



Semi-tensor product approach to controllability, reachability, and stabilizability of extended finite state machines*

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Abstract: This paper uses the semi-tensor product (STP) of matrices and adopts algebraic methods to study the controllability, reachability, and stabilizability of extended finite state machines (EFSMs). First, we construct the bilinear dynamic system model of the EFSM, laying the foundation for further research. Second, combined with this bilinear dynamic system model, we propose theorems for the controllability, reachability, and stabilizability of the bilinear dynamic system model of the EFSM. Finally, we design an algorithm to determine the controllability and stabilizability of the EFSM. The correctness of the main results is verified through examples.

Key words: Semi-tensor product (STP); Matrix approach; Algebraic method; Finite-valued systems

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1 Introduction

In finite state machines (FSMs), the system's controllability, reachability, and stabilizability are of great significance and often reflect the essential characteristics of the system. Controllability refers to whether the state variables in the system can be controlled. Reachability concerns whether the specified state in the system can be reached in a limited number of state transitions. Stabilizability relates to the stable state of the entire system. By studying and discussing these properties, we can achieve optimal control strategies for FSMs.

As the research on FSMs deepens, many scholars have been exploring the controllability, reachability,

and stabilizability of FSMs while studying other problems. This has laid the foundation for further dealing with research objectives. Zoubeyr et al. (2010) used reachability graphs to represent the execution states of protocols and to verify consistency. A new verification technology based on backtracking for complex FSM modeling to detect all deadlock errors has been proposed. This provides a new method for detecting FSM system deadlocks. Yao and Li (2017) conducted a stabilizability analysis by combining the method of multiple Lyapunov functions with pattern-dependent average dwell time method, and derived the corresponding sufficient conditions. Based on this, a state feedback controller has been designed to achieve stabilizability for hybrid systems. This provides a new approach for studying multiple systems. Similarly, many other scholars have made outstanding contributions to the controllability (Dridi et al., 2022; Henry et al., 2022; Glück, 2023), reachability (Adak et al., 2021; Köcher, 2021; Pilch et al., 2021), and stabilizability (Han and Chen, 2018; Han et al., 2018; Zhang et al., 2020; Boukerrou et al., 2022)

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of FSMs, further demonstrating their importance. However, we found that the studies mentioned above all used traditional formal language methods to introduce the concepts of controllability, reachability, and stabilizability for conventional FSMs. Additionally, these methods do not apply to extended finite state machines (EFSMs). Therefore, we have identified a new mathematical tool, the semi-tensor product (STP), and have used algebraic methods to study the controllability, reachability, and stabilizability of EFSMs.

The advantage of STP is that it automatically finds the hierarchy of data and effectively processes multi-dimensional arrays while generalizing matrix multiplication to two matrices of any dimension and preserving the main properties of ordinary matrix multiplication. This method has wide applications in solving complex mathematical and engineering problems, such as FSMs (Dou et al., 2019; Deng et al., 2022; Yan et al., 2022, 2023), information systems (Li HT et al., 2018; Wang B and Feng, 2019; Wang YH et al., 2019), Boolean networks (Kong et al., 2023; Li WR and Li, 2023; Sun and Li, 2023), engineering electronics (Wu and Shen, 2017; Lv et al., 2022), and fuzzy control systems (Wang SL and Li, 2019; Chang et al., 2022; Wang XJ et al., 2022).

Most of the research results on FSMs using traditional formal language methods have been applied to conventional FSMs. Moreover, the results of these studies do not apply to EFSMs. Therefore, we introduce the concepts of controllability, reachability, and stabilizability from control theory into the bilinear dynamic system model of EFSMs under the framework of STP. These concepts enable us to study EFSMs like studying control theory systems. First, we construct the bilinear dynamic system model of the EFSM and achieve the purpose of studying the EFSM using algebraic methods. Second, we obtain the concept of controllability, reachability, and stabilizability for the bilinear dynamic system model of the EFSM, and establish a foundation for analyzing controllability, reachability, and stabilizability from a mathematical computation perspective. Finally, we design an algorithm based on mathematical computation to determine the controllability and stabilizability of the bilinear dynamic system model of the EFSM. Compared with traditional methods, our algorithm does not require complex logical symbolic operations and can easily find the needed matrices according to definitions. It requires only algebraic calculations to

determine the controllability and stabilizability of the dynamic system model of the EFSM.

2 Preliminaries

Drawing inspiration from previous research in the field of EFSMs (Foster et al., 2018; Goorden et al., 2020; Lin et al., 2020), we present the main objective of this paper.

