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Solution and stability of continuous-time cross-dimensional linear systems*

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Abstract: We investigate the solution and stability of continuous-time cross-dimensional linear systems (CCDLSs) with dimension bounded by V-addition and V-product. Using the integral iteration method, the solution to CCDLSs can be obtained. Based on the new algebraic expression of the solution and the Jordan decomposition method of matrix, a necessary and sufficient condition is derived for judging whether a CCDLS is asymptotically stable with a given initial state. This condition demonstrates a method for finding the domain of attraction and its relationships. Then, all the initial states that can be stabilized are studied, and a method for designing the corresponding controller is proposed. Two examples are presented to illustrate the validity of the theoretical results.

Key words: Cross-dimensional; V-addition; V-product; Asymptotic stability; Stabilization
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1 Introduction

The cross-dimensional system, also known as the dimension varying system, is an extension of the classical linear system. It is often used to study complex systems such as departure and joining of spacecrafts, vehicle clutch systems, and modeling of biological systems (Pan et al., 2014). Because of the change of state dimension, a cross-dimensional system is usually regarded as a switching system with more general state jump behaviors. Therefore, switching is usually used to deal with cross-dimensional systems (Yang H et al., 2014). Since some states with different dimensions may be closely related or completely independent of states with other dimensions, the factors affecting the dimension change need to be considered when building the model. This provides the switching system, to some extent, with the ability to reflect some properties of the cross-dimensional systems. However, this method does not fully consider the dynamics of the system in the process of dimension change. The period of state transition may occur quite frequently in practice, so the dynamics in this process cannot be ignored. For this reason, it is necessary to establish a theory to describe and study cross-dimensional linear systems.

Semi-tensor product is an extension of the traditional matrix product (Cheng et al., 2011). It breaks through the dimension limitation of matrix product such that many systems can be modeled as a bilinear dynamic equation:

$$\boldsymbol{x}(t+1)=\boldsymbol{A}\boldsymbol{u}(t)\boldsymbol{x}(t),$$

such as game theory (Cheng, 2014; Zhao and Wang, 2016), Boolean networks (Lu et al., 2016; Liu et al., 2017), logical systems (Wu and Shen, 2018a, 2018b; Li HT and Ding, 2019; Li YL et al., 2020), and asynchronous sequential machines (Wang B et al., 2017,

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2019). Then two other operations, V-addition and V-product, which break through the dimension limitation, are put forward one after another. The former is an extension of the traditional vector addition, and the latter is an extension of the traditional product of matrix and vector (Cheng, 2019). Theoretically, these three operations not only maintain some properties of traditional operations, but also provide a new research framework for the analysis of dimension-free matrix theory. Then, cross-dimensional linear systems have been studied based on this theory (Cheng et al., 2017; Cheng, 2019).

Cross-dimensional linear systems are usually divided into two types: dimension-bounded and dimension-unbounded. The controllability and observability of the former were studied in Cheng et al. (2017). The least-dimension projective realization of the latter was introduced in Cheng et al. (2018), and more detailed properties of the projection were given in Feng et al. (2019a). Zhang KZ and Johansson (2018) investigated long-term behavior of cross-dimensional linear dynamical systems. Feng et al. (2019b) discussed the variation of cross-dimensional linear dynamical systems' dimensionality.

In a system, its trajectory can not only express the dynamic evolution law, but also serve as an important tool to study controllability, observability, and stability (Zhang Y and Zhou, 2017; Li XD et al., 2018; Yang XY et al., 2018; Wang ZC et al., 2019) . Stability, as an important property of the system, is the ability to describe whether a system can operate stably. Thus, trajectory description and stability analysis are crucial research topics. Cheng et al. (2017) obtained the trajectory of continuous-time cross-dimensional linear systems (CCDLSs) using the integration method. The mathematical expression of the CCDLS trajectory contains a multi-integral term, which makes it difficult to calculate the expressions. This term is difficult to deal with when analyzing the characteristics of the system. Therefore, it is necessary to simplify the trajectory of CCDLSs. In this study, based on the calculation formula of the trajectory given in Cheng et al. (2017), we find the inhomogeneous linear differential equations that the trajectory satisfies using the integral iteration method. A trajectory formula without multi-integral terms is obtained by solving the equations. This process is actually a variation of the process obtaining previous results, so the mathematical expression of the trajectory is convenient for theoretical study. Then, asymptotic stability and stabilization of CCDLSs are analyzed according to this trajectory.

The main contributions of this paper are as follows:

1. The trajectory of CCDLSs is described in the form of a single integral. Compared with previous results (Cheng et al., 2017; Cheng, 2019), its mathematical expression is simpler and is more suitable for analyzing the stability of CCDLSs.

2. A necessary and sufficient condition is given for judging whether a CCDLS is asymptotically stable. With this condition, the domain of attraction for CCDLSs in a given initial space can be obtained.

3. For stabilization, we give all the initial states that can be stabilized. Inspired by the results of classical linear systems, a method for computing the corresponding controllers is provided.

Some notations mentioned in this paper are listed as follows:

1. $\mathbb{R}/\mathbb{C}$: the set of real/complex numbers.

2. $\mathbf{1}_n := [1, 1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^n$.

3. $\boldsymbol{f}_{[n]}(t) := \left[1, t, \dfrac{t^2}{2!}, \ldots, \dfrac{t^{n-1}}{(n-1)!}\right]^{\mathrm{T}} \in \mathbb{R}^n$.

4. $m \vee n$: the least common multiple of m and n.

5. $\mathrm{Re}(\lambda)/\mathrm{Im}(\lambda)$: the real/complex part of $\lambda \in \mathbb{C}$.

6. $\mathbf{0}_{m\times n}$: the $m \times n$ null matrix, especially $\mathbf{0}_m := \mathbf{0}_{m\times 1}$.

7. $\mathrm{Row}_i(\boldsymbol{A})/\mathrm{Col}_i(\boldsymbol{A})$: the i^{th} row/column of matrix $\boldsymbol{A}$.

8. $\mathcal{V}_r$: the r-dimensional column vector space, especially $\mathcal{V} := \bigcup_{i\in\mathbb{Z}^+} \mathcal{V}_i$.

9. $\mathbb{Z}_{z_1}^{z_2} := [z_1, z_2] \cap \mathbb{N}$ and $\mathbb{Z}^+ := [1, +\infty) \cap \mathbb{N}$, where $\mathbb{N}$ is the set of integers.

10. $\boldsymbol{\delta}_n^i$: the i^{th} column of identity matrix $\boldsymbol{I}_n$, especially $\boldsymbol{\delta}_n^0 := [0, 0, \ldots, 0]^{\mathrm{T}} \in \mathbb{R}^n$.

