1 \leq k_2$, and hence, $I_{p_1} \subseteq I_{p_2}$. Conversely, suppose that I_{p_1} and I_{p_2} are nested. We may assume that

$$I_{p_1} = [j_1, k_1] \subseteq I_{p_2} = [j_2, k_2].$$

Then define

$$p = z_{j_2} \cdots z_{j_1-1} \star z_{k_1+1} \cdots z_{k_2},$$

with the convention that

$$p = \begin{cases} \star z_{k_1+1} \cdots z_{k_2}, & j_1 = j_2, \\ z_{j_2} \cdots z_{j_1-1} \star, & k_1 = k_2. \end{cases}$$

We have $p_2|_p = p_1$. Hence, (u_1, p_1) and (u_2, p_2) are nested.

Finally, suppose that (u_1, p_1) and (u_2, p_2) are intersecting. Then there exist p in $M^\star(Z)$ and a, b, c in $S(Z)$ such that

$$w = p|_{abc}, \quad p_1 = p|_{\star c}, \quad p_2 = p|_{a\star},$$

or

$$w = p|_{abc}, \quad p_1 = p|_{a\star}, \quad p_2 = p|_{\star c}.$$

We just need to consider the first case since the proof of the second case is similar. Denote

$$p = z_1 \cdots z_{j-1} \star z_{k+1} \cdots z_n.$$

From

$$z_1 \cdots z_{j-1} \star z_{k_1+1} \cdots z_n = p_1 = p|_{\star c} = z_1 \cdots z_{j-1} \star c z_{k+1} \cdots z_n,$$

we obtain

$$j_1 = j \quad c = z_{k_1+1} \cdots z_k.$$

Since $c \neq 1$, we have $k_1 < k$. Similarly, from

$$z_1 \cdots z_{j_2-1} \star z_{k_2+1} \cdots z_n = p_2 = p|_{a\star} = z_1 \cdots z_{j-1} a \star z_{k+1} \cdots z_n,$$

we obtain

$$k_2 = k, \quad a = z_j \cdots z_{j_2-1}.$$

Since $a \neq 1$, we have $j < j_2$. Consequently,

$$w = p|_{abc} = z_1 \cdots z_{j-1} abc z_{k+1} \cdots z_n = z_1 \cdots z_{j_2-1} b z_{k_1+1} \cdots z_n.$$

Since $b \neq 1$, we have $j_2 \leq k_1$. Then

$$j_1 = j < j_2 \leq k_1 < k = k_2.$$

Thus,

$$I_{p_1} \cap I_{p_2} = [j_2, k_1] \neq \emptyset, \quad I_{p_1} \setminus I_{p_2} = [j_1, j_2 - 1] \neq \emptyset, \quad I_{p_2} \setminus I_{p_1} = [k_1 + 1, k_2] \neq \emptyset.$$

Hence,

$$I_{p_1} \not\subseteq I_{p_2}, \quad I_{p_2} \not\subseteq I_{p_1}.$$

Thus, I_{p_1} and I_{p_2} are intersecting.

Conversely, suppose that

$$I_{p_1} \cap I_{p_2} \neq \emptyset, \quad I_{p_1} \not\subseteq I_{p_2}, \quad I_{p_2} \not\subseteq I_{p_1}.$$

From $I_{\mathbf{p}_1} \cap I_{\mathbf{p}_2} \neq \emptyset$, we have $j_1 \leq k_2$ and $j_2 \leq k_1$. Then from $I_{\mathbf{p}_1} \not\subseteq I_{\mathbf{p}_2}$ and $I_{\mathbf{p}_2} \not\subseteq I_{\mathbf{p}_1}$, we have $k_1 > k_2$ and $j_1 > j_2$, or $k_1 < k_2$ and $j_1 < j_2$. We just consider the first case with the second case being similar. Then we have $j_2 < j_1 \leq k_2 < k_1$. Define

$$p = z_1 \cdots z_{j_2-1} \star z_{k_1+1} \cdots z_n,$$

$$a = z_{j_2} \cdots z_{j_1-1}, \quad b = z_{j_1} \cdots z_{k_2}, \quad c = z_{k_2+1} \cdots z_{k_1}.$$

Then we have

$$a, b, c \neq 1, \quad w = p|_{abc}, \quad p_2 = p|_{\star c}, \quad p_1 = p|_{a\star}.$$

This shows that (u_1, p_1) and (u_2, p_2) are intersecting. □

The following result is the classification of the relative locations of two placements in the free monoid $M(Z)$. Its generalization to operated monoids will be treated in the subsequent sections.

Theorem 2.11 *Let $w \neq 1$ be a word in $M(Z)$. For any two placements $\mathbf{p}_1 = (u_1, p_1)$ and $\mathbf{p}_2 = (u_2, p_2)$ in w , exactly one of the following is true:*

- (a) (u_1, p_1) and (u_2, p_2) are separated;
- (b) (u_1, p_1) and (u_2, p_2) are nested;
- (c) (u_1, p_1) and (u_2, p_2) are intersecting.

