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Exact price of anarchy for a two-uniform-machine scheduling game under parallel processing

Key words: Scheduling game; Hierarchy; Parallel processing; Price of anarchy (PoA); Nash equilibrium

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Motivation

Distributed scheduling (scheduling game) and the price of anarchy

- Scheduling game: motivated by practical engineering scenarios in which heterogeneous resources serve users with different privileges.
- In distributed scheduling environments, decisions are made by selfish jobs (i.e., players in scheduling games) that are only interested in achieving their objectives without any concern for the global optimal outcome for the game.
- The objective of each job: minimizing its completion time.
- Global social objective: minimizing the maximum completion time across all machines.
- Mechanism design: provides a powerful framework for mitigating equilibrium inefficiency by guiding selfish job behaviors toward desirable system outcomes.
- Nash equilibrium: a schedule where no job has a unilateral incentive to switch to another machine.
- Price of anarchy (PoA): the ratio between the social objective value of the worst-case Nash equilibrium and the optimal social objective value.

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Motivation

Parallel processing policy v.s. makespan policy

- Several classical and well-studied policies: ShortestFirst, LongestFirst, randomized policy, and makespan policy.
- Parallel processing hybridizes two classic policies: ShortestFirst and makespan.
- Under the makespan policy, the completion time of a job is equal to the completion time of the machine on which it is processed.
- Under parallel processing, only the largest job on a machine incurs a completion time equal to the machine load, which substantially complicates equilibrium analysis.
- Intuitively, the parallel processing policy should be superior to the makespan policy because it divides the time horizon into tiny time slots of equal length and processes all currently unfinished jobs in parallel.
- **Key question:** Whether the parallel processing policy reduces PoA under the makespan policy.

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Problem statement

Scheduling game $Q2|GoS|C_{max}$ under parallel processing

- Uniform machines: m machines M_t with speed s_t , certificate $g(M_t), 1 \le t \le m$.

two machines $M_1$with speed 1, $g(M_1) = 1$ and M_2 with speed s, $g(M_2)$=2 (*s* is the speed ratio)

- Independent selfish jobs: n jobs, $J_i = (p_i, g_i)$ where p_i and $g_i \in \{1,2\}$ denote the size and class of J_i, $1 \le i \le n$, respectively.
- Hierarchical scheduling: job J_i can be executed on machine M_t only when $g_i \ge g(M_t)$.

Class 1 only on M_1 and class 2 on either machine

- Global social objective: Minimize makespan $C_{\max}$

Parallel processing (EQUI) policy

Each machine splits its timeline into equal slots and serves all unfinished jobs round-robin.

Completion time of the k^{th} job on machine t:

$$C_{kt} = \frac{(n_i - k)p_{kt} + \sum_{j=1}^{k} p_{jt}}{s_t}.$$

Key property: machine load = completion time of the largest job

Lemma 1 (Yu et al., 2010) A Nash equilibrium always exists under the parallel processing policy.

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Upper bound of PoA

PoA upper bound derived via case analysis on Nash equilibria

- Theorem 1 (upper bound)

  Under parallel processing, the PoA for $Q2|\mathrm{GoS}|C_{\max}$ is at most:

$$f(s) = \min\left\{1 + s,\ \max\left\{\frac{2+s}{1+s}, \frac{1+2s}{1+s}\right\}\right\}.$$

- Proof sketch: classify by which machine is the bottleneck, then consider subcases on whether class-1 load exceeds the largest class-2 job, and further classify by $s \geq 1$ or $s < 1$.

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Key Lemma: In any Nash equilibrium, suppose that $J_{k1} = (p_{k1}, 2)$ is the last-ranked maximum-size job in the job set $\alpha \subseteq G_{21}$ and $t(G_1) > p_{B1}$. Then, the completion time of job J_{k1} satisfies $C_{k1} > t(\alpha) + p_{k1}$.

$J_{B1} = (p_{B1}, 2)$: the largest class 2 job scheduled on machine M_1;

$t(\delta)$: the total size of the jobs in any job set δ;

G_1: the set of class 1 jobs;

G_{21}: the set of class 2 jobs assigned to machine M_1.

Lower bound of PoA

Tight instance constructions match the upper bound $f(s)$

- Theorem 2 (lower bound)

  For every speed ratio $s > 0$, there exists a scheduling game instance under parallel processing whose PoA equals $f(s)$.

Each construction is verified as a Nash equilibrium: no job can reduce its completion time by unilaterally switching machines. Combined with Theorem 1, this proves that the bound is exact.

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Table 1 Tight instances for $Q2|\mathrm{GoS}|C_{\max}$

