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Derivation of the multi-model generalized labeled multi-Bernoulli filter: a solution to multi-target hybrid systems*

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Abstract: In this study, we extend traditional (single-target) hybrid systems to multi-target hybrid systems with a focus on the multi-maneuvering-target tracking system. This system consists of a continuous state, a discrete and switchable state, and a discrete, time-constant, and unique state. By defining a new generalized labeled multi-Bernoulli density, we prove that it is closed under the Chapman-Kolmogorov prediction and Bayes update for multi-target hybrid systems. In other words, we provide the exact derivation of a solution to this system, i.e., the multi-model generalized labeled multi-Bernoulli filter, which has been developed without strict proof.

Key words: Multi-maneuvering-target tracking; Multi-model; Generalized labeled multi-Bernoulli filter; Multi-target hybrid systems

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1 Introduction

A hybrid system consists of continuous state evolutions and discrete state (or mode) transitions (Seah and Hwang, 2009). It has attracted considerable interest in the field of state estimation and control from researchers in academic and industrial communities, because it can model complicated behaviors of various estimation and control systems, such as robotic, transportation, and process control systems (Hwang et al., 2006). Hybrid estimation involves the continuous and discrete states of a hybrid system. Many estimation algorithms have been used in different applications (Seah and Hwang, 2009), such as target tracking, signal processing, and fault diagnosis. In this study, we focus on the application of target tracking, more specifically, multi-maneuvering-target tracking (MMTT).

The prevailing approaches in hybrid estimation are based on multi-model (MM) algorithms, for instance, the generalized pseudo-Bayesian (GPB) (Chang and Athans, 1978) and the interacting multi-model (IMM) estimator (Bar-Shalom et al., 2005; Li and Jilkov, 2005). These algorithms, especially the IMM algorithm, have proven to have excellent performance at low computational cost. Nevertheless, they are limited to a single-maneuvering-target scenario. To track multiple maneuvering targets, the MM approach is combined with traditional multi-target tracking (MTT) techniques, e.g., multi-hypothesis tracking (MHT) (Reid, 1979) and joint probabilistic data association (JPDA) (Fortmann et al., 1980). However, these algorithms are based on a conventional random vector (RV) framework.

As promising alternative approaches for MTT, random finite set (RFS) based algorithms have attracted significant attention. The RFS framework is a natural extension of the RV framework. It is a system-level, top-down, and direct generalization of ordinary single-sensor single-target engineering statistics in

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the realm of multi-sensor multi-target detection and tracking, and provides powerful new conceptual and computational methods for dealing with multi-sensor multi-target detection and tracking problems (Mahler, 2014). For example, RFS algorithms can solve the problem of simultaneous estimation of target states and the target number. Hence, it is suitable for the MTT problem. However, conventional RV-based methods need to heuristically decompose the MTT problem into track management and state filtering. The target states are estimated through state filtering, and the number of targets can be estimated through track management.

Under the RFS framework, many well-known MTT algorithms have been developed, including the probability hypothesis density (PHD) (Vo BN and Ma, 2006), cardinalized PHD (CPHD) (Vo BT et al., 2007), multi-Bernoulli (MB) filter (the cardinality balanced multi-target multi-Bernoulli (CBMeMBer) filter) (Vo BT et al., 2009), labeled MB (LMB) (Reuter et al., 2014), marginalized GLMB (M-GLMB) (the marginalized δ-GLMB (Mδ-GLMB)) (Papi et al., 2015), and generalized LMB (GLMB) filter (also named the Vo-Vo filter) (Vo BT and Vo, 2013; Vo BN et al., 2014, 2017). Meanwhile, many studies combine the MM and RFS algorithms to solve the MMTT problem, e.g., MM-PHD (Punithakumar et al., 2008; Pasha et al., 2009; Wood, 2011; Sithiravel et al., 2016), MM-CPHD (Georgescu and Willett, 2012; Mahler, 2012), MM-MB (Dunne and Kirubarajan, 2013; Yang et al., 2013), MM-LMB (Reuter et al., 2015), and MM-GLMB (Jiang et al., 2016; Punchihewa et al., 2016; Punchihewa, 2017; Yi et al., 2017). The MM-GLMB filter is expected to have better tracking performance than the MM versions of other approximate RFS approaches, because the GLMB filter is the first analytical solution to the multi-target Bayesian filter (Mahler, 2007).

In these approaches, the continuous kinematic state is augmented with an additional discrete mode state, and the mode state evolves as a Markov process with constant mode transition probabilities independent of the continuous state. Applications of the augmented state method to PHD, CPHD, and MB filters are straightforward since the structures of these filters are relatively simple. For example, the intensity of the PHD filter is a function only on the single-target state space. However, the GLMB filter is more complex and involves a labeled multi-target state space. One of the important properties of the GLMB filter is that the GLMB density is closed under the Chapman-Kolmogorov prediction and Bayes update (Vo BT and Vo, 2013). It is unknown whether directly incorporating an augmented discrete state will change the closure property of the GLMB filter. Furthermore, this question has not been answered (Jiang et al., 2016; Punchihewa et al., 2016; Punchihewa, 2017; Yi et al., 2017) although the MM-GLMB filter was proposed. The motivation of this study is to address this problem. By defining a new H-GLMB (a GLMB for hybrid systems) density, we will prove that the H-GLMB density is closed under the Chapman-Kolmogorov prediction and Bayes update for multi-target hybrid systems.