11. For $[\boldsymbol{\delta}_n^{i_1}, \boldsymbol{\delta}_n^{i_2}, \ldots, \boldsymbol{\delta}_n^{i_r}] \in \mathbb{R}^{n\times r}$, it is briefly denoted by $\boldsymbol{\delta}_n[i_1, i_2, \ldots, i_r]$.

12. $\|\boldsymbol{x}(t)\| \to 0$ means the limit of $\boldsymbol{x}(t)$ as t approaches infinity, i.e., $\lim\limits_{t\to+\infty} \boldsymbol{x}(t) = 0$, where $\boldsymbol{x}(t) \in \mathbb{R}^n$, and $\|\boldsymbol{x}(t)\|$ represents the standard norm on $\mathbb{R}^n$.

13. Let $\boldsymbol{A} = (a_{ij}) \in \mathbb{R}^{m\times n}$ and $\boldsymbol{B} \in \mathbb{R}^{p\times q}$. The

Kronecker product of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined as

$$\boldsymbol{A}\otimes\boldsymbol{B}:=\begin{bmatrix} a_{11}\boldsymbol{B} & a_{12}\boldsymbol{B} & \cdots & a_{1n}\boldsymbol{B} \\ a_{21}\boldsymbol{B} & a_{22}\boldsymbol{B} & \cdots & a_{2n}\boldsymbol{B} \\ \vdots & \vdots & & \vdots \\ a_{m1}\boldsymbol{B} & a_{m2}\boldsymbol{B} & \cdots & a_{mn}\boldsymbol{B} \end{bmatrix}.$$

2 Preliminaries and system description

2.1 V-product and V-addition

First, the definitions of V-product and V-addition are introduced.

Definition 1 (Cheng et al., 2011) Let $\boldsymbol{A}\in\mathbb{R}^{m\times n}$ and $\boldsymbol{B}\in\mathbb{R}^{p\times q}$. The semi-tensor product of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined as

$$\boldsymbol{A}\ltimes\boldsymbol{B}:=[\boldsymbol{A}\otimes\boldsymbol{I}_{(n\vee p)/n}][\boldsymbol{B}\otimes\boldsymbol{I}_{(n\vee p)/p}],$$

where "$\otimes$" represents the Kronecker product.

Clearly, if $n=p$, then $\boldsymbol{A}\ltimes\boldsymbol{B}=\boldsymbol{A}\boldsymbol{B}$. Thus, it can be seen that the semi-tensor product is an extension of the traditional matrix product. In this paper, we omit the symbol "$\ltimes$" if there is no confusion.

Using semi-tensor product, we can construct the swap matrix, denoted by $\boldsymbol{W}_{[m,n]}$, as follows:

$$\begin{aligned}\boldsymbol{W}_{[m,n]}:=&[\boldsymbol{\delta}_n^1\boldsymbol{\delta}_m^1,\boldsymbol{\delta}_n^2\boldsymbol{\delta}_m^1,\ldots,\boldsymbol{\delta}_n^n\boldsymbol{\delta}_m^1,\boldsymbol{\delta}_n^1\boldsymbol{\delta}_m^2,\boldsymbol{\delta}_n^2\boldsymbol{\delta}_m^2,\ldots,\\ &\boldsymbol{\delta}_n^n\boldsymbol{\delta}_m^2,\ldots,\boldsymbol{\delta}_n^1\boldsymbol{\delta}_m^m,\boldsymbol{\delta}_n^2\boldsymbol{\delta}_m^m,\ldots,\boldsymbol{\delta}_n^n\boldsymbol{\delta}_m^m].\end{aligned}$$

Lemma 1 (Cheng, 2019) The swap matrix has the following properties:

(1) $\boldsymbol{W}_{[m,n]}$ is an orthogonal matrix and

$$\boldsymbol{W}_{[m,n]}=\boldsymbol{W}_{[n,m]}^{-1}=\boldsymbol{W}_{[n,m]}^{\mathrm{T}}.$$

(2) Let $\boldsymbol{A}\in\mathbb{R}^{m\times n}$ and $\boldsymbol{B}\in\mathbb{R}^{p\times q}$. Then

$$\boldsymbol{W}_{[m,p]}(\boldsymbol{A}\otimes\boldsymbol{B})\boldsymbol{W}_{[q,n]}=\boldsymbol{B}\otimes\boldsymbol{A}.$$

Definition 2 (Cheng, 2019) Let $\boldsymbol{A}\in\mathbb{R}^{m\times n}$, $\boldsymbol{B}\in\mathbb{R}^{r\times q}$, $\boldsymbol{x}\in\mathbb{R}^r$, and $\boldsymbol{y}\in\mathbb{R}^s$.

(1) V-product of $\boldsymbol{A}$ with $\boldsymbol{x}$ is defined as

$$\boldsymbol{A}\vec{\ltimes}\boldsymbol{x}:=[\boldsymbol{A}\otimes\boldsymbol{I}_{(n\vee r)/n}][\boldsymbol{x}\otimes\boldsymbol{1}_{(n\vee r)/r}].$$

(2) V-product of $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined as

$$\boldsymbol{A}\vec{\ltimes}\boldsymbol{B}:=[\boldsymbol{A}\otimes\boldsymbol{I}_{(n\vee r)/n}][\boldsymbol{B}\otimes\boldsymbol{1}_{(n\vee r)/r}].$$

(3) V-addition of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as

$$\boldsymbol{x}\vec{\pm}\boldsymbol{y}:=\boldsymbol{x}\otimes\boldsymbol{1}_{(r\vee s)/r}+\boldsymbol{y}\otimes\boldsymbol{1}_{(r\vee s)/s}.$$

Semi-tensor product, V-product, and V-addition have the following properties:

Lemma 2 (Cheng, 2019) Let $\boldsymbol{A}\in\mathbb{R}^{m\times n}$, $\boldsymbol{B}\in\mathbb{R}^{p\times q}$, $\boldsymbol{x}\in\mathbb{R}^r$, and $\boldsymbol{y}\in\mathbb{R}^s$.

(1) $\boldsymbol{A}\vec{\ltimes}(\boldsymbol{B}\vec{\ltimes}\boldsymbol{x})=(\boldsymbol{A}\ltimes\boldsymbol{B})\vec{\ltimes}\boldsymbol{x}$.

(2) $\boldsymbol{A}\vec{\ltimes}(a\boldsymbol{x}\vec{\pm}b\boldsymbol{y})=a\boldsymbol{A}\vec{\ltimes}\boldsymbol{x}\vec{\pm}b\boldsymbol{A}\vec{\ltimes}\boldsymbol{y},\ a,\ b\in\mathbb{R}$.