Definition 1 An EFSM is a seven-tuple: $A = (L, D, \Sigma, A_s, L_0, D_0, L_m)$, where $L = \{l_1, l_2, \dots, l_n\}$ is a finite set of states, $D = D_1 \times D_2 \times \dots \times D_p$ is a domain of p one-dimensional data variables, Σ is a finite set of events, $L_0 \subseteq L$ is a set of initial states, $D_0 = D_0^1 \times D_0^2 \times \dots \times D_0^p$ is a set of initial data values, $L_m \subseteq L$ is a set of marked states, A_s is a finite set of transitions, and each transition $\alpha \in A_s$ is a five-tuple $\alpha = (o_\alpha, t_\alpha, \sigma_\alpha, g_\alpha, f_\alpha)$ with $o_\alpha \in L$ being the origin state of α , $t_\alpha \in L$ being the terminal state of α , $\sigma_\alpha \in \Sigma$ being the transition label, $g_\alpha \subseteq D$ being the enabling guard of string e , and $f_\alpha: D \rightarrow D$ being the data update function.

If the current state is o_α and g_α is judged to be true, $\alpha = (o_\alpha, t_\alpha, \sigma_\alpha, g_\alpha, f_\alpha)$ will transition. At this point, the state t_α is reachable, and the data update function f_α updates the current value.

Definition 2 (Zahn et al., 2022) A state $l_i \in L$ is said to be controllable to $l_j \in L$, if for $l_0 = l_i$ there is an input string $e = e_1 e_2 \dots e_t \in \Sigma$, so that $l_i(t+1) = l_j$, where l_i and l_j are different states in L , $l_i(t+1)$ is the state at time $t+1$, e is called the input sequence, and $l_i \xrightarrow{e_1} l_{i+1} \xrightarrow{e_2} \dots \xrightarrow{e_t} l_j$ is the state transition path.

Then, if the state $l_i \in L$ is controllable to any state $l_j \in L$, the state $l_i \in L$ is said to be controllable. In the same way, if the state $l_j \in L$ is reachable from any state $l_i \in L$, the state $l_j \in L$ is said to be reachable.

Next, we give the concepts of controllability, reachability, and stabilizability of nonempty sets of states.

Given two nonempty sets $L_1 \subseteq L$ and $L_2 \subseteq L$, we assume that $L_1 \cup L_2 = L$, $L_1 \cap L_2 = \emptyset$, with $l_\alpha \in L_1$ and $l_\beta \in L_2$.

Then, the nonempty set of states $L_1 \subseteq L$ is said to be controllable if for any state $l_\beta \in L_2$, there exists a state $l_\alpha \in L_1$ such that l_α is controllable to l_β . In the same way, the nonempty set of states $L_2 \subseteq L$ is said to be reachable if for any state $l_\alpha \in L_1$, there exists a state $l_\beta \in L_2$ such that l_β is reachable from l_α .

The nonempty set of states $L_1 \subseteq L$ is said to be one-step returnable if for any state $l_{\alpha_1} \in L_1$, there exists an input sequence $e \in \Sigma$ and a state $l_{\alpha_2} \in L_1$ such that $l_{\alpha_1} \in L_1$ and $l_{\alpha_2} \in L_1$ are reachable from each other, where l_{α_1} and l_{α_2} represent different states in L_1 .

The nonempty set of states $L_1 \subseteq L$ is said to be stabilizable if L_1 is controllable and one-step returnable.

Definition 3 (Cheng et al., 2011) For $M \in \mathbb{R}^{m \times n}$ and $N \in \mathbb{R}^{p \times q}$, their STP is expressed as $M \times N$, defined as

$$M \times N := (M \otimes I_{s/n})(N \otimes I_{s/p}), \quad (1)$$

where s is the least common multiple of n and p , $\mathbb{R}^{m \times n}$ is the set of $m \times n$ real matrices, and “ \otimes ” is the Kronecker product.

Remark 1 The STP of a matrix is an extension of traditional matrix multiplication. When n is equal to p , it becomes a traditional matrix multiplication. In this paper, all matrix products are STPs, and we omit the sign \times unless otherwise specified. Although STP breaks through the dimension limitation of traditional matrix multiplication, it still retains the basic properties of the traditional matrix.

Definition 4 (Yan et al., 2015) (Swap matrix) A swap matrix $W_{[m,n]}$ an $mn \times mn$ matrix, is defined as follows: Its rows and columns are labeled by double index (i, j) , the rows are arranged by the ordered multi-index $\text{Id}[j, i; n, m]$, and the columns are arranged by the ordered multi-index $\text{Id}[i, j; m, n]$. Then, the element at the position $[(I, J), (i, j)]$ is

$$W_{[(I,J),(i,j)]} = \begin{cases} 1, & I=i \text{ and } J=j, \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

Then, for any $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$, we have

$$\begin{cases} W_{[m,n]} \times X \times Y = Y \times X, \\ W_{[n,m]} \times Y \times X = X \times Y. \end{cases} \quad (3)$$

3 Bilinear dynamic system model of EFSMs

In this section, we will construct a bilinear dynamic system model of EFSMs and compare it with the conventional FSM.