2 Notations and definitions

Symbols commonly used in the labeled RFS context are adopted throughout this paper (Vo BT and Vo, 2013; Vo BN et al., 2014). A finite set of interesting states is denoted as $X=\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{|X|}\}$, and its cardinality is denoted by $|X|$. For each $\boldsymbol{x}_i \in X$ (i=1, 2, …, $|X|$), $\boldsymbol{x}_i \triangleq [\boldsymbol{\xi}, \mu, l]$, where $\boldsymbol{\xi} \in \mathbb{X}$, $\mu \in \mathbb{U}$, and $l \in \mathbb{L}$ denote a continuous vector (e.g., a kinematic state), a discrete and switchable mode variable (e.g., a mode state), and a discrete, time-constant, and unique label variable (e.g., an ID state), respectively, and $\mathbb{X}$, $\mathbb{U}$, and $\mathbb{L}$ denote the continuous state space, discrete mode space, and discrete and unique label space, respectively. $F(\mathbb{X})$ is the collection of finite subsets of $\mathbb{X}$, and $F_n(\mathbb{X})$ denotes $F(\mathbb{X})$ with $\mathbb{X}$ having exactly n elements. $\Delta(X) = \delta_{|X|}(|L(X)|)$ denotes the distinct label indicator, where $L: \mathbb{X} \times \mathbb{U} \times \mathbb{L} \to \mathbb{L}$ is the projection of $L(\boldsymbol{\xi}, \mu, l)=l$ and $L(X) \triangleq \{l : (\boldsymbol{\xi}, \mu, l) \in X\}$. δ is the generalized Kronecker delta (i.e., $\delta_Y(X)=1$ if $X=Y$, and $\delta_Y(X)=0$ otherwise), and supports arbitrary arguments such as sets, vectors, and integers. $1_{\mathbb{L}}(\cdot)$ represents the indicator function of $\mathbb{L}$. The multi-target exponential is defined by $f^X = \prod_{\boldsymbol{x} \in X} f(\boldsymbol{x})$ with $h^{\varnothing}=1$.

For any real-valued functions $h: \mathbb{X} \times \mathbb{L} \to \mathbb{R}$ and $f: F(\mathbb{X} \times \mathbb{L}) \to \mathbb{R}$, the conventional vector integral and set integral are defined by (Vo BT and Vo, 2013)

$$\int h(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \triangleq \sum_{l\in\mathbb{L}}\int_{\mathbb{X}} h(\boldsymbol{\xi},l)\mathrm{d}\boldsymbol{\xi}, \tag{1}$$

$$\begin{aligned}&\int f(X)\delta X\\ &\triangleq \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{(l_1,\cdots,l_n)\in\mathbb{L}^n}\int_{\mathbb{X}^n} f(\{(\boldsymbol{\xi}_1,l_1),\cdots,(\boldsymbol{\xi}_n,l_n)\})\mathrm{d}(\boldsymbol{\xi}_1,\cdots,\boldsymbol{\xi}_n).\end{aligned} \tag{2}$$

To derive the MM-GLMB filter, the standard inner product, which is defined by $\langle\alpha,\beta\rangle=\int\alpha(x)\beta(x)\mathrm{d}x$ for real-valued functions α and β, or defined by $\langle\alpha,\beta\rangle=\sum_{i=0}^{\infty}\alpha(i)\beta(i)$ for real sequences α and β, is generalized for functions α and β whose arguments consist of continuous and discrete variables. The generalized inner product is defined by

$$\langle\alpha,\beta\rangle=\langle\alpha(\cdot,\cdot),\beta(\cdot,\cdot)\rangle \triangleq \sum_{\mu}\int\alpha(\boldsymbol{\xi},\mu)\beta(\boldsymbol{\xi},\mu)\mathrm{d}\boldsymbol{\xi}. \tag{3}$$

Accordingly, the conventional vector integral and set integral need to be extended, and the extended vector integral and set integral including the mode variables are defined as follows:

$$\int h(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \triangleq \sum_{l\in\mathbb{L}}\sum_{\mu\in\mathbb{U}}\int_{\mathbb{X}} h(\boldsymbol{\xi},l,\mu)\mathrm{d}\boldsymbol{\xi}, \tag{4}$$

$$\begin{aligned}&\int f(X)\delta X\\ &\triangleq \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{(l_1,\cdots,l_n)\in\mathbb{L}^n}\sum_{(\mu_1,\cdots,\mu_n)\in\mathbb{U}^n}\int_{\mathbb{X}^n} f(\{(\boldsymbol{\xi}_1,l_1,\mu_1),\cdots,\\ &\quad(\boldsymbol{\xi}_n,l_n,\mu_n)\})\mathrm{d}(\boldsymbol{\xi}_1,\cdots,\boldsymbol{\xi}_n)\end{aligned} \tag{5}$$

for $h:\mathbb{X}\times\mathbb{U}\times\mathbb{L}\rightarrow\mathbb{R}$ and $f:F(\mathbb{X}\times\mathbb{U}\times\mathbb{L})\rightarrow\mathbb{R}$. In addition, the following two definitions are introduced:

Definition 1 H-GLMB RFS is an RFS on $F(\mathbb{X}\times\mathbb{U}\times\mathbb{L})$ and is distributed according to

$$\pi(X)=\Delta(X)\sum_{c\in\mathbb{C}}w^{(c)}(L(X))[p^{(c)}]^X, \tag{6}$$

where $\mathbb{C}$ is a discrete index space, and non-negative weight $w^{(c)}(L)$ and single-target density $p^{(c)}$ satisfy the normalization property:

$$\sum_{L\subseteq\mathbb{L},c\in\mathbb{C}}w^{(c)}(L)=\sum_{(L,c)\in F(\mathbb{L})\times\mathbb{C}}w^{(c)}(L)=1, \tag{7}$$

$$\sum_{\mu}\int p^{(c)}(\boldsymbol{\xi},\mu,l)\mathrm{d}\boldsymbol{\xi}=1. \tag{8}$$

Definition 2 H-LMB RFS is an RFS on $F(\mathbb{X}\times\mathbb{U}\times\mathbb{L})$ and is distributed according to

$$\pi(X)=\Delta(X)w(L(X))p^X. \tag{9}$$

Remark 1 Although the H-GLMB and H-LMB densities are in the same form as the standard GLMB and LMB ones (Vo BT and Vo, 2013), repectively, the H-GLMB and H-LMB densities are defined on $F(\mathbb{X}\times\mathbb{U}\times\mathbb{L})$, while the GLMB and LMB densities are defined on $F(\mathbb{X}\times\mathbb{L})$. Hence, GLMB/LMB is a special case of H-GLMB/H-LMB when $|\mathbb{U}|=1$. Moreover, as LMB is a special case of GLMB (Vo BT and Vo, 2013), H-LMB is also a special case of H-GLMB with one term (Wu et al., 2020).

3 Multi-target hybrid system model

As mentioned earlier, many applications fall within the state estimation of the hybrid system category. We use the MMTT problem to explain hybrid estimation since MMTT is a typical hybrid system. Specifically, MMTT involves the joint estimation of continuous vectors (i.e., kinematic states), discrete and switchable mode variables (i.e., mode states), and discrete, time-constant, and unique label variables (i.e., identity or ID states).

Consider an MMTT scenario. Suppose that at time $k-1$, there are $|X|$ targets whose states are $X=\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{|X|}\}$ (for clarity, the subscript k for the current time is omitted and for the next time is indicated by the subscript "+"). Next time, new targets may appear and some of the previous targets may die with probability $1-p_S(\boldsymbol{x})$ or continue to survive with survival probability $p_S(\boldsymbol{x})$. If a target with state $\boldsymbol{x}$ survives, then it evolves to a new state $\boldsymbol{x}_+$ with the following transition probability:

$$\begin{aligned}\phi(\boldsymbol{x}_+|\boldsymbol{x})&=\phi(\boldsymbol{\xi}_+,\mu_+,l_+|\boldsymbol{\xi},\mu,l)\\ &=\delta_l(l_+)\phi(\boldsymbol{\xi}_+|\boldsymbol{\xi},\mu_+,l)\tau(\mu_+|\mu),\end{aligned} \tag{10}$$

where $\delta_l(l_+)$ indicates that label l is time-constant, and $\phi(\xi_+|\xi,\mu_+,l)$ denotes the state transition probability density. $\tau(\mu_+|\mu)$ is the mode transition probability from μ to μ_+ and satisfies

$$\sum_{\mu_+} \tau(\mu_+ \mid \mu) = 1. \tag{11}$$

Remark 2 In some applications, mode transition depends on the continuous state and can be described by a continuous-state-dependent mode transition probability (Seah and Hwang, 2009). Nevertheless, the mode transition probability is assumed to be independent of the continuous state in an IMM algorithm and MM versions of RFS methods.

If the newborn targets and surviving targets are independent, then the multi-target transition kernel (incorporating the mode variable, the target spawning case is omitted here) can be obtained by (Vo BT and Vo, 2013; Vo BN et al., 2014)

$$\begin{aligned}&\phi(X_+ \mid X)\\ &= \pi_{\mathrm{S}}(X_+ \cap (\mathbb{X}\times\mathbb{U}\times\mathbb{L}) \mid X)\pi_\gamma(X_+ - (\mathbb{X}\times\mathbb{U}\times\mathbb{L})),\end{aligned} \tag{12}$$

where the density of set B of the newborn targets is assumed to be H-LMB, i.e.,

$$\pi_\gamma(B) = \Delta(B) w_\gamma(L(B)) p_\gamma^B, \tag{13}$$

and the set S of the surviving targets is distributed according to

$$\pi_{\mathrm{S}}(S \mid X) = \Delta(S)\Delta(X) 1_{L(X)}(L(S))[\Phi(S;\cdot)]^X \tag{14}$$

with

$$\begin{aligned}&\Phi(S;\boldsymbol{\xi},\mu,l)\\ &= [1-1_{L(S)}(l)]q_{\mathrm{S}}(\boldsymbol{\xi},\mu,l)\\ &+ \sum_{(\boldsymbol{\xi}_+,\mu_+,l_+)\in S} \delta_l(l_+) p_{\mathrm{S}}(\boldsymbol{\xi},\mu,l)\phi(\boldsymbol{\xi}_+ \mid \boldsymbol{\xi},l,\mu_+)\tau(\mu_+ \mid \mu).\end{aligned} \tag{15}$$