Proof Let

$$w = z_1 \cdots z_n, \quad z_1, \dots, z_n \in Z, \quad n \geq 1.$$

By Theorem 2.10, we only need to prove that the same conclusion holds for $I_{\mathbf{p}_1}$ and $I_{\mathbf{p}_2}$. But this follows from the simple fact that, for the two subsets $I_{\mathbf{p}_1}$ and $I_{\mathbf{p}_2}$ of $[n]$, exactly one of the following is true:

- (a) $I_{\mathbf{p}_1}$ and $I_{\mathbf{p}_2}$ have empty intersection ($\iff I_{\mathbf{p}_1}$ and $I_{\mathbf{p}_2}$ are separated);
- (b) $I_{\mathbf{p}_1}$ and $I_{\mathbf{p}_2}$ have nonempty intersection, and $I_{\mathbf{p}_1} \subseteq I_{\mathbf{p}_2}$ or $I_{\mathbf{p}_2} \subseteq I_{\mathbf{p}_1}$ ($\iff I_{\mathbf{p}_1}$ and $I_{\mathbf{p}_2}$ are nested); or
- (c) $I_{\mathbf{p}_1}$ and $I_{\mathbf{p}_2}$ have nonempty intersection, and $I_{\mathbf{p}_1} \not\subseteq I_{\mathbf{p}_2}$ and $I_{\mathbf{p}_2} \not\subseteq I_{\mathbf{p}_1}$ ($\iff I_{\mathbf{p}_1}$ and $I_{\mathbf{p}_2}$ are intersecting).

3 Bracketed words and Motzkin words

We recall the concepts of bracketed words and Motzkin words [1,13,15,18] before generalizing Theorem 2.11 to this context.

3.1 Bracketed words

We first recall the concept and construction of free operated monoids, following [13,15].

Definition 3.1 An *operated monoid* is a monoid U together with a map $P: U \rightarrow U$. A *morphism* from an operated monoid U with a map $P: U \rightarrow U$

to an operated monoid V with a map $Q: V \rightarrow V$ is a monoid homomorphism $f: U \rightarrow V$ such that $f \circ P = Q \circ f$.

Definition 3.2 A *free operated monoid* on a set Y is an operated monoid (U_Y, P_Y) together with a map $j_Y: Y \rightarrow U_Y$ with the property that, for any operated monoid (V, Q) and any map $f: Y \rightarrow V$, there is a unique morphism $\bar{f}: (U_Y, P_Y) \rightarrow (V, Q)$ of operated monoids such that $f = \bar{f} \circ j_Y$.

For any set Y , let $[Y] := \{[y] \mid y \in Y\}$ denote a set indexed by, but disjoint from Y . Let X be a given set. We will construct a direct system $\{\mathfrak{M}_n := \mathfrak{M}_n(X)\}_{n \geq 0}$ of monoids with natural embeddings $\tilde{i}_{n-1}: \mathfrak{M}_{n-1} \hookrightarrow \mathfrak{M}_n$ for $n \geq 1$ by induction on n . The free operated monoid $\mathfrak{M}(X)$ on X is the direct limit of the system, after we equip it with a natural operator.

First, let $\mathfrak{M}_0 := M(X)$. Then define the monoid

$$\mathfrak{M}_1 := M(X \cup [\mathfrak{M}_0]) = M(X \cup [M(X)])$$

and denote the natural embedding of monoids induced by the inclusion $X \hookrightarrow X \cup [\mathfrak{M}_0]$ by

$$\tilde{i}_0: \mathfrak{M}_0 = M(X) \hookrightarrow M(X \cup [\mathfrak{M}_0]) = \mathfrak{M}_1.$$

Inductively assuming that \mathfrak{M}_n and $\tilde{i}_{n-1}: \mathfrak{M}_{n-1} \hookrightarrow \mathfrak{M}_n$ have been defined for $n \geq 1$, we define the monoid

$$\mathfrak{M}_{n+1} := M(X \cup [\mathfrak{M}_n]). \tag{3}$$

From the embedding of monoids $\tilde{i}_{n-1}: \mathfrak{M}_{n-1} \hookrightarrow \mathfrak{M}_n$, we have an embedding $[\mathfrak{M}_{n-1}] \hookrightarrow [\mathfrak{M}_n]$. By the freeness of $\mathfrak{M}_n = M(X \cup [\mathfrak{M}_{n-1}])$, we obtain a natural embedding of monoids:

$$\tilde{i}_n: \mathfrak{M}_n = M(X \cup [\mathfrak{M}_{n-1}]) \hookrightarrow M(X \cup [\mathfrak{M}_n]) = \mathfrak{M}_{n+1}.$$

This completes our inductive definition of the direct system. Let

$$\mathfrak{M}(X) := \bigcup_{n \geq 0} \mathfrak{M}_n = \varinjlim \mathfrak{M}_n$$

be the direct limit of the system. We note that $\mathfrak{M}(X)$ is a monoid. By taking direct limit on both sides of $\mathfrak{M}_n = M(X \cup [\mathfrak{M}_{n-1}])$, we obtain

$$\mathfrak{M}(X) = M(X \cup [\mathfrak{M}(X)]), \tag{4}$$

whose elements are called *bracketed words*.

The *depth* of $f \in \mathfrak{M}(X)$ is defined to be

$$\text{depth}(f) := \min \{n \mid f \in \mathfrak{M}_n\}. \tag{5}$$

The following result shows that $\mathfrak{M}(X)$ is the equivalence of free monoids in the category of operated monoids.

