Hande Nur DALKILIÇ, Yusuf YAYLI, 2025. Pseudo-evolute curves and caustic surfaces. *Frontiers of Information Technology & Electronic Engineering,* 26(9):1733-1741. https://doi.org/10.1631/FITEE.2400930

Pseudo-evolute curves and caustic surfaces

Key words: Pseudo-evolute curves; Caustic surfaces; Developable surfaces; Rectifying caustic developable surfaces; Osculating caustic developable surfaces; Normal caustic developable surfaces

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Motivation and Method

The envelope of normals of a surface is defined as a *caustic surface*. Caustic surface along a space curve γ is expressed as follows:

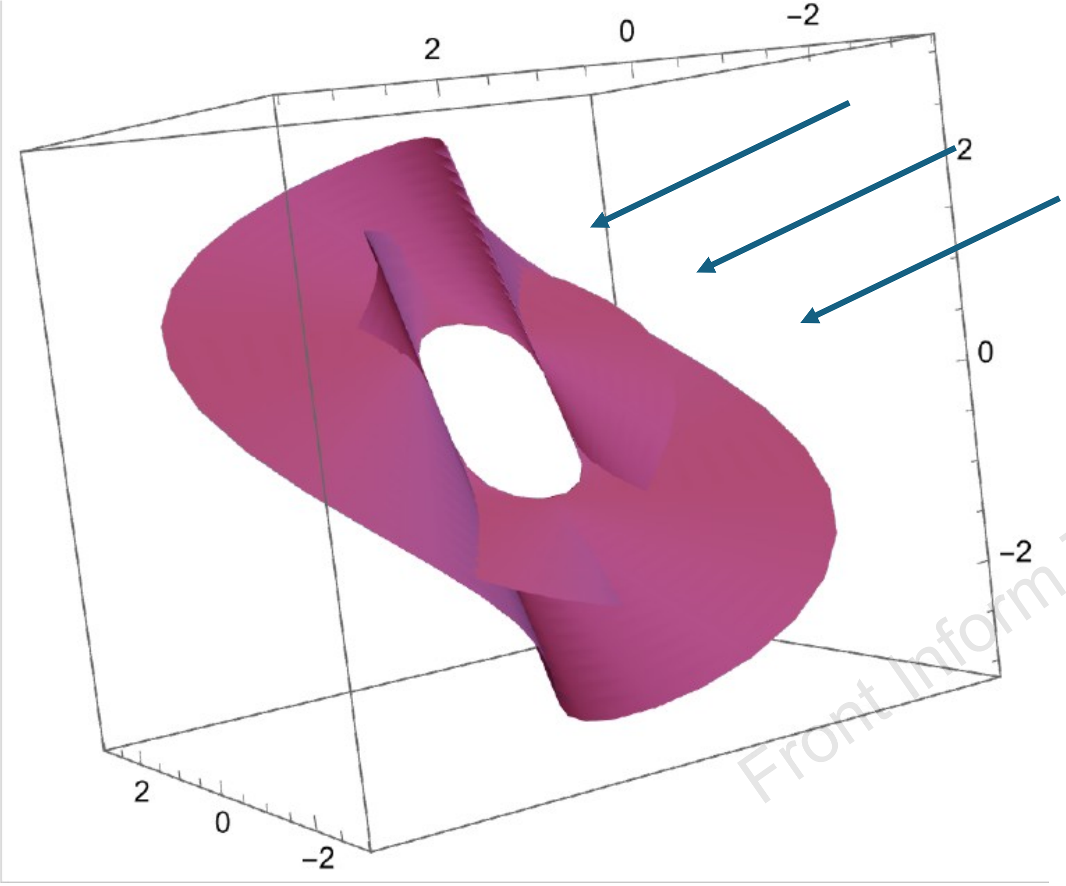
$$\boldsymbol{\varepsilon}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{f}(s), \qquad u \in \mathbb{R}$$

where s and u are the parameters of the caustic surface, $\boldsymbol{f}$ is the direction of the generators of the caustic envelope surface.

$$\boldsymbol{f}(s) = \langle \boldsymbol{\gamma}'(s) \times \boldsymbol{d}(s), \boldsymbol{d}'(s) \rangle\, \boldsymbol{\gamma}'(s) - \langle \boldsymbol{\gamma}'(s) \times \boldsymbol{d}(s), \boldsymbol{\gamma}''(s) \rangle \boldsymbol{d}(s),$$

where $\boldsymbol{d}$ is the reflected vector.