For multiple targets with a multi-target state $X=\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{|X|}\}$ at time k, the measurements received by a sensor are represented as $Z=\{\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_{|Z|}\}$. Some of these measurements may come from Poisson clutter with intensity κ and some may originate from the detected targets. Each target with a single-target state $\boldsymbol{x} = [\boldsymbol{\xi}, \mu, l] \in X$ is detected with the detection probability $p_{\mathrm{D}}(\boldsymbol{x})$. Conditional on detection, the probability density function of observation $\boldsymbol{z}\in Z$ originating from this target is denoted by the single-target likelihood $g(\boldsymbol{z}|\boldsymbol{x})$.

If measurements are independent of clutter and each detection from detected targets, the multi-target likelihood (like the single-target transition and multi-target transition kernel $\phi(\cdot|\cdot)$, the single-target likelihood and multi-target likelihood use the same symbol $g(\cdot|\cdot)$, but they can be easily distinguished by their arguments), including the mode variable, can be obtained by (Vo BT and Vo, 2013; Vo BN et al., 2014)

$$g(Z \mid X) = \mathrm{e}^{-\langle \kappa, 1\rangle} \kappa^Z \sum_{\theta\in\Theta} \delta_{\theta^{-1}(\{0:|Z|\})}(L(X))[\varphi_Z(\cdot;\theta)]^X, \tag{16}$$

where Θ is the set of positive 1-1 maps and $\theta: \mathbb{L} \to \{0:|Z|\} \triangleq \{0,1,\cdots,|Z|\}$ (Vo BN et al., 2017), satisfying that if $\theta(l)=\theta(l')>0$, then $l=l'$ (i.e., no two distinct labels are mapped to the same positive measurement index).

$$\begin{aligned}\varphi_Z(\boldsymbol{x};\theta) &= \delta_0(\theta(l))(1-p_{\mathrm{D}}(\boldsymbol{x}))\\ &+ [1-\delta_0(\theta(l))]\, p_{\mathrm{D}}(\boldsymbol{x}) g(z_{\theta(l)} \mid \boldsymbol{x}) / \kappa(z_{\theta(l)}).\end{aligned} \tag{17}$$

4 MM-GLMB filter

For the above multi-target hybrid system, a general solution is the MM-GLMB filter, which is described by the following two propositions (Wu et al., 2020):

Proposition 1 $\mathbb{L}$ and $\mathbb{B}$ are denoted by the label space for (survived) targets at time k (including those born prior to k) and the label space for targets born at time k. If the birth density is H-LMB (with birth label space $\mathbb{B}$) as given in Eq. (13), the prior multi-target density is H-GLMB (with label space $\mathbb{L}$) as given in Eq. (6), and the multi-target transition kernel is as given in Eq. (12), then the predicted multi-target density is also H-GLMB given by

$$\pi_+(X_+) = \Delta(X_+) \sum_{c\in\mathbb{C}} w_+^{(c)}(L(X_+))[p_+^{(c)}]^{X_+} \tag{18}$$

with label space $\mathbb{L}_+ = \mathbb{L} \cup \mathbb{B}$ and

$$w_+^{(c)}(L) = w_{\mathrm{S}}^{(c)}(L \cap \mathbb{L}) w_\gamma(L - \mathbb{L}), \tag{19}$$

$$p_+^{(c)}(\xi_+,\mu_+,l) = 1_{\mathbb{L}}(l)p_{+,\mathrm{S}}^{(c)}(\xi_+,\mu_+,l)+1_{\mathbb{B}}(l)p_\gamma(\xi_+,\mu_+,l), \tag{20}$$

$$w_{\mathrm{S}}^{(c)}(L)=[\eta_{\mathrm{S}}^{(c)}]^L\sum\nolimits_{I\supseteq L}[1-\eta_{\mathrm{S}}^{(c)}]^{I-L}w^{(c)}(I), \tag{21}$$

$$\eta_{\mathrm{S}}^{(c)}(l)=\langle p_{\mathrm{S}}(\cdot,\cdot,l),p^{(c)}(\cdot,\cdot,l)\rangle, \tag{22}$$

$$p_{+,\mathrm{S}}^{(c)}(\xi_+,\mu_+,l) =\frac{\langle p_{\mathrm{S}}(\cdot,\cdot,l)\phi(\xi_+|\cdot,\mu_+,l)\tau(\mu_+|\cdot),p^{(c)}(\cdot,\cdot,l)\rangle}{\eta_{\mathrm{S}}^{(c)}(l)}, \tag{23}$$

where $w_\gamma(L)$ and p_γ are given in the birth model (13).