Lemma 3.3 [13] *Define the map $[\]: \mathfrak{M}(X) \rightarrow \mathfrak{M}(X)$ by taking $w \in \mathfrak{M}(X)$ to $[w]$. Let $j_X: X \rightarrow \mathfrak{M}(X)$ be the natural embedding. Then the triple $(\mathfrak{M}(X), [\], j_X)$ is the free operated monoid on X .*

By [13, Theorem 4.2], another representation of free operated semigroups is given by rooted trees. See [9,17] for the application of rooted trees in quantum field theory.

3.2 Motzkin words

We now recall the definition of Motzkin words [1,12,18], which acquired its name since it encodes Motzkin paths [11]. Motzkin words give another construction of free operated monoids [13] and in this paper serve as the bridge between bracketed words and the usual words.

Let X be a set. Let \llbracket and \rrbracket be symbols not in X .

Definition 3.4 An element of the free monoid $M(X \cup \{\llbracket, \rrbracket\})$ is called a *Motzkin word* on X if it has the properties that

- (a) the number of \llbracket in the word equals the number of \rrbracket in the word;
- (b) counting from the left, the number of occurrence of \llbracket is always greater or equal to the number of occurrence of \rrbracket .

The set of all Motzkin words is denoted by $\mathscr{W}(X)$.

Example 3.5 Let x, y, z be elements in X .

- (a) The word $\llbracket \llbracket x \rrbracket \llbracket y \rrbracket \rrbracket z$ is a Motzkin word.
- (b) The word $\llbracket x \rrbracket \llbracket yz$ is not a Motzkin word since it does not satisfy the first property.
- (c) The word $x \llbracket \llbracket y \rrbracket \llbracket z \rrbracket$ is not a Motzkin word since it does not satisfy the second property.

Define

$$P: \mathscr{W}(X) \rightarrow \mathscr{W}(X), \\ m \mapsto \llbracket m \rrbracket.$$

Then $\mathscr{W}(X)$ is an operated monoid.

Theorem 3.6 [13, Theorem 3.4] *Let Y be a set. Let*

$$\phi := \phi_Y: \mathfrak{M}(Y) \rightarrow \mathscr{W}(Y), \quad \phi(u) = u, \quad u \in Y,$$

be the homomorphism of operated monoid defined by the universal property of the free operated monoid $\mathfrak{M}(Y)$. Then ϕ is an isomorphism.

3.3 $\vec{\star}$ -bracketed words and $\vec{\star}$ -Motzkin words

Let X be a set. For $k \geq 1$, let \star_1, \dots, \star_k be distinct symbols not in X , and let

$$\vec{\star} = (\star_1, \dots, \star_k).$$

Denote

$$X^{\vec{\star}} = X \cup \{\star_1, \dots, \star_k\}.$$

Definition 3.7 A $\vec{\star}$ -bracketed word on X is defined to be a bracketed word in the operated monoid $\mathfrak{M}(X^{\vec{\star}})$ with exactly one occurrence of \star_i for each $i = 1, \dots, k$. The set of all $\vec{\star}$ -bracketed words on X is denoted by $\mathfrak{M}^{\vec{\star}}(X)$. When $k = 1$ and 2 , we denote $\mathfrak{M}^{\vec{\star}}(X)$ by $\mathfrak{M}^{\star}(X)$ and $\mathfrak{M}^{\star_1, \star_2}(X)$, respectively.

Definition 3.8 For $q \in \mathfrak{M}^{\vec{\star}}(X)$ and $s_1, \dots, s_k \in \mathfrak{M}(X)$, we define

$$q|_{\vec{s}} := q|_{s_1, \dots, s_k} \quad (6)$$

to be the bracketed word in $\mathfrak{M}(X)$ obtained by replacing the letter \star_i in q by s_i for $1 \leq i \leq k$. An (s_1, \dots, s_k) -bracketed word on X is a bracketed word of the form Eq. (6) for some $q \in \mathfrak{M}^{\vec{\star}}(X)$.

We next introduce the concept of a $\vec{\star}$ -Motzkin word on X .

Definition 3.9 A word in $M(X^{\vec{\star}} \cup \{\llbracket, \rrbracket\})$ is called a $\vec{\star}$ -Motzkin word on X if it is in the intersection $M^{\vec{\star}}(X \cup \{\llbracket, \rrbracket\}) \cap \mathscr{W}(X^{\vec{\star}})$ or, more precisely, if it has the properties that

- (a) the number of \llbracket in the word equals the number of \rrbracket in the word;
- (b) counting from the left, the number of occurrence of \llbracket is always greater or equal to the number of occurrence of \rrbracket ;
- (c) for each $1 \leq i \leq k$, the letter \star_i appears exactly once in the word.

The set of all $\vec{\star}$ -Motzkin words is denoted by $\mathscr{W}^{\vec{\star}}(X)$.

Taking $Y = X^{\vec{\star}}$ in Theorem 3.6, we obtain

Corollary 3.10 *There is a unique isomorphism*

$$\phi_{X^{\vec{\star}}}: \mathfrak{M}(X^{\vec{\star}}) \rightarrow \mathscr{W}(X^{\vec{\star}})$$

of operated monoids such that

$$\phi_{X^{\vec{\star}}}(a) = a, \quad \forall a \in X^{\vec{\star}}.$$

In particular, $\phi_{X^{\vec{\star}}}(\star_i) = \star_i$ for $1 \leq i \leq k$.