Given an EFSM, $A=(L, D, \Sigma, A_s, L_0, D_0, L_m)$, where $L=\{l_1, l_2, \dots, l_n\}$, $\Sigma=\{e_1, e_2, \dots, e_m\}$, and $D=\{d_1, d_2, \dots,$

$d_p\}$, identify l_i with $\delta_n^i (1 \leq i \leq n)$, expressed as $l_i \sim \delta_n^i$; for simplicity, δ_n^i is the vector form of l_i . Then, L can be denoted as Δ_n ; that is, $L=\{\delta_n^1, \delta_n^2, \dots, \delta_n^n\}$. Similarly, for Σ , identify e_j with $\delta_m^j (1 \leq j \leq m)$ (denoted as $e_j \sim \delta_m^j$), δ_m^j is the vector form of e_j , and Δ_m is used to denote Σ , i.e., $\Sigma=\{\delta_m^1, \delta_m^2, \dots, \delta_m^m\}$. We do the same with $d_z (1 \leq z \leq p)$; that is, identify d_z with δ_p^z , denoted as $d_z \sim \delta_p^z$, so δ_p^z is the vector form of d_z . Then D can be denoted as Δ_p , i.e., $D=\{\delta_p^1, \delta_p^2, \dots, \delta_p^p\}$.

In the EFSM, a transition $\alpha=(o_\alpha, t_\alpha, \sigma_\alpha, g_\alpha, f_\alpha)$ is enabled if the current state is o_α and the guard condition g_α is judged to be true. When the transition α is executed, the state t_α is reached, and the variables are updated by applying f_α . Then, the essence of g_α is the judgment of the input sequence. In our dynamic system model, when the judgment is false, the system output is 0; otherwise, the system output is normal. Identify g_α with $\delta_2^k (k=1 \text{ or } 2)$, denoted as $g_\alpha \sim \delta_2^k$, that is, $\delta_2^k (k=1 \text{ or } 2)$ is the vector form of g_α , i.e., $g_\alpha=\{\delta_2^1, \delta_2^2\}$.

Definition 5 $F=[F_1, F_2, \dots, F_m] \in \mathbb{R}^{n \times 2nm}$ is called the state transition structure matrix (STSM) of an EFSM $A=(L, D, \Sigma, A_s, L_0, D_0, L_m)$, defined as

$$F_{j(s,t)} := \begin{cases} 1, & \text{if } \delta_2^k = \delta_2^1 \text{ and } \delta_n^s \in f(\delta_n^t, \delta_m^j), \\ 0, & \text{if } \delta_2^k = \delta_2^2 \text{ or otherwise,} \end{cases} \quad (4)$$

where f indicates that the current state can be transferred when receiving a new event.

Remark 2 We obtain the STSM F which can describe all dynamic behaviors of an EFSM. Furthermore, we can use this property to compute the new state of the original state after receiving an event.

Definition 6 $H=[H_1, H_2, \dots, H_m] \in \mathbb{R}^{p \times 2pm}$ is called the state output structure matrix (SOSM) of an EFSM $A=(L, D, \Sigma, A_s, L_0, D_0, L_m)$, defined as

$$H_{i(s,t)} := \begin{cases} 1, & \text{if } \delta_2^k = \delta_2^1 \text{ and } \delta_p^s \in f_\alpha(\delta_n^t, \delta_m^j), \\ 0, & \text{if } \delta_2^k = \delta_2^2 \text{ or otherwise.} \end{cases} \quad (5)$$

The following are the STSM and SOSM of conventional FSMs (Yan et al., 2015), compared with the STSM and SOSM of EFSMs.

A conventional FSM is a seven-tuple $A=(X, E, Y, f_1, g_1, x^0, X^m)$, where f_1 and g_1 are the transition and output functions respectively, which are defined as

$$f_1: X \times E \rightarrow X, g_1: X \times E \rightarrow Y.$$

For an FSM $A=(X, E, Y, f, g, x^0, X^m)$, where $X=\{x_1, x_2, \dots, x_z\}$, $E=\{e_1, e_2, \dots, e_c\}$, and $Y=\{y_1, y_2, \dots, y_v\}$, identify x_f with $\delta_z^f (1 \leq f \leq z)$, expressed as $x_f \sim \delta_z^f$, and δ_z^f is the vector form of $x_f (1 \leq f \leq z)$. Then, X can be denoted as Δ_z ; that is, $X=\{\delta_z^1, \delta_z^2, \dots, \delta_z^z\}$. Similarly, for E , identify e_g with $\delta_c^g (1 \leq g \leq c)$ (denoted as $e_g \sim \delta_c^g$), δ_c^g is the vector form of e_g , and we use Δ_c to denote E , i.e., $E=\{\delta_c^1, \delta_c^2, \dots, \delta_c^c\}$. We do the same with $y_h (1 \leq h \leq v)$; that is, identify y_h with $\delta_v^h (1 \leq h \leq v)$, denoted as $y_h \sim \delta_v^h$; δ_v^h is the vector form of y_h . Then Y can be denoted as Δ_v , i.e., $Y=\{\delta_v^1, \delta_v^2, \dots, \delta_v^v\}$.