Proposition 2 If the predicted multi-target density is H-GLMB (18) and the multi-target likelihood is given by Eq. (16), then the posterior density is also H-GLMB in the form of

$$\pi(X|Z)=\Delta(X)\sum_{c\in\mathbb{C}}\sum_{\theta\in\Theta}w_Z^{(c,\theta)}(L(X))[p^{(c,\theta)}(\cdot|Z)]^X, \tag{24}$$

where

$$w_Z^{(c,\theta)}(L)=\frac{\delta_{\theta^{-1}(\{0:|Z|\})}(L)[\eta_Z^{(c,\theta)}]^L w^{(c)}(L)}{\sum_{c\in\mathbb{C}}\sum_{\theta\in\Theta}\sum_{J\subseteq\mathbb{L}}\delta_{\theta^{-1}(\{0:|Z|\})}(J)[\eta_Z^{(c,\theta)}]^J w^{(c)}(J)}, \tag{25}$$

$$p^{(c,\theta)}(\xi,\mu,l\,|\,Z)=\frac{p^{(c)}(\xi,\mu,l)\varphi_Z(\xi,\mu,l;\theta)}{\eta_Z^{(c,\theta)}(l)}, \tag{26}$$

$$\eta_Z^{(c,\theta)}(l)=\langle p^{(c)}(\cdot,\cdot,l),\varphi_Z(\cdot,\cdot,l;\theta)\rangle. \tag{27}$$

Remark 3 Propositions 1 and 2 complete the MM-GLMB filter, which was used in tracking multiple maneuvering targets hidden in the Doppler blind zone (Wu et al., 2020). These two propositions are consistent with the results in Jiang et al. (2016), Punchihewa et al. (2016), Punchihewa (2017), and Yi et al. (2017). Formally, Propositions 1 and 2 are similar to Propositions 8 and 7 (which complete the GLMB filter) in Vo BT and Vo (2013), respectively. The key difference between the MM-GLMB and GLMB filters is the incorporation of the mode variable in the former. Actually, the GLMB filter is a special case of the MM-GLMB filter when $|\mathbb{U}|=1$. Intuitively, the MM-GLMB filter can be obtained from the GLMB filter through variable substitution (i.e., replacing the original state ξ in the GLMB filter with the augmented state (ξ,μ)). The above-mentioned variable substitution is partially the idea adopted in Jiang et al. (2016), Punchihewa et al. (2016), Punchihewa (2017), and Yi et al. (2017). Nevertheless, the soundproof or specific derivation of the MM-GLMB filter was not provided.

5 Exact derivation

5.1 Motivation of the derivation

To clarify this work, i.e., why the existing MM-GLMB filter should be derived, we use the PHD filter as an example for comparison and elaboration.

The PHD filter includes the following prediction and update equations:

$$v_+(\xi)=v_\gamma(\xi)+\int p_{\mathrm{S}}(\xi')\phi(\xi\,|\,\xi')v(\xi')\mathrm{d}\xi', \tag{28}$$

$$v(\xi)=[1-p_{\mathrm{D}}(\xi)]v_+(\xi) +\sum_{z\in Z}\frac{p_{\mathrm{D}}(\xi)g(z\,|\,\xi)v_+(\xi)}{\kappa(z)+\int p_{\mathrm{D}}(\tilde{\xi})g(z\,|\,\tilde{\xi})v_+(\tilde{\xi})\mathrm{d}\tilde{\xi}}, \tag{29}$$

where v denotes the prior intensity if its argument is ξ' or the posterior intensity if its argument is ξ, v_+ the predicted intensity at time k, v_γ the newborn intensity at time k, Z the measurement set, p_{S} the survival probability, $\phi(\cdot|\cdot)$ the transition density, p_{D} the detection probability, and $g(\cdot|\cdot)$ the measurement likelihood.

Through variable substitution and using the generalized inner product (3), an MM extension of the PHD filter (i.e., the MM-PHD filter) can be obtained as follows:

$$v_+(\xi,\mu)=v_\gamma(\xi,\mu)+\sum\nolimits_{\mu'}\Big[\int p_{\mathrm{S}}(\xi',\mu')\cdot\phi(\xi,\mu\,|\,\xi',\mu')v(\xi',\mu')\Big]\mathrm{d}\xi', \tag{30}$$

$$v(\xi,\mu) =[1-p_{\mathrm{D}}(\xi,\mu)]v_+(\xi,\mu) +\sum_{z\in Z}\frac{p_{\mathrm{D}}(\xi,\mu)g(z\,|\,\xi,\mu)v_+(\xi,\mu)}{\kappa(z)+\sum_{\tilde{\mu}}\int p_{\mathrm{D}}(\tilde{\xi},\tilde{\mu})g(z\,|\,\tilde{\xi},\tilde{\mu})v_+(\tilde{\xi},\tilde{\mu})\mathrm{d}\tilde{\xi}}. \tag{31}$$

The above process of using variable substitution to extend the PHD filter to the MM-PHD filter is straightforward because, for the PHD filter, the state variable involves only a random vector ξ. However, the generalization from the GLMB filter to the MM-GLMB filter is not straightforward, because the state variable of the MM-GLMB filter involves a labeled

RFS X_+ according to Eq. (18) or X according to Eq. (24).