Motivation and Method



Miror surface $\boldsymbol{\phi}(s,u)$

What is the precise meaning of the term “reflected vector” in this context?

The reflected vector represents the direction of reflected light rays transmitted from a light source to a mirror surface, which is a developable surface.

In this study, we follow the methodology proposed by Hoffmann et al. (2022) and fix the directions of the light sources and the reflected vectors at different angles.

Motivation and Method

The question is how the reflected vector can be obtained. If the mirror surface Φ is derived from the surface $\boldsymbol{M}(s,u)$ based on the curve $\boldsymbol{\gamma}(s)$, and the normal vector field of the mirror surface is $\boldsymbol{S} = (s_1, s_2, s_3)$, we can find the reflected vector $\boldsymbol{d}$ of the transmitted light ray in the X direction as follows:

$$\boldsymbol{d} = -(\boldsymbol{I}_3 + 2\boldsymbol{S}^2)\boldsymbol{X},$$

$$(s_1, s_2, s_3) \cong \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix},$$

where $\boldsymbol{I}$ is a unit matrix, and $s_1, s_2, s_3 \in \mathbb{R}$ are the vector components of the normal vector.

Motivation and Method

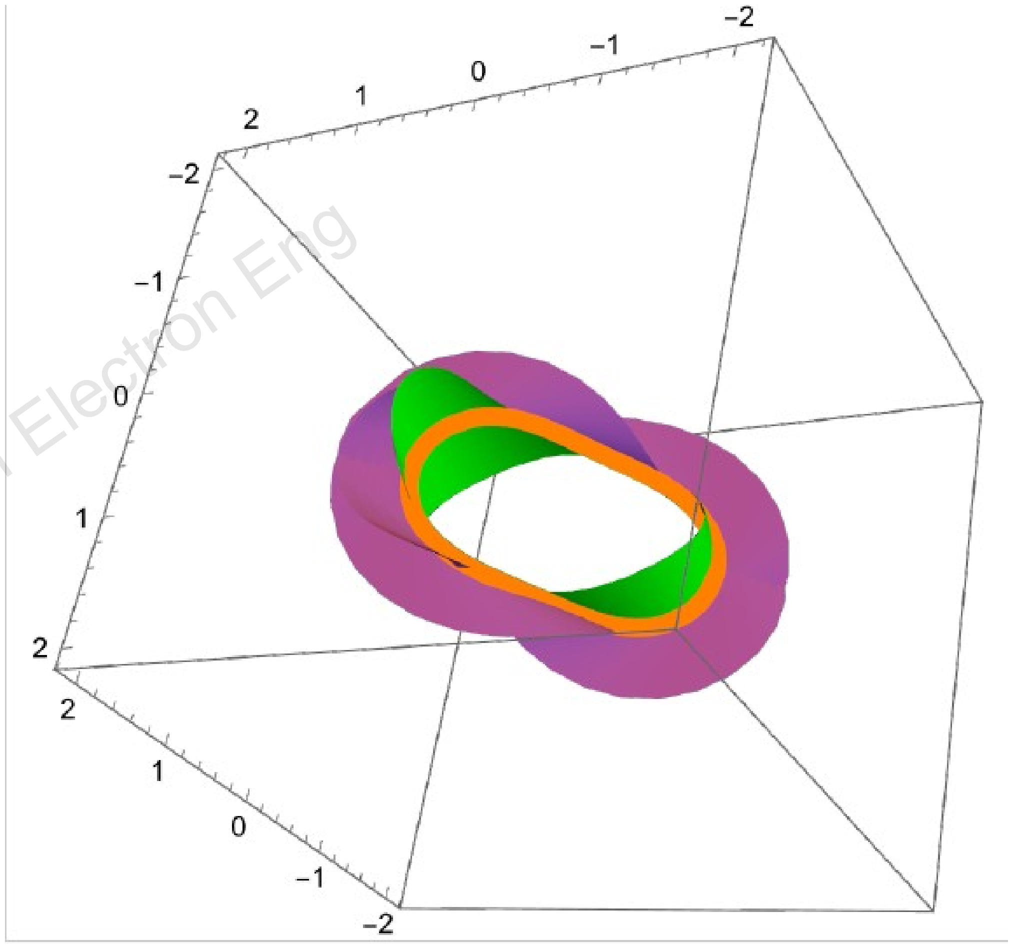
The curve formed by the motion of the rectifying planes of a space curve $\boldsymbol{\gamma}(s)$ is defined as a pseudo-evolute curve.