Definition 7 (Yan et al., 2015) $F'=[F'_1, F'_2, \dots, F'_m]$ is called the STSM of conventional FSM $A=(X, E, Y, f, g, x^0, X^m)$, defined as

$$F'_{i(s,f)} = \begin{cases} 1, & \delta_z^s \in f_1(\delta_z^f, \delta_c^i), \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

Definition 8 (Yan et al., 2015) $H'=[H'_1, H'_2, \dots, H'_m]$ is called the SOSM of conventional FSM $A=(X, E, Y, f, g, x^0, X^m)$, defined as

$$H'_{i(h,f)} = \begin{cases} 1, & \delta_v^h \in g_1(\delta_z^f, \delta_c^i), \\ 0, & \text{otherwise.} \end{cases} \quad (7)$$

Remark 3 In the framework of STP, F' is uniquely determined by e_i in conventional FSMs, and F is uniquely determined by e_j and g_α in EFSMs, which is the essential difference. At the same time, F' and F represent the dynamic logical evolution of all states of conventional FSMs and EFSMs, respectively. This distinction implies that some research findings and properties applicable to FSMs may not be directly applicable to EFSMs. Similarly, H' and H are the case.

Theorem 1 Let $F=[F_1, F_2, \dots, F_m]$ be the STSM of an EFSM $A=(L, D, \Sigma, A_s, L_0, D_0, L_m)$, where $L=\{l_1, l_2, \dots, l_n\}$, $\Sigma=\{e_1, e_2, \dots, e_m\}$, and $g_\alpha=\{\delta_2^1, \delta_2^2\}$. Then, after the EFSM reads input e_j at any l_i , the new state is

$$l_{i+1}=F \times \delta_m^j \times \delta_n^i \times \delta_2^k, \quad (8)$$

where δ_m^j is the vector form of the input $e_j (1 \leq j \leq m)$, δ_n^i is the vector form of state $l_i (1 \leq i \leq n)$, and δ_2^k is the judgment of the input sequence of the EFSM.

Proof The transition function of an EFSM A is $f: L_0 \times \Sigma \rightarrow L_m$, which can be expressed as

$$f(\delta_n^i, \delta_m^j)=F \times \delta_m^j \times \delta_n^i. \quad (9)$$

The EFSM A reads input $e_j (1 \leq j \leq m)$ at the state l_i and moves to the state l_{i+1} after being judged to be true, i.e., $f(l_i, e_j)=l_{i+1}$.

According to Definition 5, the column judged true by $\text{col}_i(F_j)$ contains one element equal to 1; all the other elements are 0.

Now, we can calculate $F \times \delta_m^j \times \delta_n^i$, but we should be careful. $F \times \delta_m^j \times \delta_n^i$ contains the vectors under two different decisions between state l_i and state l_{i+1} after reading input $e_j (1 \leq j \leq m)$; i.e., state l_i through state l_{i+1} is judged true or false. To make clear the state after reading the input, the true state after reading the input can be obtained by multiplying the judgment vector after $F \times \delta_m^j \times \delta_n^i$, i.e., $F \times \delta_m^j \times \delta_n^i \times \delta_2^k$.

Then, after the EFSM reads input $e_j (1 \leq j \leq m)$ at any state l_i , the new state is $l_{i+1}=F \times \delta_m^j \times \delta_n^i \times \delta_2^k$.

Remark 4 We have obtained a method for receiving events in any initial state and obtaining the new state, which provides a solid mathematical foundation for determining the controllability, reachability, and stabilizability of the EFSM. For details, we can refer to Examples 1 and 2 in Section 4.

Theorem 2 Let $H=[H_1, H_2, \dots, H_m]$ be the SOSM of an EFSM $A=(L, D, \Sigma, A_s, L_0, D_0, L_m)$, where $L=\{l_1, l_2, \dots, l_n\}$, $\Sigma=\{e_1, e_2, \dots, e_m\}$, and $g_\alpha=\{\delta_2^1, \delta_2^2\}$. Then, after the EFSM reads input $e_j (1 \leq j \leq m)$ at any state value d_i , the new state value is

$$d_{i+1}=H \times \delta_m^j \times \delta_p^i \times \delta_2^k. \quad (10)$$

The proof is omitted since it is similar to that of Theorem 1.

4 Controllability, reachability, and stabilizability of EFSMs

In this section, we study the controllability, reachability, and stabilizability of the EFSM by using the bilinear dynamic system model of the EFSM in Section 3.