Next, the definition of a dimension-bounded operator is introduced.

Definition 3 (Cheng et al., 2017) Let $\boldsymbol{A}\in\mathbb{R}^{m\times n}$.

(1) $\mathcal{V}_r$ is $\boldsymbol{A}$-invariant if for any $\boldsymbol{x}\in\mathcal{V}_r$, $\boldsymbol{A}\vec{\ltimes}\boldsymbol{x}\in\mathcal{V}_r$.

(2) $\boldsymbol{A}$ is called a dimension-bounded operator if for any $\boldsymbol{x}\in\mathcal{V}_{r_0}$, there exist $t_0\geq 0$ and $r\in\mathbb{Z}^+$ such that for any $t\geq t_0$, $\boldsymbol{A}^t\vec{\ltimes}\boldsymbol{x}\in\mathcal{V}_r$ holds.

This definition shows that $\mathcal{V}_r$ is $\boldsymbol{A}$-invariant if and only if $r\vee n=(rn)/m$, and that $\boldsymbol{A}$ is a dimension-bounded operator if and only if $m|n$. If $\mathcal{V}_r$ is $\boldsymbol{A}$-invariant, then $\boldsymbol{A}|_{\mathcal{V}_r}$ is a linear mapping on $\mathcal{V}_r$. Thus, we can find a square matrix, denoted by $\boldsymbol{A}_r$, such that $\boldsymbol{A}|_{\mathcal{V}_r}=\boldsymbol{A}_r$. According to Cheng et al. (2017), it is easy to calculate

$$\boldsymbol{A}_r=(\boldsymbol{A}\otimes\boldsymbol{I}_{r/m})(\boldsymbol{I}_r\otimes\boldsymbol{1}_{n/m}).$$

2.2 Continuous-time cross-dimensional linear system

A CCDLS can be described as (Cheng et al., 2017)

$$\begin{cases}\dot{\boldsymbol{x}}(t)=\boldsymbol{A}\vec{\ltimes}\boldsymbol{x}(t),\\ \boldsymbol{A}\in\mathbb{R}^{m\times n},\ \boldsymbol{x}(0)=\boldsymbol{x}_0\in\mathcal{V}_{r_0},\end{cases}\tag{1}$$

where $\boldsymbol{x}(t)$ represents the trajectory of CCDLS (1). Then the solution to CCDLS (1) starting from the initial state $\boldsymbol{x}_0$ is expressed as

$$\boldsymbol{x}(t)=\mathrm{e}^{\boldsymbol{A}t}\vec{\ltimes}\boldsymbol{x}_0=\sum_{k=0}^{\infty}\frac{t^k}{k!}[\boldsymbol{A}^k\vec{\ltimes}\boldsymbol{x}_0].\tag{2}$$

Assume that $\boldsymbol{A}$ is a dimension-bounded operator. That is, CCDLS (1) is a dimension-bounded system.

To better understand CCDLSs, we briefly introduce the formal derivative of $\boldsymbol{x}(t)$, defined as

$$\dot{\boldsymbol{x}}(t)=\boldsymbol{A}\mathrm{e}^{\boldsymbol{A}t}\vec{\ltimes}\boldsymbol{x}_0=\sum_{k=0}^{\infty}\frac{t^k}{k!}[\boldsymbol{A}^{k+1}\vec{\ltimes}\boldsymbol{x}_0],$$

where $\boldsymbol{A}\mathrm{e}^{\boldsymbol{A}t}$ is the principal formal polynomial. Clearly, if $\boldsymbol{A}$ is a square matrix, then $\dot{\boldsymbol{x}}(t)=\boldsymbol{x}'(t)$, where $\boldsymbol{x}'(t)$ is the classical derivative of $\boldsymbol{x}(t)$.

Remark 1 For $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $P(\boldsymbol{A}) = \sum_{k=0}^{\infty} c_k \boldsymbol{A}^k t^k$ is called a principal formal polynomial. Note that this addition is not a concrete operation. In other words, the above addition has no practical meaning, just formal addition. Details can be found in Cheng et al. (2017).

The corresponding continuous-time cross-dimensional linear control system (CCDLCS) is defined as

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A} \vec{\ltimes} \boldsymbol{x}(t) \vec{\boxplus} \boldsymbol{B}\boldsymbol{u}(t), \\ \boldsymbol{y}(t) = \boldsymbol{C} \vec{\ltimes} \boldsymbol{x}(t), \end{cases} \tag{3}$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{B} \in \mathbb{R}^{m\times p}$, $\boldsymbol{C} \in \mathbb{R}^{q\times m}$, and $\boldsymbol{u}(t) \in \mathbb{R}^p$ and $\boldsymbol{y}(t)$ represent the input and output of CCDLCS (3), respectively. The trajectory of CCDLCS (3) starting from the initial state $\boldsymbol{x}_0$ can be expressed as

$$\boldsymbol{x}(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}[\boldsymbol{A}^k \vec{\ltimes} \boldsymbol{x}_0] \vec{\boxplus} \int_0^t \mathrm{e}^{\boldsymbol{A}(t-\tau)} \vec{\ltimes} \boldsymbol{B} u(\tau)\mathrm{d}\tau,$$

where $\mathrm{e}^{\boldsymbol{A}(t-\tau)}$ is the principal formal polynomial.

3 Analysis of continuous-time cross-dimensional linear systems

In this section, we propose a new method to analyze the trajectory of CCDLSs. First, two preparations are made for the following discussion:

Proposition 1 Let $k \in \mathbb{Z}^+$, $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, and $\boldsymbol{B} \in \mathbb{R}^{n\times p}$. Then, the following statements are true:

(1) $\boldsymbol{AB} = \frac{1}{k}(\boldsymbol{A} \otimes \mathbf{1}_k^{\mathrm{T}})(\boldsymbol{B} \otimes \mathbf{1}_k)$.

(2) $(\boldsymbol{AB}) \otimes \mathbf{1}_k^{\mathrm{T}} = \boldsymbol{A}(\boldsymbol{B} \otimes \mathbf{1}_k^{\mathrm{T}})$.

(3) $(\boldsymbol{AB}) \otimes \mathbf{1}_k = (\boldsymbol{A} \otimes \mathbf{1}_k)\boldsymbol{B}$.

Proof We prove only statements (1) and (2). The proof of statement (3) is quite similar to that of statement (2), so it is omitted here.