Essentially, the GLMB filter is derived from the following multi-target Bayesian filter (Mahler, 2007):

$$\pi_+(X_+) = \int \phi(X_+ \mid X)\pi(X)\delta X, \tag{32}$$

$$\pi(X \mid Z) = \frac{g(Z \mid X)\pi(X)}{\int g(Z \mid X)\pi(X)\delta X}. \tag{33}$$

Note that for the standard GLMB filter, the labeled multi-target states X_+ and X are defined on $F(\mathbb{X} \times \mathbb{L})$; i.e., the labeled multi-target states do not contain the mode variable.

If we use variable substitution to extend the GLMB filter to the MM-GLMB filter, it is unclear how to define the set integral of the labeled RFS in Eqs. (32) and (33). In addition, one of the important properties of the GLMB filter is that the GLMB density is closed under the Chapman-Kolmogorov equation and Bayes update (Vo BT and Vo, 2013). Therefore, it is unclear whether directly incorporating the augmented discrete state changes the closure property of the GLMB filter. Unfortunately, these two problems have not been answered in Jiang et al. (2016), Punchihewa et al. (2016), Punchihewa (2017), and Yi et al. (2017). To solve the first problem, we introduce the definition in Eq. (5), which is extended from the conventional set integral (2) that does not contain the mode variable. Next, we will solve the second problem, which is to strictly derive the MM-GLMB filter.

5.2 Lemma required in the derivation

In the derivation of both Propositions 1 and 2, the following lemma is deployed:

Lemma 1 For $h: F(\mathbb{L}) \to \mathbb{R}$ and $g: \mathbb{X} \times \mathbb{U} \times \mathbb{L} \to \mathbb{R}$ being integrable on $\mathbb{X}$, we have

$$\begin{aligned} &\int \Delta(X) h(L(X)) g^X \delta X \\ &= \sum_{L \subseteq \mathbb{L}} h(L) \Big[\sum_{\mu} \int g(\boldsymbol{\xi}, \mu, \cdot) \mathrm{d}\boldsymbol{\xi} \Big]^L. \end{aligned} \tag{34}$$

Proof From Eq. (5), we have Eq. (35), where the second to the last lines follow the symmetry of $h(\{l_1,\cdots,l_n\})\prod_{i=1}^{n}\sum_{\mu_i\in\mathbb{U}}\int g(\boldsymbol{\xi}_i,\mu_i,l_i)\mathrm{d}\boldsymbol{\xi}_i$ in (l_1, …, l_n) and Lemma 12 in Vo BT and Vo (2013). Furthermore, the double sums in the second to the last lines of Eq. (35) can be combined as a sum over the subset of $\mathbb{L}$ and the last line.

$$\begin{aligned} &\int \Delta(X) h(L(X)) g^X \delta X \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(l_1,\cdots,l_n)\in\mathbb{L}^n} \delta_n(|\{l_1,\cdots,l_n\}|) h(\{l_1,\cdots,l_n\}) \\ &\quad \cdot \sum_{(\mu_1,\cdots,\mu_n)\in\mathbb{U}^n} \prod_{i=1}^{n} \int g(\boldsymbol{\xi}_i,\mu_i,l_i)\mathrm{d}\boldsymbol{\xi}_i \\ &= \sum_{n=0}^{\infty} \sum_{\{l_1,\cdots,l_n\}\in F_n(\mathbb{L})} h(\{l_1,\cdots,l_n\}) \prod_{i=1}^{n} \sum_{\mu_i\in\mathbb{U}} \int g(\boldsymbol{\xi}_i,\mu_i,l_i)\mathrm{d}\boldsymbol{\xi}_i \\ &= \sum_{L\subseteq\mathbb{L}} h(L) \Big[\sum_{\mu} \int g(\boldsymbol{\xi},\mu,\cdot)\mathrm{d}\boldsymbol{\xi} \Big]^L. \end{aligned} \tag{35}$$

Remark 4 Lemma 1 is an extension of Lemma 3 in Vo BT and Vo (2013), and the mode variable is considered. Using Lemma 1, we derive Propositions 1 and 2.

5.3 Derivation of Proposition 1

By $B = X_+ - \mathbb{X}\times\mathbb{U}\times\mathbb{L}$ and $S = X_+ \cap \mathbb{X}\times\mathbb{U}\times\mathbb{L}$ and by substituting multi-target transition kernel (12) into the Chapman-Kolmogorov equation, we have the following predicted multi-target density:

$$\begin{aligned} \pi_+(X_+) &= \int \phi(X_+ \mid X)\pi(X)\delta X \\ &= \pi_\gamma(B) \int \pi_{\mathrm{S}}(S \mid X)\pi(X)\delta X \\ &= \pi_\gamma(B)\pi_{\mathrm{S}}(S), \end{aligned} \tag{36}$$

where the birth density $\pi_\gamma(B)$ is given by Eq. (13). Hence, the key lies in the calculation of the density $\pi_{\mathrm{S}}(S)$ of the surviving multi-target state, which is calculated by Eq. (37):