Proposition 3.11 *The isomorphism $\phi_{X^{\vec{\star}}}$ restricts to a bijection*

$$\phi_{\vec{\star}}: \mathfrak{M}^{\vec{\star}}(X) \rightarrow \mathscr{W}^{\vec{\star}}(X).$$

Proof Since the bijection $\phi_{X^{\vec{\star}}}$ sends \star_i to \star_i , $1 \leq i \leq k$, an element $w \in \mathfrak{M}(X^{\vec{\star}})$ is in $\mathfrak{M}^{\vec{\star}}(X)$ if and only if $\phi_{X^{\vec{\star}}}(w) \in \mathscr{W}(X^{\vec{\star}})$ is in $\mathscr{W}^{\vec{\star}}(X)$. \square

4 Relative locations in bracketed words and Motzkin words

In this section, we first define the relative locations of bracketed words and Motzkin words by using $\vec{\star}$ -bracketed words and $\vec{\star}$ -Motzkin words in Section 4.1.

The relationship between the two kinds of relative locations is established in Section 4.2 and is used to prove the main theorem in Section 4.3.

4.1 Placements in bracketed words and Motzkin words

Definition 4.1 Let X be a set. Let f, s be in $\mathfrak{M}(X)$. A *placement* of s in f (by q) is a pair $\mathfrak{p} := (s, q)$ with $q \in \mathfrak{M}^*(X)$ such that $f = q|_s$. We then call s a *bracketed subword* of f .

The relative locations between two placements in a bracketed word can now be defined in the same way as those in an ordinary word.

Definition 4.2 Two placements (s_1, q_1) and (s_2, q_2) in $f \in \mathfrak{M}(X)$ are called

(a) *separated*, if there exists an element q in $\mathfrak{M}^{*1,*2}(X)$ such that

$$q_1|_{*1} = q|_{*1,s_2}, \quad q_2|_{*2} = q|_{s_1,*2}, \quad f = q|_{s_1,s_2};$$

(b) *nested*, if there exists an element q in $\mathfrak{M}^*(X)$ such that $q_1 = q_2|_q$ or $q_2 = q_1|_q$;

(c) *intersecting*, if there exist an element q in $\mathfrak{M}^*(X)$ and elements a, b, c in $\mathfrak{M}(X) \setminus \{1\}$ such that either

(i) $f = q|_{abc}, q_1 = q|_{*c}, q_2 = q|_{a*}$; or

(ii) $f = q|_{abc}, q_1 = q|_{a*}, q_2 = q|_{*c}$.

A bracketed word s might appear in a bracketed word f with multiple placements with different relative locations with respect to another bracketed subword.

Example 4.3 Let

$$f = \llbracket abc \rrbracket ab \in \mathfrak{M}(X).$$

Let $s_1 = \llbracket abc \rrbracket$ and let $s_2 = ab$. Then we obtain the placements $(s_1, q_1), (s_2, q_{21})$, and (s_2, q_{22}) , where

$$q_1 = \llbracket *ab \rrbracket, \quad q_{21} = \llbracket [*c]ab \rrbracket, \quad q_{22} = \llbracket [abc]* \rrbracket.$$

Then the placements (s_1, q_1) and (s_2, q_{21}) are nested since $q_{21} = q_1|_q$, where $q = \llbracket *c \rrbracket$. However, the placements (s_1, q_1) and (s_2, q_{22}) are separated.

We next describe the relative locations of two placements of bracketed words in terms of Motzkin words. By Definition 3.4, we obtain the following result.

Lemma 4.4 Let u be in $\mathscr{W}(X)$, and let p be in $\mathscr{W}^*(X)$. Then $p|_u$ is in $\mathscr{W}(X)$.

Taking $Z = X \cup \{\llbracket, \rrbracket\}$ in Definition 2.3, we have the following definition.

Definition 4.5 Let w and u be Motzkin words in $\mathscr{W}(X)$. A *placement* of u in w (by p) is a pair (u, p) with $p \in \mathscr{W}^*(X)$ such that $p|_u = w$. We then call u a *Motzkin subword* of w .

By Definition 3.4, the set $\mathscr{W}^{*1,*2}(X)$ of $(*1, *2)$ -Motzkin words is a subset of $M^{*1,*2}(X \cup \{\llbracket, \rrbracket\})$. Thus, as a special case of Definition 2.5, we obtain the

definition of two placements (u_1, p_1) and (u_2, p_2) in w being *separated* or *nested* or *intersecting*.

4.2 Relationship between relative locations

We now establish the relationship between placements in bracketed words and placements in Motzkin words.