Theorem 3 The state $l_i \sim \delta_n^i \in L$ is controllable to the state $l_j \sim \delta_n^j \in L$, if and only if there exists a finite input sequence $e_j \sim \delta_m^j \in \Sigma$ such that g_α is judged to be true.

Proof (Necessity) If state $l_i \sim \delta_n^i \in L$ is controllable to state $l_j \sim \delta_m^j \in L$, it means that there exists a finite sequence of inputs $e_j \sim \delta_m^j \in \Sigma$ such that the EFSM operates properly, then g_α must be judged true at this point.

(Sufficiency) We assume that there is a finite input sequence $e_j \sim \delta_m^j \in \Sigma$ that makes g_α judged to be true, and the EFSM operates properly at this time. Then, we can obtain that $l_i \sim \delta_n^i \in L$ is controllable to $l_j \sim \delta_m^j \in L$.

Example 1 Suppose that there is an EFSM A_1 as shown in Fig. 1, where IN represents the input of the EFSM, \emptyset represents that the output is 0 when g_α is judged to be false, $L = \{l_1, l_2, l_3\}$, and $\Sigma = \{e_1, e_2\}$.

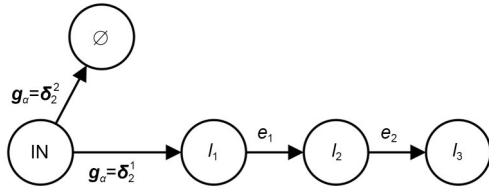


Fig. 1 State transition diagram of A_1

It can be seen from Fig. 1 that state l_3 in L can be obtained by receiving events e_1 and e_2 from states l_1 and l_2 , with a state transition path of $l_1 \xrightarrow{e_1} l_2 \xrightarrow{e_2} l_3$. At this time, state l_1 is controllable to state l_3 . We can also think of the state transition path as $l_2 \xrightarrow{e_2} l_3$, so that state l_2 is controllable to state l_3 .

Under the STP framework, the STSM of the EFSM in Fig. 1 is $F = \delta_3 [2, 0, 0, 0, 0, 0, 0, 0, 3, 0, 0, 0]$.

Then, the state transition path $l_1 \xrightarrow{e_1} l_2 \xrightarrow{e_2} l_3$ involves two transitions $F \times \delta_2^1 \times \delta_3^1 \times \delta_2^1 = \delta_3^2$ and $F \times \delta_2^2 \times \delta_3^2 \times \delta_2^2 = \delta_3^3$, and the state transition $l_2 \xrightarrow{e_2} l_3$ is $F \times \delta_3^2 \times \delta_2^2 \times \delta_3^2 = \delta_3^3$. According to Theorem 3, we can know that state l_1 is controllable to state l_3 , and that state l_2 is also controllable to state l_3 .

According to the contents and proof of Theorem 3, we can make the following corollary:

Corollary 1 The state $l_j \sim \delta_m^j \in L$ is reachable from the state $l_i \sim \delta_n^i \in L$ if and only if there exists a finite input sequence $e_j \sim \delta_m^j \in \Sigma$ such that g_α is judged to be true.

The proof of Corollary 1 is similar to the proof of Theorem 3 and is thus omitted.

Then, we give the concepts of controllability, reachability, and stabilizability of nonempty sets of states.

Theorem 4 $L_1 \subseteq L$ is controllable if and only if there exists an input sequence $e_j \sim \delta_m^j \in \Sigma$ and g_α is judged to be true such that all $l_\alpha \sim \delta_n^\alpha$ ($1 \leq \alpha \leq n$) in L_1 are controllable.

Proof (Necessity) Suppose that $L_1 \subseteq L$ is controllable. We can know for any state $l_\beta \sim \delta_m^\beta \in L_2$ ($1 \leq \beta \leq m$), and there exists a state $l_\alpha \sim \delta_n^\alpha \in L_1$ such that all $l_\alpha \sim \delta_n^\alpha$ ($1 \leq \alpha \leq n$) are controllable to $l_\beta \sim \delta_m^\beta$. Based on the conclusion of Theorem 3, there exists an input sequence $e_j \sim \delta_m^j \in \Sigma$, and g_α is judged to be true.

(Sufficiency) Since there exists an input sequence $e_j \sim \delta_m^j \in \Sigma$ and g_α is judged to be true such that all $l_\alpha \sim \delta_n^\alpha$ ($1 \leq \alpha \leq n$) in L_1 are controllable, then we can obtain that $L_1 \subseteq L$ is controllable.

Example 2 Suppose that there is an EFSM A_2 as shown in Fig. 2, where IN represents the input of the EFSM, \emptyset represents that the output is 0 when g_α is judged to be false, $L = \{l_1, l_2, l_3, l_4, l_5\}$, $L_1 = \{l_1, l_2, l_3\}$, $L_2 = \{l_4, l_5\}$, and $\Sigma = \{e_1, e_2\}$.