In analyzing caustic surfaces, we are interested in observing the generation of three types of caustic developable surfaces: the rectifying, osculating, and normal caustic developable surfaces.

In our investigation of rectifying caustic surfaces, two key questions emerge. The first concerns determining the optimal direction for the light source to generate these caustic surfaces. The second involves exploring potential tangent surfaces of curves that could be used to generate these surfaces.

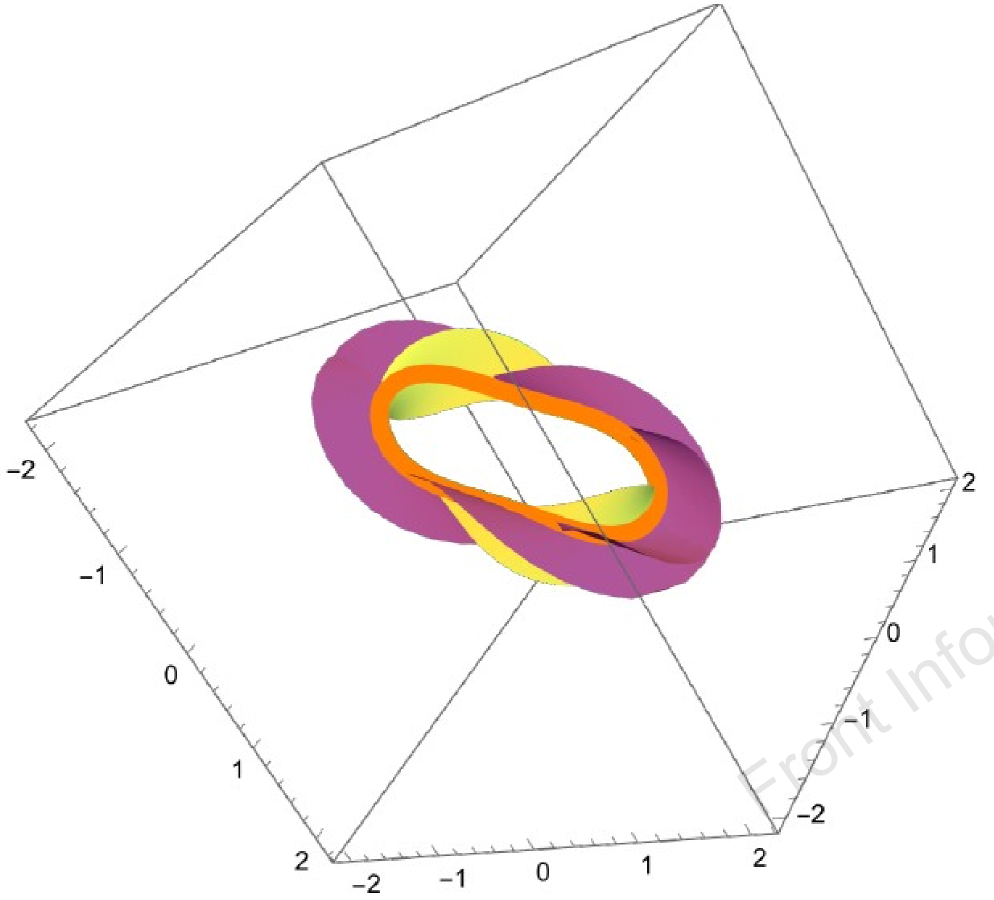
Results

The reflected surface is obtained as rectifying caustic developable surface when the light source is positioned along the $\boldsymbol{e}_3$ direction, where the Frenet frame of a space curve $\boldsymbol{\gamma}$ is given by the set $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$.