1. Let $\boldsymbol{C} = \frac{1}{k}(\boldsymbol{A} \otimes \mathbf{1}_k^{\mathrm{T}})(\boldsymbol{B} \otimes \mathbf{1}_k)$. Then we obtain

$$\begin{aligned}(\boldsymbol{C})_{ij} &= \frac{1}{k}[\mathrm{Row}_i(\boldsymbol{A}) \otimes \mathbf{1}_k^{\mathrm{T}}][\mathrm{Col}_j(\boldsymbol{B}) \otimes \mathbf{1}_k] \\ &= \mathrm{Row}_i(\boldsymbol{A})\mathrm{Col}_j(\boldsymbol{B}) \\ &= (\boldsymbol{AB})_{ij}.\end{aligned}$$

Clearly, this is true for any $i \in \mathbb{Z}_1^m$ and $j \in \mathbb{Z}_1^p$. Consequently, statement (1) holds.

2. It is easy to see that

$$\begin{aligned}\mathrm{Row}_i[\boldsymbol{A}(\boldsymbol{B} \otimes \mathbf{1}_k^{\mathrm{T}})] &= \mathrm{Row}_i(\boldsymbol{A})(\boldsymbol{B} \otimes \mathbf{1}_k^{\mathrm{T}}) \\ &= \mathrm{Row}_i[(\boldsymbol{AB}) \otimes \mathbf{1}_k^{\mathrm{T}}].\end{aligned}$$

Clearly, this is true for any $i \in \mathbb{Z}_1^m$. Thus, statement (2) holds.

According to Lemma 1, we can obtain Proposition 2 about the similar decomposition of a square matrix.

Proposition 2 Given two positive integers k and n, there exists a nonsingular matrix $\boldsymbol{P} \in \mathbb{R}^{nk\times nk}$ such that for any $\boldsymbol{A} \in \mathbb{R}^{n\times n}$,

$$\boldsymbol{P}\left[\frac{1}{k}(\boldsymbol{A} \otimes \mathbf{1}_k) \otimes \mathbf{1}_k^{\mathrm{T}}\right]\boldsymbol{P}^{-1} = \mathrm{diag}(\boldsymbol{A}, \underbrace{\mathbf{0}_{n\times n}, \ldots, \mathbf{0}_{n\times n}}_{k-1}).$$

Proof By Lemma 1, we have

$$\begin{aligned}&\boldsymbol{W}_{[k,n]}^{-1}\left[\frac{1}{k}(\boldsymbol{A} \otimes \mathbf{1}_k) \otimes \mathbf{1}_k^{\mathrm{T}}\right]\boldsymbol{W}_{[k,n]} \\ &= \frac{1}{k}\boldsymbol{W}_{[n,k]}[\boldsymbol{A} \otimes (\mathbf{1}_k \otimes \mathbf{1}_k^{\mathrm{T}})]\boldsymbol{W}_{[k,n]} \\ &= \frac{1}{k}(\mathbf{1}_k \otimes \mathbf{1}_k^{\mathrm{T}}) \otimes \boldsymbol{A}.\end{aligned} \tag{4}$$

Let $\boldsymbol{P} = \boldsymbol{U}\boldsymbol{Q}^{-1}\boldsymbol{W}_{[n,k]}$, where $\boldsymbol{Q} = (\boldsymbol{I}_k - \boldsymbol{L}) \otimes \boldsymbol{I}_n$, $\boldsymbol{U} = (\boldsymbol{I}_k - \frac{1}{k}\boldsymbol{L}^{\mathrm{T}}) \otimes \boldsymbol{I}_n$, and $\boldsymbol{L} = \boldsymbol{\delta}_k[0,1,\ldots,1] \in \mathbb{R}^{k\times k}$. Then from Eq. (4), we can obtain

$$\begin{aligned}&\boldsymbol{P}\left[\frac{1}{k}(\boldsymbol{A} \otimes \mathbf{1}_k) \otimes \mathbf{1}_k^{\mathrm{T}}\right]\boldsymbol{P}^{-1} \\ &= \boldsymbol{U}\boldsymbol{Q}^{-1}\left[\frac{1}{k}(\mathbf{1}_k \otimes \mathbf{1}_k^{\mathrm{T}}) \otimes \boldsymbol{A}\right]\boldsymbol{Q}\boldsymbol{U}^{-1} \\ &= \boldsymbol{U}\boldsymbol{Q}^{-1}\left\{\frac{1}{k}[(\boldsymbol{\delta}_k^1)^{\mathrm{T}} \otimes \mathbf{1}_k] \otimes \boldsymbol{A}\right\} \\ &= \mathrm{diag}(\boldsymbol{A}, \underbrace{\mathbf{0}_{n\times n}, \ldots, \mathbf{0}_{n\times n}}_{k-1}).\end{aligned}$$

From the proof above, it can be seen that $\boldsymbol{P}$ is dependent on n and k, denoted by $\boldsymbol{P}_{[n,k]}$. Then, $\boldsymbol{P}_{[n,k]} = \left[\left(\boldsymbol{I}_k + \boldsymbol{L}_k - \frac{1}{k}\boldsymbol{L}_k^{\mathrm{T}} - \frac{1}{k}\boldsymbol{L}_k^{\mathrm{T}}\boldsymbol{L}_k\right) \otimes \boldsymbol{I}_n\right]\boldsymbol{W}_{[n,k]}$ with $\boldsymbol{L}_k = \boldsymbol{\delta}_k[0, 1, \ldots, 1] \in \mathbb{R}^{k\times k}$.

Propositions 1 and 2 play a foreshadowing role for the subsequent discussion. The former shows that if some conditions are satisfied, the operation order of the Kronecker product and traditional matrix product can be commutative. The latter indicates that the square matrix is similar to a quasi diagonal matrix after taking the Kronecker product. Based on this, we can use Proposition 2 to simplify the square matrix that satisfies some specific conditions.

Next, the solution to CCDLS (1) is preliminarily analyzed based on Propositions 1 and 2.

For CCDLS (1), given its initial state $\boldsymbol{x}_0 \in \mathcal{V}_{r_0}$, r_j denotes the dimension of $\boldsymbol{A}^j \vec{\ltimes} \boldsymbol{x}_0$, that is, $\boldsymbol{A}^j \vec{\ltimes} \boldsymbol{x}_0 \in \mathcal{V}_{r_j}$. Since $\boldsymbol{A}$ is a dimension-bounded operator, there exists an integer s such that for any $j \in [s, +\infty) \cap \mathbb{N}$, $r_j = r_s$. This means that $\mathcal{V}_{r_s}$ is $\boldsymbol{A}$-invariant. Then we can find a square matrix, denoted by $\boldsymbol{A}_{r_s}$, such that $\boldsymbol{A}|_{\mathcal{V}_{r_s}} = \boldsymbol{A}_{r_s}$.