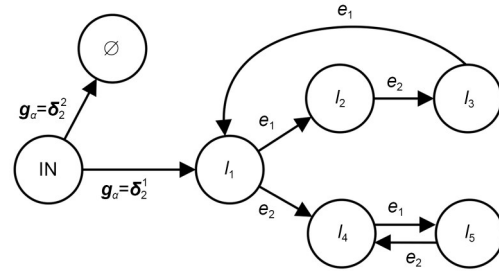


Fig. 2 State transition diagram of A_2

As shown in Fig. 2, there is a state transition path $l_1 \xrightarrow{e_1} l_2 \xrightarrow{e_2} l_3 \xrightarrow{e_1} l_1$ in L_1 , indicating that L_1 is controllable. Similarly, there is a state transition path $l_4 \xrightarrow{e_1} l_5 \xrightarrow{e_2} l_4$ in L_2 , indicating that L_2 is controllable.

Under the STP framework, the STSM of the EFSM in Fig. 2 is $F = \delta_5 [2, 0, 0, 0, 1, 0, 5, 0, 0, 0, 4, 0, 3, 0, 0, 0, 0, 0, 4, 0]$.

Then, the state transition path $l_1 \xrightarrow{e_1} l_2 \xrightarrow{e_2} l_3 \xrightarrow{e_1} l_1$ involves three transitions $F \times \delta_2^1 \times \delta_3^1 \times \delta_2^1 = \delta_5^2$, $F \times \delta_2^2 \times \delta_5^2 \times \delta_2^2 = \delta_5^3$, and $F \times \delta_2^3 \times \delta_5^3 \times \delta_2^3 = \delta_5^4$, and the state transition path $l_4 \xrightarrow{e_1} l_5 \xrightarrow{e_2} l_4$ involves two transitions $F \times \delta_4^1 \times \delta_5^1 \times \delta_4^1 = \delta_5^5$ and $F \times \delta_4^2 \times \delta_5^2 \times \delta_4^2 = \delta_5^6$. According to Theorem 4, we can know that if L_1 is controllable, then L_2 is also controllable.

Remark 5 Through Examples 1 and 2, it can be seen that under the STP framework, we only need to find

out the STSM of the EFSM according to Definition 5 and perform simple mathematical calculations to determine whether the corresponding state and state sets are controllable. The method is simple and effective.

Corollary 2 $L_2 \subseteq L$ is reachable if and only if there exists an input sequence $e_j \sim \delta_m^j \in \Sigma$ and g_a is judged to be true such that all $l_\beta \sim \delta_m^\beta$ ($1 \leq \beta \leq m$) in L_2 are reachable.

The proof of Corollary 2 is similar to the proof of Theorem 4 and is thus omitted.

Theorem 5 $L_1 \subseteq L$ is one-step returnable if and only if there exists an input sequence $e_j \sim \delta_m^j \in \Sigma$ and g_a is judged to be true such that all $l_\alpha \sim \delta_n^\alpha$ ($1 \leq \alpha \leq n$) in L_1 are mutually reachable.

Proof (Necessity) Suppose that $L_1 \subseteq L$ is one-step returnable. We can know that all $l_\alpha \sim \delta_n^\alpha$ ($1 \leq \alpha \leq n$) in L_1 are reachable from one another. Based on the content of Section 2, there exists an input sequence $e_j \sim \delta_m^j \in \Sigma$, and g_a is judged to be true.

(Sufficiency) Since there exists an input sequence $e_j \sim \delta_m^j \in \Sigma$ and g_a is judged to be true such that all $l_\alpha \sim \delta_n^\alpha$ ($1 \leq \alpha \leq n$) in L_1 are mutually reachable, then we can obtain that $L_1 \subseteq L$ is one-step returnable.

Based on the content of Section 2, Theorem 4, and Theorem 5, we have the following results on the stabilizability of the EFSM:

Theorem 6 $L_1 \subseteq L$ is stabilizable if and only if Theorem 4 and Theorem 5 hold.

Next, we will design an algorithm to judge the controllability, one-step returnability, and stabilizability of the bilinear dynamic system model of EFSMs.

5 An illustrative example

In this section, we provide an illustrative example to confirm our results' correctness and effectiveness.

Example 3 Suppose that there is an EFSM A_3 as shown in Fig. 3, where IN represents the input of the EFSM, \emptyset represents the output being 0 when g_a is judged to be false, $L = \{l_1, l_2, l_3, l_4, l_5, l_6\}$, $L_1 = \{l_1, l_2, l_3\}$, $L_2 = \{l_4, l_5, l_6\}$, and $\Sigma = \{e_1, e_2\}$.