$$\begin{aligned} &\pi_{\mathrm{S}}(S) \\ &= \int \pi_{\mathrm{S}}(S \mid X)\pi(X)\delta X \\ &= \Delta(S) \int 1_{L(X)}(L(S)) [\Phi(S;\cdot)]^X \Delta(X) \\ &\quad \cdot \sum_{c\in\mathbb{C}} w^{(c)}(L(X)) [p^{(c)}]^X \delta X \\ &= \Delta(S) \sum_{c\in\mathbb{C}} \int \Delta(X) 1_{L(X)}(L(S)) \\ &\quad \cdot w^{(c)}(L(X)) [\Phi(S;\cdot), p^{(c)}]^X \delta X \\ &= \Delta(S) \sum_{c\in\mathbb{C}} \sum_{I\subseteq\mathbb{L}} 1_I(L(S)) w^{(c)}(I) \prod_{\ell\in I} \langle \Phi(S;\cdot,\cdot,l), p^{(c)}(\cdot,\cdot,l) \rangle, \end{aligned} \tag{37}$$

where $\pi(X)$ and $\pi_{\mathrm{S}}(S|X)$ are given by Eqs. (6) and (14), respectively. The last line of Eq. (37) follows Lemma 1. Due to the existence of the $1_I(L(S))$ term, we consider only $I\supseteq L(S)$. In Eq. (37), the term $\prod_{l\in I}\langle \Phi(S;\cdot,\cdot,l), p^{(c)}(\cdot,\cdot,l)\rangle$ is calculated as Eq. (38):

$$\begin{aligned}
&\prod_{l\in I}\langle \Phi(S;\cdot,\cdot,l), p^{(c)}(\cdot,\cdot,l)\rangle\\
&=\prod_{l\in L(S)}\langle \Phi(S;\cdot,\cdot,l), p^{(c)}(\cdot,\cdot,l)\rangle \prod_{l\in I-L(S)}\langle \Phi(S;\cdot,\cdot,l), p^{(c)}(\cdot,\cdot,l)\rangle\\
&=\prod_{l\in L(S)}\sum_{(\boldsymbol{x}_+,\mu_+,l_+)\in S}\langle p_{\mathrm{S}}(\cdot,\cdot,l)\phi(\boldsymbol{\xi}_+|\cdot,\mu_+,l)\tau(\mu_+|\cdot),p^{(c)}(\cdot,\cdot,l)\rangle\\
&\quad\cdot\delta_l(l_+)\prod_{\ell\in I-L(S)}\langle q_{\mathrm{S}}(\cdot,\cdot,l), p^{(c)}(\cdot,\cdot,l)\rangle\\
&=\prod_{l\in L(S)}\sum_{(\boldsymbol{x}_+,\mu_+,l_+)\in S}\delta_l(l_+)p_{+,\mathrm{S}}^{(c)}(\boldsymbol{\xi}_+,\mu_+,l)\eta_{\mathrm{S}}^{(c)}(l)\prod_{l\in I-L(S)}q_{\mathrm{S}}^{(c)}(l)\\
&=\prod_{(\boldsymbol{x}_+,\mu_+,l)\in S}p_{+,\mathrm{S}}^{(c)}(\boldsymbol{\xi}_+,\mu_+,l)\eta_{\mathrm{S}}^{(c)}(l)\prod_{l\in I-L(S)}q_{\mathrm{S}}^{(c)}(l)\\
&=[p_{+,\mathrm{S}}^{(c)}]^S[\eta_{\mathrm{S}}^{(c)}]^{L(S)}[q_{\mathrm{S}}^{(c)}]^{I-L(S)},
\end{aligned}\tag{38}$$

where $p_{+,\mathrm{S}}^{(c)}(\boldsymbol{\xi}_+,\mu_+,l)$ is given by Eq. (23) and

$$\begin{aligned}
&\eta_{\mathrm{S}}^{(c)}(l)\\
&=\sum_{\mu_+}\int\langle p_{\mathrm{S}}(\cdot,\cdot,l)\phi(\boldsymbol{\xi}_+|\cdot,\mu_+,l)\tau(\mu_+|\cdot), p^{(c)}(\cdot,\cdot,l)\rangle\mathrm{d}\boldsymbol{\xi}_+\\
&=\langle p_{\mathrm{S}}(\cdot,\cdot,l), p^{(c)}(\cdot,\cdot,l)\rangle,
\end{aligned}\tag{39}$$

$$q_{\mathrm{S}}^{(c)}(l)=\langle q_{\mathrm{S}}(\cdot,\cdot,l), p^{(c)}(\cdot,\cdot,l)\rangle=1-\eta_{\mathrm{S}}^{(c)}(l).\tag{40}$$

In Eq. (39), Eq. (11) is used.

Hence, substituting Eq. (38) into Eq. (37) yields

$$\begin{aligned}
&\pi_{\mathrm{S}}(S)\\
&=\Delta(S)\sum_{c\in\mathbb{C}}\sum_{I\subseteq\mathbb{L}}1_I(L(S))w^{(c)}(I)[p_{+,\mathrm{S}}^{(c)}]^S[\eta_{\mathrm{S}}^{(c)}]^{L(S)}[q_{\mathrm{S}}^{(c)}]^{I-L(S)}\\
&=\Delta(S)\sum\nolimits_{c\in\mathbb{C}}w_{\mathrm{S}}^{(c)}(L(S))[p_{+,\mathrm{S}}^{(c)}]^S,
\end{aligned}\tag{41}$$

where $w_{\mathrm{S}}^{(c)}(L)$ is given by Eq. (21).