According to Cheng et al. (2017), the solution to CCDLS (1) is as follows:

$$\begin{aligned}\boldsymbol{x}(t) =& \boldsymbol{x}_0 \vec{\boxplus} t\boldsymbol{A} \vec{\ltimes} \boldsymbol{x}_0 \vec{\boxplus} \ldots \vec{\boxplus} \frac{t^{s-1}}{(s-1)!}\boldsymbol{A}^{s-1} \vec{\ltimes} \boldsymbol{x}_0 \\ & \vec{\boxplus} \int_0^t \mathrm{d}\tau_1 \int_0^{\tau_1} \mathrm{d}\tau_2 \ldots \int_0^{\tau_{s-1}} \mathrm{e}^{\boldsymbol{A}_{r_s}\tau_s}(\boldsymbol{A}^s \vec{\ltimes} \boldsymbol{x}_0)\mathrm{d}\tau_s.\end{aligned} \tag{5}$$

However, its mathematical expression is complicated and computational complexity is high. Therefore, we present a new method to calculate it. Compared with Eq. (5), it is more convenient to obtain the solution by this method and this solution is suitable for studying asymptotic stability.

Theorem 1 If the initial state of CCDLS (1) is $\boldsymbol{x}_0 \in \mathcal{V}_{r_0}$, then the solution to CCDLS (1) is

$$\begin{aligned}\boldsymbol{x}(t) =& \mathrm{e}^{\boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]}t}(\boldsymbol{x}_0 \otimes \boldsymbol{1}_{\frac{\alpha}{r_0}}) \\ & + \int_0^t \mathrm{e}^{\boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]}(t-\tau)} \boldsymbol{\Xi} \boldsymbol{f}_{[s]}(\tau)\mathrm{d}\tau,\end{aligned} \tag{6}$$

where

$$\begin{cases}\alpha = \vee_{j=0}^{s} r_j, \\ \boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]} = \dfrac{r_s}{\alpha}(\boldsymbol{A}_{r_s} \otimes \boldsymbol{1}_{\frac{\alpha}{r_s}}) \otimes \boldsymbol{1}^{\mathrm{T}}_{\frac{\alpha}{r_s}}, \\ \boldsymbol{\Xi} = \Big\{(\boldsymbol{A} \vec{\ltimes} \boldsymbol{x}_0) \otimes \boldsymbol{1}_{\frac{\alpha}{r_1}} - \boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]}(\boldsymbol{x}_0 \otimes \boldsymbol{1}_{\frac{\alpha}{r_0}}), \\ \quad (\boldsymbol{A}^2 \vec{\ltimes} \boldsymbol{x}_0) \otimes \boldsymbol{1}_{\frac{\alpha}{r_2}} - \boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]}[(\boldsymbol{A} \vec{\ltimes} \boldsymbol{x}_0) \otimes \boldsymbol{1}_{\frac{\alpha}{r_1}}], \\ \quad \vdots \\ \quad (\boldsymbol{A}^s \vec{\ltimes} \boldsymbol{x}_0) \otimes \boldsymbol{1}_{\frac{\alpha}{r_s}} - \boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]}[(\boldsymbol{A}^{s-1} \vec{\ltimes} \boldsymbol{x}_0) \otimes \boldsymbol{1}_{\frac{\alpha}{r_{s-1}}}]\Big\}.\end{cases}$$

Proof Let $\boldsymbol{x}_j = \boldsymbol{A}^j \vec{\ltimes} \boldsymbol{x}_0 \in \mathcal{V}_{r_j}$ and $\hat{\boldsymbol{x}}_j = \boldsymbol{x}_j \otimes \boldsymbol{1}_{\frac{\alpha}{r_j}}$, $j = 0, 1, \ldots, s$. By the definitions of V-product and V-addition, Eq. (2) can be expressed as

$$\begin{aligned}\boldsymbol{x}(t) =& \boldsymbol{x}_0 \vec{\boxplus} t\boldsymbol{A} \vec{\ltimes} \boldsymbol{x}_0 \vec{\boxplus} \ldots \vec{\boxplus} \frac{t^{s-1}}{(s-1)!}\boldsymbol{A}^{s-1} \vec{\ltimes} \boldsymbol{x}_0 \\ & \vec{\boxplus} \left[\frac{t^s}{s!}\boldsymbol{A}^s \vec{\ltimes} \boldsymbol{x}_0 + \frac{t^{s+1}}{(s+1)!}\boldsymbol{A}^{s+1} \vec{\ltimes} \boldsymbol{x}_0 + \ldots\right] \\ =& \boldsymbol{x}_0 \vec{\boxplus} t\boldsymbol{x}_1 \vec{\boxplus} \frac{t^2}{2!}\boldsymbol{x}_2 \vec{\boxplus} \ldots \vec{\boxplus} \frac{t^{s-1}}{(s-1)!}\boldsymbol{x}_{s-1} \vec{\boxplus} \left[\frac{t^s}{s!}\boldsymbol{I}_{r_s}\right. \\ & \left. + \frac{t^{s+1}}{(s+1)!}\boldsymbol{A}_{r_s} + \frac{t^{s+2}}{(s+2)!}\boldsymbol{A}^2_{r_s} + \ldots\right]\boldsymbol{x}_s \\ =& \sum_{j=0}^{s-1}\frac{t^j}{j!}\hat{\boldsymbol{x}}_j + \left[\sum_{j=0}^{\infty}\frac{t^{s+j}}{(s+j)!}\boldsymbol{A}^j_{r_s}\right]\hat{\boldsymbol{x}}_s.\end{aligned} \tag{7}$$

Then, it can be seen that $\boldsymbol{x}(t)$ is invariant given the initial state $\boldsymbol{x}_0$. Hence, computing the s-order derivative of Eq. (7), we have

$$\boldsymbol{x}^{(s)}(t) = \left(\sum_{j=0}^{\infty}\frac{t^j}{j!}\boldsymbol{A}^j_{r_s}\boldsymbol{x}_s\right) \otimes \boldsymbol{1}_{\frac{\alpha}{r_s}} = (\mathrm{e}^{\boldsymbol{A}_{r_s}t} \otimes \boldsymbol{1}_{\frac{\alpha}{r_s}})\boldsymbol{x}_s. \tag{8}$$