Normal caustic developable surface (green), mirror surface (purple), and base curve $\boldsymbol{\gamma}$ (orange)

Results

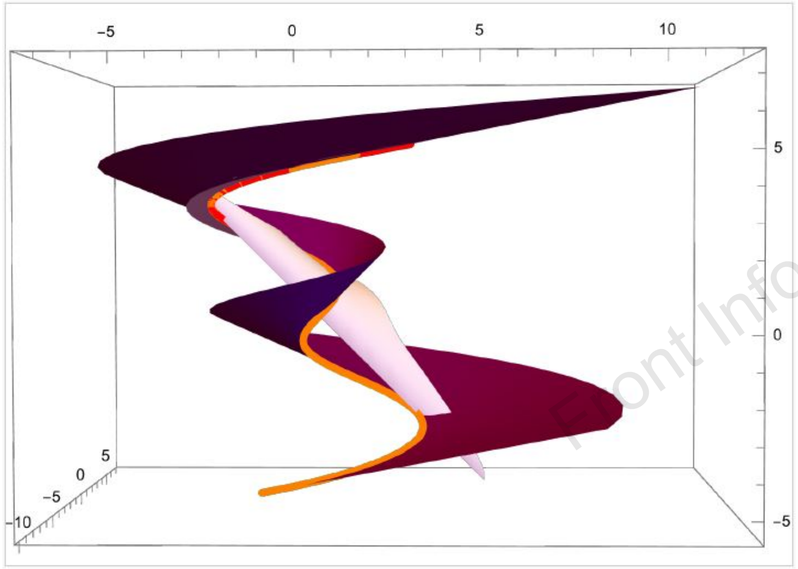


Osculating caustic developable surface (yellow), mirror surface (purple), and base curve $\boldsymbol{\gamma}$ (orange)

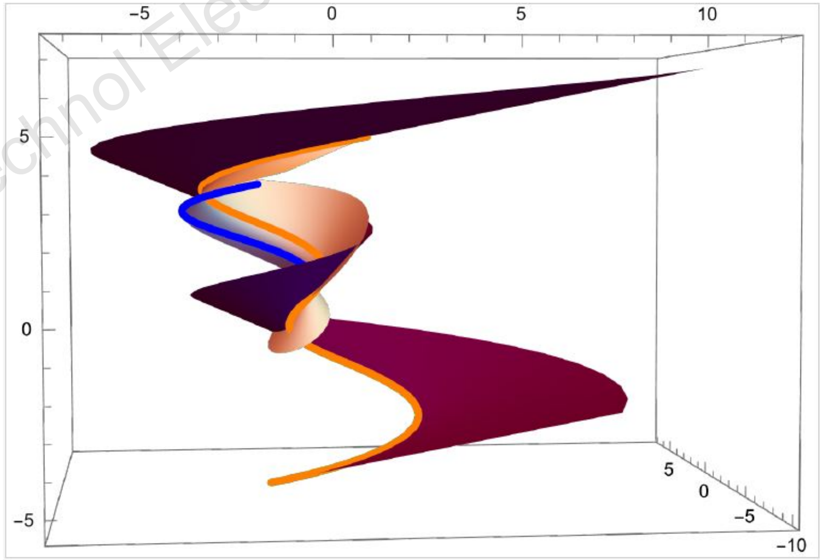
The reflected surfaces are obtained as osculating caustic developable surfaces when the light source is positioned along the y direction, and as normal caustic developable surfaces when the light source is positioned along the u direction, where the Darboux frame of a surface $\boldsymbol{M}$ along a space curve $\boldsymbol{\gamma}$ is given by the set $\{\boldsymbol{e}_1, \boldsymbol{y}, \boldsymbol{u}\}$.

Results

In the later part of our study, we realised that the osculating caustic developable surface is also a flat approximation surface of the surface $\boldsymbol{M}$. Similarly, the normal caustic developable surface is a normal approximation surface of the surface $\boldsymbol{M}$.