In Fig. 3, when the state starts at l_1 , reads input e_1 , and obtains result l_2 , l_1 receives signal e_2 and obtains result l_4 ; l_2 receives signal e_1 and obtains result l_3 , and l_2 receives signal e_2 and obtains result l_5 ; l_3 receives signal e_1 and obtains result l_1 ; l_4 receives signal e_1 and obtains result l_6 , and l_4 receives signal e_2 and obtains result l_5 ;

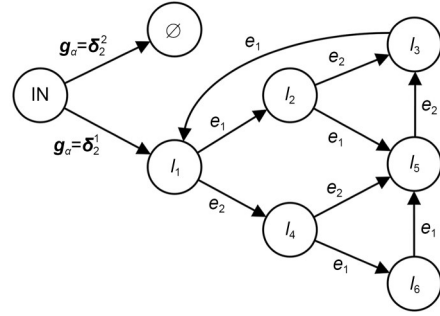


Fig. 3 State transition diagram of A_3

l_3 receives signal e_2 and obtains result l_3 ; l_6 receives signal e_1 and obtains result l_5 .

In the dynamic system model of the EFSM, $L = \{\delta_6^1, \delta_6^2, \delta_6^3, \delta_6^4, \delta_6^5, \delta_6^6\}$, $L_1 = \{\delta_6^1, \delta_6^2, \delta_6^3\}$, $L_2 = \{\delta_6^4, \delta_6^5, \delta_6^6\}$, and $e = \{\delta_2^1, \delta_2^2\}$.

Then, the state starts at $l_1 \sim \delta_6^1$, reads input $e_1 \sim \delta_2^1$, and obtains result $l_2 \sim \delta_6^2$; $l_1 \sim \delta_6^1$ receives signal $e_2 \sim \delta_2^2$ and obtains result $l_4 \sim \delta_6^4$; $l_2 \sim \delta_6^2$ receives signal $e_1 \sim \delta_2^1$ and obtains result $l_3 \sim \delta_6^3$, and $l_2 \sim \delta_6^2$ receives signal $e_2 \sim \delta_2^2$ and obtains result $l_5 \sim \delta_6^5$; $l_3 \sim \delta_6^3$ receives signal $e_1 \sim \delta_2^1$ and obtains result $l_1 \sim \delta_6^1$; $l_4 \sim \delta_6^4$ receives signal $e_1 \sim \delta_2^1$ and obtains result $l_6 \sim \delta_6^6$, and $l_4 \sim \delta_6^4$ receives signal $e_2 \sim \delta_2^2$ and obtains result $l_5 \sim \delta_6^5$; $l_5 \sim \delta_6^5$ receives signal $e_2 \sim \delta_2^2$ and obtains result $l_3 \sim \delta_6^3$; $l_6 \sim \delta_6^6$ receives signal $e_1 \sim \delta_2^1$ and obtains result $l_5 \sim \delta_6^5$.

Next, we use Algorithm 1 to judge the controllability, one-step returnability, and stabilizability of the EFSM A_3 .

We find the STSM F , i.e., $F = \delta_6 [2, 0, 5, 0, 1, 0, 6, 0, 0, 0, 5, 0, 4, 0, 3, 0, 0, 0, 5, 0, 3, 0, 0, 0]$.

Step 1: based on Theorem 3, we can determine whether the state of the EFSM A_3 is controllable: $F \times \delta_2^1 \times \delta_6^1 \times \delta_2^1 = \delta_6^2$, $F \times \delta_2^2 \times \delta_6^1 \times \delta_2^1 = \delta_6^4$, $F \times \delta_2^2 \times \delta_6^2 \times \delta_2^1 = \delta_6^3$, $F \times \delta_2^1 \times \delta_6^2 \times \delta_2^1 = \delta_6^5$, $F \times \delta_2^1 \times \delta_6^3 \times \delta_2^1 = \delta_6^1$, $F \times \delta_2^2 \times \delta_6^4 \times \delta_2^1 = \delta_6^5$, $F \times \delta_2^1 \times \delta_6^4 \times \delta_2^1 = \delta_6^6$, $F \times \delta_2^2 \times \delta_6^5 \times \delta_2^1 = \delta_6^3$, $F \times \delta_2^1 \times \delta_6^6 \times \delta_2^1 = \delta_6^5$.

As shown above, we test the controllability of all states using algebraic methods, and we can conclude that states $l_1 - l_6$ are controllable.

Step 2: based on Theorem 4, combined with step 1, we know that nonempty sets of states $S_{C1} = L_1$ and $S_{C2} = L_2$ are controllable.