Computing the derivative of Eq. (8), we obtain

$$\begin{aligned}\boldsymbol{x}^{(s+1)}(t) &= [(\boldsymbol{A}_{r_s}\mathrm{e}^{\boldsymbol{A}_{r_s}t}) \otimes \boldsymbol{1}_{\frac{\alpha}{r_s}}]\boldsymbol{x}_s \\ &= \left[\frac{r_s}{\alpha}(\boldsymbol{A}_{r_s} \otimes \boldsymbol{1}_{\frac{\alpha}{r_s}}) \otimes \boldsymbol{1}^{\mathrm{T}}_{\frac{\alpha}{r_s}}\right](\mathrm{e}^{\boldsymbol{A}_{r_s}t} \otimes \boldsymbol{1}_{\frac{\alpha}{r_s}})\boldsymbol{x}_s \\ &= \boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]}\boldsymbol{x}^{(s)}(t).\end{aligned}$$

Then we have

$$\begin{aligned}\boldsymbol{x}^{(s)}(t) &= \int_0^t \boldsymbol{x}^{(s+1)}(\tau)\mathrm{d}\tau + \hat{\boldsymbol{x}}_s \\ &= \boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]}\left(\int_0^t \boldsymbol{x}^{(s)}(\tau)\mathrm{d}\tau + \hat{\boldsymbol{x}}_{s-1} - \hat{\boldsymbol{x}}_{s-1}\right) + \hat{\boldsymbol{x}}_s \\ &= \boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]}\boldsymbol{x}^{(s-1)}(t) + (\hat{\boldsymbol{x}}_s - \boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]}\hat{\boldsymbol{x}}_{s-1}).\end{aligned}$$

Furthermore, we can obtain

$$\begin{aligned}\boldsymbol{x}^{(s-1)}(t) =& \int_0^t \boldsymbol{x}^{(s)}(\tau)\mathrm{d}\tau + \hat{\boldsymbol{x}}_{s-1} \\ =& \boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]}\left(\int_0^t \boldsymbol{x}^{(s-1)}(\tau)\mathrm{d}\tau + \hat{\boldsymbol{x}}_{s-2} - \hat{\boldsymbol{x}}_{s-2}\right) \\ & + t(\hat{\boldsymbol{x}}_s - \boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]}\hat{\boldsymbol{x}}_{s-1}) + \hat{\boldsymbol{x}}_{s-1} \\ =& \boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]}\boldsymbol{x}^{(s-2)}(t) + (\hat{\boldsymbol{x}}_{s-1} - \boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]}\hat{\boldsymbol{x}}_{s-2}) \\ & + t(\hat{\boldsymbol{x}}_s - \boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]}\hat{\boldsymbol{x}}_{s-1}).\end{aligned}$$

Repeating the above integral iterative process, it can be concluded that

$$\boldsymbol{x}'(t) = \boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]}\boldsymbol{x}(t) + \sum_{j=1}^{s}\frac{t^{j-1}}{(j-1)!}(\hat{\boldsymbol{x}}_j - \boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]}\hat{\boldsymbol{x}}_{j-1}). \tag{9}$$

In addition, Eq. (9) can be expressed alternatively as

$$\begin{cases} \boldsymbol{x}'(t) = \boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]}\boldsymbol{x}(t) + \boldsymbol{\Xi}\boldsymbol{f}_{[s]}(t), \\ \boldsymbol{x}(0) = \hat{\boldsymbol{x}}_0 = (\boldsymbol{x}_0 \otimes \mathbf{1}_{\frac{\alpha}{r_0}}). \end{cases} \tag{10}$$

Eq. (10) is an inhomogeneous linear differential equation. Thus, we can calculate its solution as Eq. (6).

Divide $\boldsymbol{P}_{[r_s, \frac{\alpha}{r_s}]}(\boldsymbol{x}_0 \otimes \mathbf{1}_{\frac{\alpha}{r_0}})$ and $\boldsymbol{P}_{[r_s, \frac{\alpha}{r_s}]}\boldsymbol{\Xi}$ into $[\bar{\boldsymbol{x}}_1^{\mathrm{T}}(0),\ \bar{\boldsymbol{x}}_2^{\mathrm{T}}(0)]^{\mathrm{T}}$ and $[\boldsymbol{\Xi}_1^{\mathrm{T}},\ \boldsymbol{\Xi}_2^{\mathrm{T}}]^{\mathrm{T}}$, respectively, where $\bar{\boldsymbol{x}}_1(0) \in \mathbb{R}^{r_s}$, $\bar{\boldsymbol{x}}_2(0) \in \mathbb{R}^{\alpha - r_s}$, $\boldsymbol{\Xi}_1 \in \mathbb{R}^{r_s \times s}$, and $\boldsymbol{\Xi}_2 \in \mathbb{R}^{(\alpha - r_s)\times s}$. By combining Theorem 1 with Proposition 2, the following corollary can be obtained:

Corollary 1 If the initial state of CCDLS (1) is $\boldsymbol{x}_0 \in \mathcal{V}_{r_0}$, then the solution to CCDLS (1) is

$$\boldsymbol{x}(t) = \boldsymbol{P}^{-1}_{[r_s, \frac{\alpha}{r_s}]} \begin{bmatrix} \exp(\boldsymbol{A}_{r_s}t)\Big(\bar{\boldsymbol{x}}_1(0) \\ \quad + \displaystyle\int_0^t \exp(-\boldsymbol{A}_{r_s}\tau)\boldsymbol{\Xi}_1\boldsymbol{f}_{[s]}(\tau)\mathrm{d}\tau\Big) \\ \bar{\boldsymbol{x}}_2(0) + \boldsymbol{\Xi}_2 \displaystyle\int_0^t \boldsymbol{f}_{[s]}(\tau)\mathrm{d}\tau \end{bmatrix}. \tag{11}$$

In this section, we analyze the solution to CCDLSs from a new perspective. By integral iteration, it can be found that CCDLS (1) is equivalent to the inhomogeneous linear differential Eq. (10) given the initial space $\mathcal{V}_{r_0}$. Based on this fact, a new expression of the solution can be found that does not contain multi-integral terms. This is crucial to the subsequent discussion of asymptotic stability.

4 Asymptotic stability

In this section, the asymptotic stability of CCDLS (1) is studied based on the solution in Theorem 1. First, to facilitate the research, some preparatory notes are given.