Proposition 4.6 *Let X be a set, and let $\{\star_1, \dots, \star_k\}$ be symbols not in X . Let $\phi: \mathfrak{M}(X) \rightarrow \mathscr{W}(X)$ be the isomorphism of operated monoids in Theorem 3.6, and let $\phi_{\vec{\star}}: \mathfrak{M}^{\vec{\star}}(X) \rightarrow \mathscr{W}^{\vec{\star}}(X)$ be the bijection in Proposition 3.11.*

(a) *Let $q \in \mathfrak{M}^{\vec{\star}}(X)$ and $s_1, \dots, s_k \in \mathfrak{M}(X)$. Then*

$$\phi(q|_{s_1, \dots, s_k}) = \phi_{\vec{\star}}(q)|_{\phi(s_1), \dots, \phi(s_k)}.$$

(b) *Let $p \in \mathscr{W}^{\vec{\star}}(X)$ and $u_1, \dots, u_k \in \mathscr{W}(X)$. Then*

$$\phi^{-1}(p|_{u_1, \dots, u_k}) = \phi_{\vec{\star}}^{-1}(p)|_{\phi^{-1}(u_1), \dots, \phi^{-1}(u_k)}.$$

Proof (a) We prove a more general result. Define

$$\begin{aligned} \overline{\mathfrak{M}}^{\vec{\star}}(X) &:= \{q \in \mathfrak{M}(X^{\vec{\star}}) \mid \text{for each } 1 \leq i \leq k, \star_i \text{ appears at most once in } q\} \\ &\subseteq \mathfrak{M}(X^{\vec{\star}}). \end{aligned}$$

For $q \in \overline{\mathfrak{M}}^{\vec{\star}}(X)$ and $s_1, \dots, s_k \in \mathfrak{M}(X)$, define $q|_{s_1, \dots, s_k}$ to be the bracketed word by replacing the \star_i in q , if there is any, by s_i , $1 \leq i \leq k$. Thus, in particular,

$$q|_{s_1, \dots, s_k} = \begin{cases} q|_{s_1}, & q \in \mathfrak{M}^{\star_1}(X), \\ q, & q \in \mathfrak{M}(X). \end{cases}$$

In general, $q|_{s_1, \dots, s_k} = q|_{s_{i_1}, \dots, s_{i_m}}$ if only $\star_{i_1}, \dots, \star_{i_m}$ from $\{\star_1, \dots, \star_k\}$ appear in q .

Similarly, define

$$\begin{aligned} \overline{\mathscr{W}}^{\vec{\star}}(X) &:= \{p \in \mathscr{W}(X^{\vec{\star}}) \mid \text{for each } 1 \leq i \leq k, \star_i \text{ appears at most once in } p\} \\ &\subseteq \mathscr{W}(X^{\vec{\star}}), \end{aligned}$$

and define $p|_{u_1, \dots, u_k}$ for $p \in \overline{\mathscr{W}}^{\vec{\star}}(X)$ and $u_1, \dots, u_k \in \mathscr{W}(X)$.

With these notations, if q is in $\overline{\mathfrak{M}}^{\vec{\star}}(X)$ and $q = q_1q_2$ with $q_1, q_2 \in \overline{\mathfrak{M}}^{\vec{\star}}(X)$, then we have

$$q|_{\vec{s}} = q_1|_{\vec{s}}q_2|_{\vec{s}}.$$

Furthermore, the bijection $\phi_{X^{\vec{\star}}}$ in Corollary 3.10 restricts to a bijection

$$\overline{\phi}_{\vec{\star}}: \overline{\mathfrak{M}}^{\vec{\star}}(X) \rightarrow \overline{\mathscr{W}}^{\vec{\star}}(X)$$

by the same argument as Proposition 3.11.

Claim 4.7 Let $q \in \overline{\mathfrak{M}}^{\vec{x}}(X)$ and $s_1, \dots, s_k \in \mathfrak{M}(X)$. Then

$$\phi(q|_{s_1, \dots, s_k}) = \overline{\phi}_{\vec{x}}(q)|_{\phi(s_1), \dots, \phi(s_k)}.$$

Proof We prove the claim by induction on the depth of $q \in \overline{\mathfrak{M}}^{\vec{x}}(X)$ defined in Equation (5).

If the depth of q is 0, then $q \in \overline{M}^{\vec{x}}(X)$, and hence,

$$q = z_1 \star_{i_1} z_2 \star_{i_2} \cdots z_m \star_{i_m} z_{m+1}$$

with $z_1, \dots, z_{m+1} \in M(X)$ and $\star_{i_1}, \dots, \star_{i_m} \in \{\star_1, \dots, \star_k\}$. Then we have

$$\begin{aligned} \phi(q|\vec{s}) &= \phi(z_1 s_{i_1} \cdots z_m s_{i_m} z_{m+1}) \\ &= \phi(z_1) \phi(s_{i_1}) \cdots \phi(z_m) \phi(s_{i_m}) \phi(z_{m+1}) \\ &= \overline{\phi}_{\vec{x}}(q)|_{\phi(\vec{s})}, \end{aligned}$$

as needed. Here, we have used the abbreviations

$$\vec{s} = (s_1, \dots, s_k), \quad \phi(\vec{s}) = (\phi(s_1), \dots, \phi(s_k)).$$

Assume that the claim has been proved for $q \in \overline{\mathfrak{M}}^{\vec{x}}(X)$ with depth less or equal to $n \geq 0$ and consider a q with depth $n + 1$.