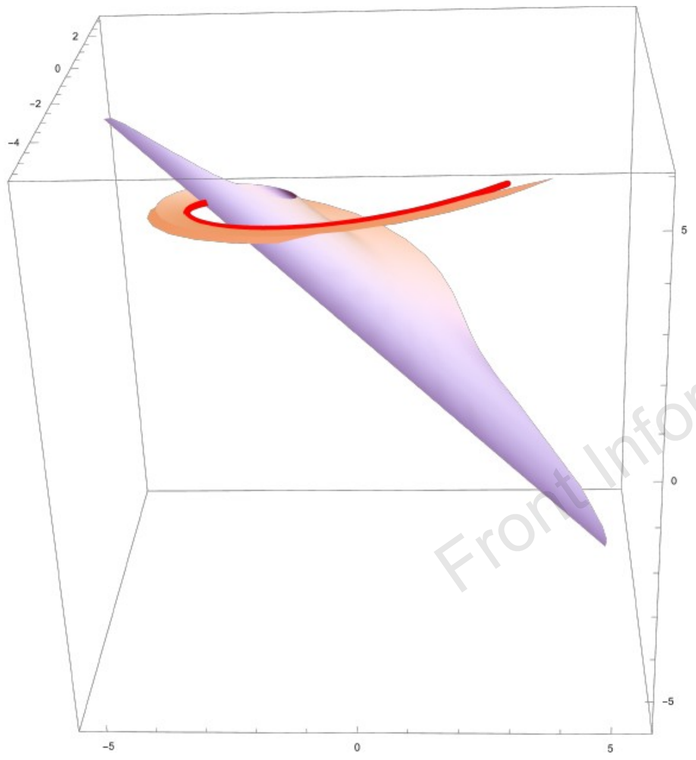
Base curve $\boldsymbol{\gamma}$ (orange), pseudo-evolute curve (red), and miror surface $\boldsymbol{\phi}(s,u)$ (purple)



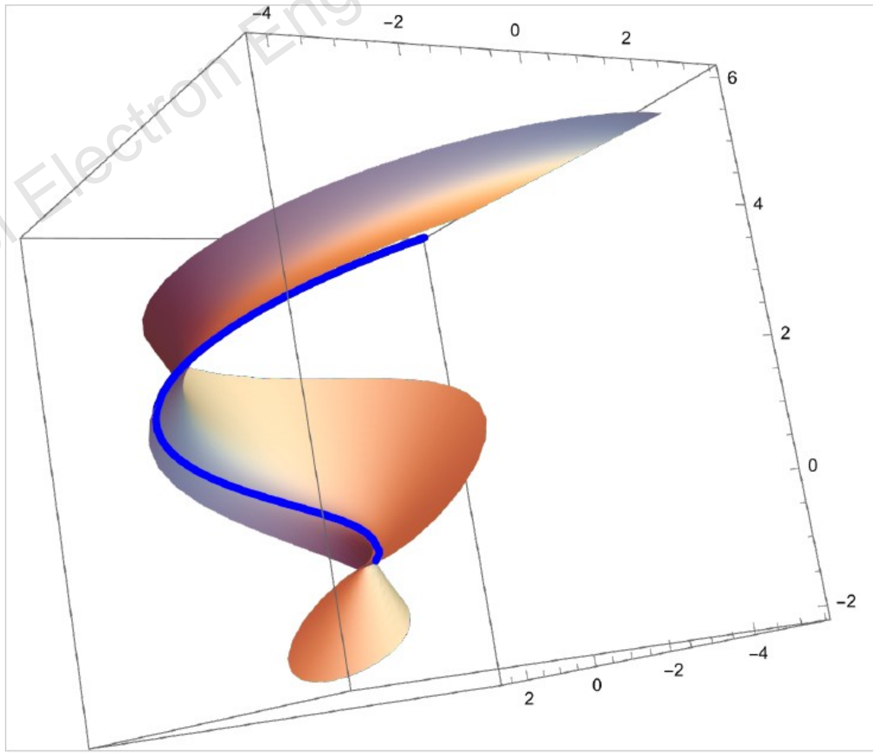
Base curve $\boldsymbol{\gamma}$ (orange), pseudo-evolute curve (blue), normal caustic developable surface (orange), and miror surface $\boldsymbol{\phi}(s,u)$ (purple)

Results

Furthermore, pseudo-evolute curves of these caustic developable surfaces were obtained as stiction curves of the surfaces.



Pseudo-evolute curve of the osculating caustic developable surface



Pseudo-evolute curve of the normal caustic developable surface

Author Bibliography

Hande Nur Dalkılıç received her bachelor's degree from the Department of Mathematics at Ankara University in 2021 and her master's degree in 2023. Since 2023, she has been continuing her doctoral studies as a research assistant at the Department of Mathematics at Ankara University. Her research interests include differential geometry, curve and surface geometry, pseudo-evolute curves, caustic surfaces, and Lorentzian geometry.

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