Furthermore, substituting Eqs. (13) and (41) into Eq. (36) yields

$$\begin{aligned}
&\pi_+(X_+)\\
&=\Delta(B)\Delta(S)\sum\nolimits_{c\in\mathbb{C}}w_\gamma(L(B))w_{\mathrm{S}}^{(c)}(L(S))[p_\gamma]^B[p_{+,\mathrm{S}}^{(c)}]^S\\
&=\Delta(X_+)\sum\nolimits_{c\in\mathbb{C}}w_\gamma(L(X_+)-\mathbb{L})w_{\mathrm{S}}^{(c)}(L(X_+)\cap\mathbb{L})[p_+^{(c)}]^{X_+}\\
&=\Delta(X_+)\sum\nolimits_{c\in\mathbb{C}}w_+^{(c)}(L(X_+))[p_+^{(c)}]^{X_+},
\end{aligned}\tag{42}$$

where $w_+^{(c)}(L)$ and $p_+^{(c)}(\boldsymbol{\xi}_+,\mu_+,l)$ are given in Eqs. (19) and (20), respectively.

5.4 Derivation of Proposition 2

The posterior multi-target density can be obtained from the Bayes rule (Eq. (33)). Given the multi-target likelihood $g(Z|X)$ (Eq. (16)) and the predicted density $\pi(X)$ (Eq. (6)) of the H-GLMB, the numerator in Eq. (33) is calculated as follows:

$$\begin{aligned}
&g(Z|X)\pi(X)\\
&=\Delta(X)\mathrm{e}^{-\langle\kappa,1\rangle}\kappa^Z\sum_{c\in\mathbb{C}}\sum_{\theta\in\Theta}\delta_{\theta^{-1}(\{0:|Z|\})}(L(X))\\
&\quad\cdot w^{(c)}(L(X))[\eta_Z^{(c,\theta)}]^{L(X)}[p^{(c,\theta)}(\cdot|Z)]^X,
\end{aligned}\tag{43}$$

where $p^{(c,\theta)}(\boldsymbol{\xi},\mu,l|Z)$ and $\eta_Z^{(c,\theta)}(l)$ are given by Eqs. (26) and (27), respectively.

Furthermore, the integral of Eq. (43), i.e., the denominator in Eq. (33), is obtained as Eq. (44):

$$\begin{aligned}
&\int g(Z|X)\pi(X)\delta X\\
&=\mathrm{e}^{-\langle\kappa,1\rangle}\kappa^Z\sum_{c\in\mathbb{C}}\sum_{\theta\in\Theta}\int\Delta(X)\,\delta_{\theta^{-1}(\{0:|Z|\})}(L(X))\\
&\quad\cdot w^{(c)}(L(X))[\eta_Z^{(c,\theta)}]^{L(X)}[p^{(c,\theta)}(\cdot|Z)]^X\delta X\\
&=\mathrm{e}^{-\langle\kappa,1\rangle}\kappa^Z\sum_{c\in\mathbb{C}}\sum_{\theta\in\Theta}\sum_{J\subseteq\mathbb{L}}\delta_{\theta^{-1}(\{0:|Z|\})}(J)w^{(c)}(J)[\eta_Z^{(c,\theta)}]^J,
\end{aligned}\tag{44}$$

where the last line follows Lemma 1.

Finally, substituting Eqs. (43) and (44) into Eq. (33) yields the posterior multi-target density:

$$\pi(X|Z)=\Delta(X)\sum_{c\in\mathbb{C}}\sum_{\theta\in\Theta}w_Z^{(c,\theta)}(L(X))[p^{(c,\theta)}(\cdot|Z)]^X,\tag{45}$$

where $w_Z^{(c,\theta)}(L)$ is given by Eq. (25).

In summary, both the predicted and updated multi-target densities are H-GLMB distributions according to Eqs. (42) and (45). In other words, the H-GLMB density is closed under the Chapman-Kolmogorov prediction and Bayes update for multi-target hybrid systems.

6 Conclusions

In this study, we have extended traditional (single-target) hybrid systems to multi-target hybrid systems with a focus on the multi-maneuvering-target tracking system, which consists of a continuous state, a discrete and switchable state, and a discrete, time-constant, and unique state. We have also provided an exact derivation of the MM-GLMB filter, which is a state estimation solution to multi-target hybrid systems.

Contributors

Weihua WU designed the research. Weihua WU, Mao ZHENG, Xun FENG, and Zewen GUAN drafted the manuscript. Yichao CAI and Hongbin JIN helped organize the manuscript. Weihua WU revised and finalized the paper.

Compliance with ethics guidelines

Weihua WU, Yichao CAI, Hongbin JIN, Mao ZHENG, Xun FENG, and Zewen GUAN declare that they have no conflict of interest.

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