Note that ϕ and $\phi_{X^{\vec{x}}}$ (and hence $\overline{\phi}_{\vec{x}}$) are multiplicative. Thus, for $q = q_1 q_2$ with $q, q_1, q_2 \in \overline{\mathfrak{M}}^{\vec{x}}(X)$, if we can prove that

$$\phi(q_i|\vec{s}) = \overline{\phi}_{\vec{x}}(q_i)|_{\phi(\vec{s})}, \quad i = 1, 2,$$

then we also have

$$\begin{aligned} \phi(q|\vec{s}) &= \phi(q_1|\vec{s}) \phi(q_2|\vec{s}) \\ &= \overline{\phi}_{\vec{x}}(q_1)|_{\phi(\vec{s})} \overline{\phi}_{\vec{x}}(q_2)|_{\phi(\vec{s})} \\ &= (\overline{\phi}_{\vec{x}}(q_1) \overline{\phi}_{\vec{x}}(q_2))|_{\phi(\vec{s})} \\ &= \overline{\phi}_{\vec{x}}(q)|_{\phi(\vec{s})}. \end{aligned}$$

Therefore, we only need to complete the induction when q is of depth $n + 1$ and is indecomposable in $\overline{\mathfrak{M}}^{\vec{x}}(X)$, that is, q is not the product of two elements in

$$\overline{\mathfrak{M}}^{\vec{x}}(X) \subseteq \mathfrak{M}(X^{\vec{x}}) = M(X^{\vec{x}} \cup [\mathfrak{M}(X^{\vec{x}})])$$

(see Eq. (4)). Then q is either in $X \cup \{\star_1, \dots, \star_k\}$ or is of the form $[\tilde{q}]$ with $\tilde{q} \in \overline{\mathfrak{M}}^{\vec{x}}(X)$. In the first case, q is of depth 0 and has been proved above. In the second case, \tilde{q} has depth n . Note that ϕ and $\phi_{\vec{x}}$ (and hence $\overline{\phi}_{\vec{x}}$) are compatible

with the brackets. Therefore, together with the induction hypothesis, we have

$$\begin{aligned} \phi(q|_{\vec{s}}) &= \phi(\lfloor \tilde{q} \rfloor |_{\vec{s}}) \\ &= \phi(\lfloor \tilde{q} \rfloor |_{\vec{s}}) \\ &= \lfloor \phi(\tilde{q} |_{\vec{s}}) \rfloor \\ &= \lfloor \overline{\phi_{\vec{x}}}(\tilde{q}) |_{\phi(\vec{s})} \rfloor \\ &= (\lfloor \overline{\phi_{\vec{x}}}(\tilde{q}) \rfloor) |_{\phi(\vec{s})} \\ &= (\overline{\phi_{\vec{x}}}(\lfloor \tilde{q} \rfloor)) |_{\phi(\vec{s})} \\ &= \overline{\phi_{\vec{x}}}(q) |_{\phi(\vec{s})}. \end{aligned}$$

This completes the inductive proof of the claim. □

Then (a) is a special case of Claim 4.7 since the restriction of $\overline{\phi_{\vec{x}}}$ to $\mathfrak{M}^{\vec{x}}(X)$ is $\phi_{\vec{x}}$.

(b) Denote

$$\vec{u} = (u_1, \dots, u_k), \quad \phi^{-1}(\vec{u}) = (\phi^{-1}(u_1), \dots, \phi^{-1}(u_k)).$$

By (a) and the bijectivity of ϕ and $\phi_{\vec{x}}$, we have

$$\begin{aligned} \phi^{-1}(p|_{\vec{u}}) &= \phi^{-1}(\phi_{\vec{x}}(\phi_{\vec{x}}^{-1}(p)) |_{\phi(\phi^{-1}(\vec{u}))}) \\ &= \phi^{-1}(\phi(\phi_{\vec{x}}^{-1}(p)) |_{\phi^{-1}(\vec{u})}) \\ &= \phi_{\vec{x}}^{-1}(p) |_{\phi^{-1}(\vec{u})}. \end{aligned}$$

This is what we need. □

By Proposition 4.6, we immediately have the following results.

Corollary 4.8 *Let f be in $\mathfrak{M}(X)$. Then (s, q) is a placement of s in f if and only if $(\phi(s), \phi_{\star}(q))$ is a placement of $\phi(s)$ in $\phi(f) \in \mathscr{W}(X)$.*

Proof The pair (s, q) is a placement of s in f if and only if $f = q|_s$, which holds if and only if $\phi(f) = \phi_{\star}(q)|_{\phi(s)}$, which holds if and only if $(\phi(s), \phi_{\star}(q))$ is a placement of $\phi(s)$ in $\phi(f)$. □

Corollary 4.9 *Let q_1, q_2 , and q be \vec{x} -bracketed words in $\mathfrak{M}^{\star}(X)$. Then $q_1 = q_2|_q$ if and only if $\phi_{\star}(q_1) = \phi_{\star}(q_2)|_{\phi_{\star}(q)}$ in $\mathscr{W}^{\star}(X)$.*

Proof To be precise, denote the isomorphism and the bijection in Proposition 4.6, in the case of $k = 1$, by

$$\phi := \phi_X: \mathfrak{M}(X) \rightarrow \mathscr{W}(X), \quad \phi_{\star_1} := \phi_{X, \star_1}: \mathfrak{M}^{\star_1}(X) \rightarrow \mathscr{W}^{\star_1}(X).$$