Step 3: based on Corollary 1, we can determine whether the state of the EFSM A_3 is reachable: $F \times \delta_2^1 \times \delta_6^1 \times \delta_2^1 = \delta_6^2$, $F \times \delta_2^2 \times \delta_6^1 \times \delta_2^1 = \delta_6^4$, $F \times \delta_2^2 \times \delta_6^2 \times \delta_2^1 = \delta_6^3$, $F \times \delta_2^1 \times \delta_6^2 \times \delta_2^1 = \delta_6^5$, $F \times \delta_2^1 \times \delta_6^3 \times \delta_2^1 = \delta_6^1$, $F \times \delta_2^2 \times \delta_6^4 \times \delta_2^1 = \delta_6^5$, $F \times \delta_2^1 \times \delta_6^4 \times \delta_2^1 = \delta_6^6$, $F \times \delta_2^2 \times \delta_6^5 \times \delta_2^1 = \delta_6^3$, $F \times \delta_2^1 \times \delta_6^6 \times \delta_2^1 = \delta_6^5$.

Algorithm 1 Finding controllable, returnable, and stabilizable states

Input: STSM F of EFSM A

Output: The controllable state set S_c , the one-step returnable state set S_o , and the stabilizable state set S

1. Let $i=1$. For l_i , if there exists a finite input sequence $e_j-\delta_m^j \in \Sigma$ such that $F \times \delta_m^j \times \delta_n^j \times \delta_2^j = \delta_n^j$, then l_i is controllable; otherwise l_i is not controllable.
2. Let $i=i+1$, and repeat step 1 until $i=n$. Check whether l_i is controllable. If so, put it into the controllable state set S_c .
3. Let $i=1$, and consider l_i . If there exists a set of states such that $F \times \delta_m^i \times \delta_n^i \times \delta_2^i = \delta_n^{i+1}$, $F \times \delta_m^i \times \delta_n^{i+1} \times \delta_2^i = \delta_n^{i+2}$, ..., $F \times \delta_m^i \times \delta_n^{i+n} \times \delta_2^i = \delta_n^i$, then these states are mutually reachable.
4. Let $i=i+1$, and repeat step 3 until $i=n$. If all $l_\alpha-\delta_n^\alpha$ ($1 \leq \alpha \leq n$) in L_1 are mutually reachable, then put them into the one-step returnable state set S_o .
5. The stabilizable state set of EFSM is $S=S_c \cap S_o$.

As shown above, we test the reachability of all states using algebraic methods, and we can conclude that states l_1-l_6 are reachable.

Step 4: in an EFSM A_3 , only $\delta_6^1 \rightarrow \delta_6^2 \rightarrow \delta_6^3 \rightarrow \delta_6^1$ ($F \times \delta_2^1 \times \delta_6^1 \times \delta_2^1 = \delta_6^2$, $F \times \delta_2^2 \times \delta_6^2 \times \delta_2^2 = \delta_6^3$, $F \times \delta_2^3 \times \delta_6^3 \times \delta_2^3 = \delta_6^1$), $\delta_6^2 \rightarrow \delta_6^3 \rightarrow \delta_6^1 \rightarrow \delta_6^2$ ($F \times \delta_2^2 \times \delta_6^2 \times \delta_2^2 = \delta_6^3$, $F \times \delta_2^3 \times \delta_6^3 \times \delta_2^3 = \delta_6^1$, $F \times \delta_2^1 \times \delta_6^1 \times \delta_2^1 = \delta_6^2$), $\delta_6^3 \rightarrow \delta_6^1 \rightarrow \delta_6^2 \rightarrow \delta_6^3$ ($F \times \delta_2^3 \times \delta_6^3 \times \delta_2^3 = \delta_6^1$, $F \times \delta_2^1 \times \delta_6^1 \times \delta_2^1 = \delta_6^2$, $F \times \delta_2^2 \times \delta_6^2 \times \delta_2^2 = \delta_6^3$) in L_1 are one-step returnable. That is, $S_o=L_1$ is one-step returnable.

Step 5: in this case, we can find that the stabilizable state set of the EFSM A_3 is $S=S_c \cap S_o=L_1$.

6 Conclusions

In this paper, controllability, reachability, and stabilizability of EFSMs were studied by using the STP method. First, we established the bilinear dynamic system model of the EFSM under the STP framework, which enabled us to analyze the dynamic behavior of the EFSM. Based on this, we studied the controllability, reachability, and stabilizability of the EFSM and proposed the relevant theorems. Further, we designed an algorithm to determine the related properties of EFSMs. In addition, the observability of the EFSM can be analyzed based on the above method. Overall, our approach contributes to a more in-depth study of EFSMs.

Contributors

Chao DONG and Yongyi YAN designed the research. Huiqin LI and Jumei YUE processed the data. Chao DONG drafted the paper. Yongyi YAN, Huiqin LI, and Jumei YUE helped organize the paper. Chao DONG, Yongyi YAN, Huiqin LI, and Jumei YUE revised and finalized the paper.

Conflict of interest

All the authors declare that they have no conflict of interest.

Data availability

The authors confirm that the data supporting the findings of this study are available within the paper.

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