Proposition 3 Let $\boldsymbol{v} \neq \mathbf{0}_p \in \mathbb{C}^p$ and $\boldsymbol{B} \in \mathbb{C}^{p\times s}$. $\boldsymbol{J} = \lambda\boldsymbol{I}_p + \boldsymbol{\Omega} \in \mathbb{C}^{p\times p}$ is a Jordan black, where $\lambda \in \mathbb{C}$ and $\boldsymbol{\Omega} = \boldsymbol{\delta}_p[0, 1, \ldots, p-1]$. If $\boldsymbol{V}(t) = \mathrm{e}^{\boldsymbol{J}t}\boldsymbol{v} + \int_0^t \mathrm{e}^{\boldsymbol{J}(t-\tau)}\boldsymbol{B}\boldsymbol{f}_{[s]}(\tau)\mathrm{d}\tau$, then $\|\boldsymbol{V}(t)\| \to 0$ if and only if $\mathrm{Re}(\lambda) < 0$ and $\boldsymbol{B} = \mathbf{0}_{p\times s}$.

Proof (Sufficiency) If $\mathrm{Re}(\lambda) < 0$ and $\boldsymbol{B} = \mathbf{0}_{p\times s}$, then $\|\boldsymbol{V}(t)\| \leq \|\mathrm{e}^{\boldsymbol{J}t}\|\|\boldsymbol{v}\| \to 0$. Therefore, the sufficiency is proved.

(Necessity) Let $\boldsymbol{V}(t) = [\boldsymbol{v}_1(t),\ \boldsymbol{v}_2(t), \ldots,\ \boldsymbol{v}_p(t)]^{\mathrm{T}}$ and $\boldsymbol{B}\boldsymbol{f}_{[s]}(t) = [f_1(t),\ f_2(t), \ldots,\ f_p(t)]^{\mathrm{T}}$, where $f_i(t) = \mathrm{Row}_i(\boldsymbol{B})\boldsymbol{f}_{[s]}(t)$ is a polynomial function.

We will prove the necessity in three steps:

Step 1: prove $f_p(t) \equiv 0$.

It is easy to calculate

$$\boldsymbol{v}_p(t) = \mathrm{e}^{\lambda t}\left[\boldsymbol{v}_p(0) + \int_0^t \mathrm{e}^{-\lambda\tau} f_p(\tau)\mathrm{d}\tau\right].$$

When $\lambda \neq 0$, we have

$$\int_0^t \mathrm{e}^{-\lambda\tau} f_p(\tau)\mathrm{d}\tau = \sum_{i=0}^{s} \frac{f_p^{(i)}(0) - f_p^{(i)}(t)\mathrm{e}^{-\lambda t}}{\lambda^{i+1}}. \tag{12}$$

Then we conclude that

$$\boldsymbol{v}_p(t) = \begin{cases} \boldsymbol{v}_p(0) + \displaystyle\int_0^t f_p(\tau)\mathrm{d}\tau,\ \lambda = 0, \\ \mathrm{e}^{\lambda t}\left[\boldsymbol{v}_p(0) + \displaystyle\sum_{i=0}^{s} \frac{f_p^{(i)}(0)}{\lambda^{i+1}}\right] \\ \quad - \displaystyle\sum_{i=0}^{s} \frac{f_p^{(i)}(t)}{\lambda^{i+1}},\ \lambda \neq 0. \end{cases} \tag{13}$$

Let $k_p = v_p(0) + \sum_{i=0}^{s} \frac{f_p^{(i)}(0)}{\lambda^{i+1}}$. Based on Eq. (13), it is easy to see that

$$|k_p|\mathrm{e}^{\mathrm{Re}(\lambda)t} - \left|\sum_{i=0}^{s} \frac{f_p^{(i)}(t)}{\lambda^{i+1}}\right| \leq |\boldsymbol{v}_p(t)| \tag{14}$$

and that

$$\left|\sum_{i=0}^{s} \frac{f_p^{(i)}(t)}{\lambda^{i+1}}\right| \leq |k_p|\mathrm{e}^{\mathrm{Re}(\lambda)t} + |\boldsymbol{v}_p(t)|. \tag{15}$$

Then, there are five cases to discuss:

(a) $\mathrm{Re}(\lambda) > 0$ and $k_p = 0$. Assume that $f_p(t)$ is not always zero; that is to say, $f_p(t)$ has at least one coefficient that is not zero. Without loss of generality, it can be denoted as

$$f_p(t) = a_r t^r + \cdots + a_1 t + a_0,$$

where $a_r \neq 0$, $0 \leq r \leq s-1$. Then we have

$$\left|\sum_{i=0}^{s} \frac{f_p^{(i)}(t)}{\lambda^{i+1}}\right| = \left|\frac{a_r}{\lambda}t^r + g(t)\right|,\ \frac{a_r}{\lambda} \neq 0,$$

where the degree of $g(t)$ is $r-1$.

Using the absolute value inequality, it is not difficult to find that

$$\left|\frac{a_r}{\lambda}\right|t^r - |g(t)| \leq \left|\sum_{i=0}^{s} \frac{f_p^{(i)}(t)}{\lambda^{i+1}}\right|.$$

Clearly,

$$\lim_{t\to+\infty}\left(\left|\frac{a_r}{\lambda}\right|t^r - |g(t)|\right) = +\infty.$$

Thus, the limit on the left side of inequality (15) is not zero as t approaches infinity. However, the limit on the right side is zero. This is impossible. Hence, $f_p(t) \equiv 0$. In addition, as $k_p = 0$, we have $v_p(0) = 0$.

(b) $\mathrm{Re}(\lambda) > 0$ and $k_p \neq 0$. In this case, the limit on the left side of inequality (14) is positive infinity; however, the limit on the right side is zero. This is also impossible.

(c) $\mathrm{Re}(\lambda) = 0$ and $\mathrm{Im}(\lambda) \neq 0$. As the limit on the right side of inequality (15) is $|k_p|$, we know that $f_p(t)$ is constant. So, we can assume that $f_p(t) = c \in \mathbb{C}$. According to inequalities (14) and (15), if t approaches infinity, then we have

$$\left|v_p(0) + \frac{c}{\lambda}\right| \leq \left|\frac{c}{\lambda}\right| \text{ and } \left|\frac{c}{\lambda}\right| \leq \left|v_p(0) + \frac{c}{\lambda}\right|.$$

It is easy to see that $v_p(0) = 0$, and then $v_p(t) = c(\mathrm{e}^{\lambda t} - 1)/\lambda$. Furthermore, we can calculate that

$$|v_p(t)| = 2\left|\frac{c}{\mathrm{Im}(\lambda)}\right|\left|\sin\left(\frac{\mathrm{Im}(\lambda)}{2}t\right)\right|.$$

Consequently, $f_p(t) = c \equiv 0$ by $|v_p(t)| \to 0$.