Also denote $X' := X \cup \{\star\}$, where \star is a symbol not in $X \cup \{\star_1\}$. Then

$$\phi_{X'}|_{\mathfrak{M}^{\star}(X)} = \phi_{X, \star}.$$

Let $q_1, q_2, q \in \mathfrak{M}^*(X)$ be given. Define

$$q'_2 := q_2|_{\star \rightarrow \star_1} \in \mathfrak{M}^{\star_1}(X) \subseteq \mathfrak{M}^{\star_1}(X').$$

We also have

$$q_1, q \in \mathfrak{M}^*(X) \subseteq \mathfrak{M}(X').$$

Then $q_1 = q_2|_q$ means $q_1 = q'_2|_q$ in $\mathfrak{M}(X')$. Then applying Proposition 4.6 (a) to the set X' with $k = 1$, we have

$$\phi_\star(q_1) = \phi_{X,\star}(q_1) = \phi_{X'}(q_1) = \phi_{X'}(q'_2|_q) = \phi_{X',\star_1}(q'_2)|_{\phi_{X'}(q)} = \phi_{\star_1}(q'_2)|_{\phi_\star(q)}.$$

But $q'_2|_\star = q_2$, so

$$\phi_{\star_1}(q'_2)|_{\phi_\star(q)} = \phi_\star(q_2)|_{\phi_\star(q)}.$$

Hence,

$$\phi_\star(q_1) = \phi_\star(q_2)|_{\phi_\star(q)}.$$

Conversely, if $\phi_\star(q_1) = \phi_\star(q_2)|_{\phi_\star(q)}$, then we have

$$\phi_\star(q_1) = \phi_{\star_1}(q'_2)|_{\phi_\star(q)},$$

where $q'_2 = q_2|_{\star_1}$. Then with the notations as above, applying Proposition 4.6 (b), we have

$$q_1 = \phi_\star^{-1}(\phi_\star(q_1)) = \phi_\star^{-1}(\phi_{\star_1}(q'_2)|_{\phi_\star(q)}) = \phi_{\star_1}^{-1}(\phi_{\star_1}(q'_2))|_{\phi_\star^{-1}(\phi_\star(q))} = q'_2|_q = q_2|_q. \quad \square$$

Theorem 4.10 *Let f be a bracketed word in $\mathfrak{M}(X)$. Let (s_1, q_1) and (s_2, q_2) be placements in f . Then*

(a) *the pairs (s_1, q_1) and (s_2, q_2) are separated in f if and only if the pairs $(\phi(s_1), \phi_\star(q_1))$ and $(\phi(s_2), \phi_\star(q_2))$ are separated in $\phi(f)$;*

(b) *the pairs (s_1, q_1) and (s_2, q_2) are nested in f if and only if the pairs $(\phi(s_1), \phi_\star(q_1))$ and $(\phi(s_2), \phi_\star(q_2))$ are nested in $\phi(f)$;*

(c) *the pairs (s_1, q_1) and (s_2, q_2) are intersecting in f if and only if the pairs $(\phi(s_1), \phi_\star(q_1))$ and $(\phi(s_2), \phi_\star(q_2))$ are intersecting in $\phi(f)$.*

Proof Let (s_1, q_1) and (s_2, q_2) be placements in f . Then by Corollary 4.8, $(\phi(s_1), \phi_\star(q_1))$ and $(\phi(s_2), \phi_\star(q_2))$ are placements in $\phi(f)$.

(a) The placements (s_1, q_1) and (s_2, q_2) in f are separated if and only if there exists an element q in $\mathfrak{M}^{\star_1, \star_2}(X)$ such that

$$q_1|_{\star_1} = q|_{\star_1, \star_2}, \quad q_2|_{\star_2} = q|_{s_1, \star_2}, \quad f = q|_{s_1, s_2}.$$

By Proposition 4.6, this is so if and only if

$$\begin{aligned} \phi_\star(q_1)|_{\star_1} &= \phi_{\star_1, \star_2}(q)|_{\star_1, \phi(s_2)}, & \phi_\star(q_2)|_{\star_2} &= \phi_{\star_1, \star_2}(q)|_{\phi(s_1), \star_2}, \\ \phi(f) &= \phi_{\star_1, \star_2}(q)|_{\phi(s_1), \phi(s_2)}. \end{aligned} \quad (7)$$

If this is true, then $(\phi(s_1), \phi_\star(q_1))$ and $(\phi(s_2), \phi_\star(q_2))$ are separated in $\phi(f)$. Conversely, if the placements $(\phi(s_1), \phi_\star(q_1))$ and $(\phi(s_2), \phi_\star(q_2))$ are separated in $\phi(f)$, then there exists an element p in $\mathscr{W}^{\star 1, \star 2}(X)$ such that

$$\phi_\star(q_1)|_{\star 1} = p|_{\star 1, \phi(s_2)}, \quad \phi_\star(q_2)|_{\star 2} = p|_{\phi(s_1), \star 2}, \quad \phi(f) = p|_{\phi(s_1), \phi(s_2)}.$$

Since

$$\phi_{\star 1, \star 2}: \mathfrak{M}^{\star 1, \star 2}(X) \rightarrow \mathscr{W}^{\star 1, \star 2}(X)$$

is bijective, there is $q \in \mathfrak{M}^{\star 1, \star 2}(X)$ such that $\phi_{\star 1, \star 2}(q) = p$. Thus, Eq. (7) holds, and hence, (s_1, q_1) and (s_2, q_2) are separated in f .