Range of s	Nash equilibrium		Optimal solution		PoA
	Machine M_1	Machine M_2	Machine M_1	Machine M_2	
$0 < s \le \frac{\sqrt{5}-1}{2}$	$(s,\ 2)$, $\lceil 1/s \rceil$ jobs $\left(\frac{1-s-s^2}{\lceil 1/s \rceil},\ 1\right)$	$(s+s^2,\ 2)^*$	$(s+s^2,\ 2)$, $\lceil 1/s \rceil$ jobs $\left(\frac{1-s-s^2}{\lceil 1/s \rceil},\ 1\right)$	$(s,\ 2)$	$1+s$
$\frac{\sqrt{5}-1}{2} < s \le 1$	3 jobs $(1/3,\ 2)$	$(s^2+s-1,\ 2)^*$, $(1+s,\ 2)^*$	$(1+s,\ 2)$	$(s^s+s-1,\ 2)$, 3 jobs $(1/3,\ 2)$	$\frac{2+s}{1+s}$
$s > 1$	$\left(\frac{ks+k+s-1}{s},\ 1\right)^*$, k jobs $(1,\ 2)^*$	$\lfloor (k+1)s \rfloor$ jobs $\left(\frac{ks+s-1}{\lfloor (k+1)s \rfloor},\ 2\right)$	$\left(\frac{ks+k+s-1}{s},\ 1\right)$	k jobs $(1,\ 2)$, $\lfloor (k+1)s \rfloor$ jobs $\left(\frac{ks+s-1}{\lfloor (k+1)s \rfloor},\ 2\right)$	$\frac{1+2s}{1+s}$

* determines the makespan

Main Results

The exact price of anarchy as a piecewise function of speed ratio s

- **Theorem 3 (exact PoA for $Q2|\mathrm{GoS}|C_{\max}$ under parallel processing)**

$$f(s)=\begin{cases}1+s, & 0<s\le\dfrac{\sqrt{5}-1}{2},\\[2ex] \dfrac{2+s}{1+s}, & \dfrac{\sqrt{5}-1}{2}<s\le 1,\\[2ex] \dfrac{1+2s}{1+s}, & s>1.\end{cases}$$

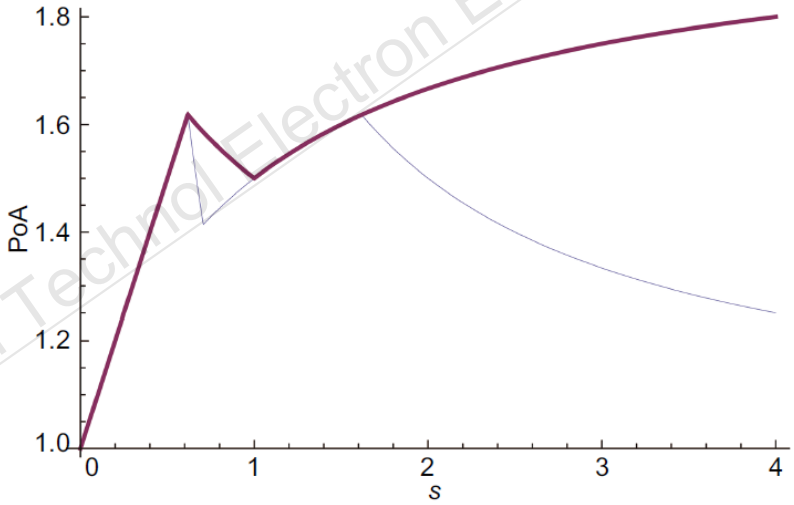


Fig. 1 PoAs for $Q2|\mathrm{GoS}|C_{\max}$ under parallel processing (thick line) and makespan (thin line) policies

Corollary 1

For identical machines $s=1$, the PoA for $P2|\mathrm{GoS}|C_{\max}$ under parallel processing is exactly:

$$\mathrm{PoA}=\frac{3}{2}.$$

The PoA curve of parallel processing always lies above or coincides with that of makespan for all $s>0$, implying that parallel processing cannot outperform makespan in terms of equilibrium efficiency loss. Therefore, equal time-slice allocation does not reduce selfish inefficiency for hierarchical scheduling games.

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Generalization

Extension to *m* uniform machines with two hierarchy levels

Generalized model

- *k* machines with service certificate 1 (speed 1): $M_1, M_2 \ldots, M_k$
- *m–k* machines with service certificate 2 (speed s): $M_{k+1}, M_{k+2} \ldots, M_m$
- Class 1 jobs only on certificate-1 machines; class 2 jobs can use any machine

Lower bound construction

We construct a concrete instance, yielding

$$\mathrm{PoA} \geq f(k,s) = \begin{cases} 1+s, & 0 < s \leq \dfrac{m-k-1}{m-k}, \\ 2 - \dfrac{1}{m-k}, & s > \dfrac{m-k-1}{m-k}. \end{cases}$$

Open problems

- The exact PoA for general *m* machines remains **open**
- Multiple hierarchy levels (more than 2) introduce substantial obstacles
- Heterogeneous machine speeds require tailored case analysis

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Conclusions

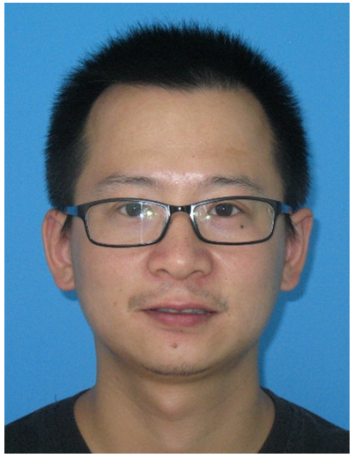
We have derived the exact PoA for a two-uniform-machine scheduling game with hierarchical service levels under the parallel processing policy. PoA follows a clean piecewise form depending on the machine speed ratio.

For identical machines, PoA is exactly 3/2. We also extend to the general *m*-machine case with a lower bound, though the exact PoA for more than two machines remains an interesting open direction.

The equal time-slice allocation does not reduce selfish inefficiency for hierarchical scheduling games.

The parallel processing policy **cannot reduce** PoA under the makespan policy.

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