(b) The placements (s_1, q_1) and (s_2, q_2) are nested in f if and only if there exists an element q in $\mathfrak{M}^\star(X)$ such that $q_1 = q_2|_q$ or $q_2 = q_1|_q$. By Corollary 4.9, this holds if and only if

$$\phi_\star(q_1) = \phi_\star(q_2)|_{\phi_\star(q)} \quad \text{or} \quad \phi_\star(q_2) = \phi_\star(q_1)|_{\phi_\star(q)}. \tag{8}$$

If this is true, then $(\phi(s_1), \phi_\star(q_1))$ and $(\phi(s_2), \phi_\star(q_2))$ are nested. Conversely, if $(\phi(s_1), \phi_\star(q_1))$ and $(\phi(s_2), \phi_\star(q_2))$ are nested, then there is $p \in \mathscr{W}^\star(X)$ such that

$$\phi_\star(q_1) = \phi_\star(q_2)|_p \quad \text{or} \quad \phi_\star(q_2) = \phi_\star(q_1)|_p.$$

Since $\phi_\star: \mathfrak{M}^\star(X) \rightarrow \mathscr{W}^\star(X)$ is bijective, there is $q \in \mathfrak{M}^\star(X)$ such that $\phi_\star(q) = p$. Thus, Eq. (8) holds, and hence, (s_1, q_1) and (s_2, q_2) are nested in f .

(c) The placements (s_1, q_1) and (s_2, q_2) are intersecting in f if and only if there exist q in $\mathfrak{M}^\star(X)$ and a, b, c in $\mathfrak{M}(X) \setminus \{1\}$ such that

$$f = q|_{abc}, \quad q_1 = q|_{\star c}, \quad q_2 = q|_{a\star},$$

or

$$f = q|_{abc}, \quad q_1 = q|_{a\star}, \quad q_2 = q|_{\star c}.$$

By Theorem 3.6, Proposition 4.6, and Corollary 4.9, this is so if and only if

$$\phi(f) = \phi_\star(q)|_{\phi(a)\phi(b)\phi(c)}, \quad \phi_\star(q_1) = \phi_\star(q)|_{\phi(a)\star}, \quad \phi_\star(q_2) = \phi_\star(q)|_{\star\phi(c)}, \tag{9}$$

or

$$\phi(f) = \phi_\star(q)|_{\phi(a)\phi(b)\phi(c)}, \quad \phi_\star(q_1) = \phi_\star(q)|_{\star\phi(c)}, \quad \phi_\star(q_2) = \phi_\star(q)|_{\phi(a)\star}. \tag{10}$$

If this is true, then $(\phi(s_1), \phi_\star(q_1))$ and $(\phi(s_2), \phi_\star(q_2))$ are intersecting since $\phi(a), \phi(b), \phi(c) \neq 1$.

Conversely, if $(\phi(s_1), \phi_\star(q_1))$ and $(\phi(s_2), \phi_\star(q_2))$ are intersecting in $\phi(f)$, then there exist p in $\mathscr{W}^\star(X)$ and α, β, γ in $\mathscr{W}(X) \setminus \{1\}$ such that

$$\phi(f) = p|_{\alpha\beta\gamma}, \quad \phi_\star(q_1) = \phi_\star(q)|_{\alpha\star}, \quad \phi_\star(q_2) = \phi_\star(q)|_{\star\gamma},$$

or

$$\phi(f) = p|_{\alpha\beta\gamma}, \quad \phi_\star(q_1) = \phi_\star(q)|_{\star\gamma}, \quad \phi_\star(q_2) = \phi_\star(q)|_{\alpha\star}.$$

By the bijectivity of ϕ and ϕ_* , there are $q \in \mathfrak{M}^*(X)$ and $a, b, c \in \mathfrak{M}(X) \setminus \{1\}$ such that

$$p = \phi_*(q), \quad \alpha = \phi(a), \quad \beta = \phi(b), \quad \gamma = \phi(c).$$

Then Eq. (9) or (10) hold. This shows that (s_1, q_1) and (s_2, q_2) are intersecting in f . \square

4.3 Relative locations of bracketed subwords

Now, we are ready to prove our main theorem on the classification of relative locations of two bracketed subwords (placements) in a bracketed word.

Theorem 4.11 (Main Theorem) *Let f be a bracketed word in $\mathfrak{M}(X)$. For any two placements (s_1, q_1) and (s_2, q_2) in f , exactly one of the following is true:*

- (a) (s_1, q_1) and (s_2, q_2) are separated;
- (b) (s_1, q_1) and (s_2, q_2) are nested;
- (c) (s_1, q_1) and (s_2, q_2) are intersecting.

Proof By Theorem 2.11, the statement of the theorem holds when (s_1, q_1) and (s_2, q_2) are replaced by the two placements $(\phi(s_1), \phi_*(q_1))$ and $(\phi(s_2), \phi_*(q_2))$ in the word

$$\phi(f) \in \mathscr{W}(X) \subseteq M(X \cup \{\llbracket, \rrbracket\}).$$

Then by Theorem 4.10, the statement holds for (s_1, q_1) and (s_2, q_2) . \square

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