(d) $\mathrm{Re}(\lambda) = \mathrm{Im}(\lambda) = 0$. Based on Eq. (13), $f_p(t) \equiv 0$ and $v_p(0) = 0$ due to the fact that $|v_p(t)| \to 0$.

(e) $\mathrm{Re}(\lambda) < 0$. Since the limit on the right side of inequality (15) is zero, we deduce that $f_p(t) \equiv 0$.

From the analysis above, we know that for any $\lambda \in \mathbb{C}$, $f_p(t) \equiv 0$ holds. Furthermore, if $\mathrm{Re}(\lambda) \geq 0$, then $v_p(0) = 0$.

Step 2: prove $f_{p-1}(t) \equiv 0$.

According to $f_p(t) \equiv 0$, it is easy to calculate that

$$\begin{aligned} v_{p-1}(t) =& \mathrm{e}^{\lambda t}[v_p(0)t + v_{p-1}(0)] \\ &+ \int_0^t \mathrm{e}^{\lambda(t-\tau)} f_{p-1}(\tau)\mathrm{d}\tau. \end{aligned} \tag{16}$$

Similar to step 1, there are two cases to discuss:

(a) $\mathrm{Re}(\lambda) \geq 0$. Under this case, we have $v_p(0) = 0$ based on step 1. Similar to the analysis process of step 1, we can obtain $f_{p-1}(t) \equiv 0$ and $v_{p-1}(0) = 0$.

(b) $\mathrm{Re}(\lambda) < 0$. Let $k_{p-1}(t) = v_p(0)t + v_{p-1}(0) + \sum_{i=0}^{s} \frac{f_{p-1}^{(i)}(0)}{\lambda^{i+1}}$. From Eqs. (12) and (16), it follows that

$$v_{p-1}(t) = \mathrm{e}^{\lambda t} k_{p-1}(t) - \sum_{i=0}^{s} \frac{f_{p-1}^{(i)}(t)}{\lambda^{i+1}}.$$

Then we can obtain

$$\left|\sum_{i=0}^{s} \frac{f_{p-1}^{(i)}(t)}{\lambda^{i+1}}\right| \leq |k_{p-1}(t)|\mathrm{e}^{\mathrm{Re}(\lambda)t} + |v_{p-1}(t)|. \tag{17}$$

As the limit on the right side of inequality (17) is zero, we have $f_{p-1}(t) \equiv 0$.

From the analysis above, we know that for any $\lambda \in \mathbb{C}$, $f_{p-1}(t) = 0$ holds. In addition, if $\mathrm{Re}(\lambda) \geq 0$, then $v_{p-1}(0) = 0$.

Step 3: prove $f_i(t) = 0$ for any $i \in \mathbb{Z}_1^{p-2}$.

For $i = p-2, p-3, \ldots, 1$, repeating the analysis similar to that in step 2, we can obtain that for any $\lambda \in \mathbb{C}$, $f_i(t) = 0$ holds, and that if $\mathrm{Re}(\lambda) \geq 0$, then $\boldsymbol{v}_i(0) = 0$.

Combining the results of steps 1–3, for any $i \in \mathbb{Z}_1^p$, we have $\boldsymbol{v}_i(0) = 0$ and $f_i(t) \equiv 0$; i.e., $\boldsymbol{V}(0) = \boldsymbol{v} = \mathbf{0}_p$ and $\boldsymbol{B} = \mathbf{0}_{p\times s}$ if $\mathrm{Re}(\lambda) \geq 0$. This contradicts $\boldsymbol{v} \neq \mathbf{0}_p$, so $\mathrm{Re}(\lambda) < 0$. Furthermore, for any $i \in \mathbb{Z}_1^p$, $f_i(t) = 0$ holds, i.e., $\boldsymbol{B} = \mathbf{0}_{p\times s}$.

Then, we take the Jordan decomposition of $\boldsymbol{A}_{r_s}$. Based on Proposition 3, a more general result is obtained.

For $\boldsymbol{A}_{r_s} \in \mathbb{R}^{r_s\times r_s}$, there exists an invertible matrix $\boldsymbol{N}$ such that

$$\boldsymbol{N}\boldsymbol{A}_{r_s}\boldsymbol{N}^{-1} = \begin{bmatrix} \bar{\boldsymbol{J}}_{\gamma_1\times\gamma_1} & \mathbf{0}_{\gamma_1\times\gamma_2} \\ \mathbf{0}_{\gamma_2\times\gamma_1} & \hat{\boldsymbol{J}}_{\gamma_2\times\gamma_2} \end{bmatrix},$$

where $\bar{\boldsymbol{J}}$ represents the Jordan matrix generated by the eigenvalues of $\boldsymbol{A}_{r_s}$ with a negative real part and $\hat{\boldsymbol{J}}$ generated by the eigenvalues of $\boldsymbol{A}_{r_s}$ with a non-negative real part, γ_1 is the sum of the geometrical multiplicities of eigenvalues with a negative real part, and $\gamma_2 = r_s - \gamma_1$.

Proposition 4 Let $\boldsymbol{v} \in \mathbb{R}^{r_s}$ and $\boldsymbol{B} \in \mathbb{R}^{r_s\times s}$. If $\boldsymbol{V}(t) = \mathrm{e}^{\boldsymbol{A}_{r_s}t}\boldsymbol{v} + \int_0^t \mathrm{e}^{\boldsymbol{A}_{r_s}(t-\tau)}\boldsymbol{B}\boldsymbol{f}_{[r_s]}(\tau)\mathrm{d}\tau$, then $\|\boldsymbol{V}(t)\| \to 0$ if and only if the following conditions hold:

(1) $\boldsymbol{B} = \mathbf{0}_{r_s\times s}$.

(2) $[\mathbf{0}_{\max(\gamma_2,1)\times\gamma_1}, \boldsymbol{I}_{\gamma_2}]\boldsymbol{N}\boldsymbol{v} = \mathbf{0}_{\max(\gamma_2,1)}$.

Proof We divide the proof into two cases:

(a) $\gamma_2 \neq 0$. Let $\bar{\boldsymbol{V}}(t) = \boldsymbol{N}\boldsymbol{V}(t) = [\bar{\boldsymbol{V}}_1^{\mathrm{T}}(t), \bar{\boldsymbol{V}}_2^{\mathrm{T}}(t)]^{\mathrm{T}}$ and $\boldsymbol{N}\boldsymbol{B} = [\boldsymbol{B}_1^{\mathrm{T}}, \boldsymbol{B}_2^{\mathrm{T}}]^{\mathrm{T}}$, where $\bar{\boldsymbol{V}}_1(t) \in \mathbb{R}^{\gamma_1}$, $\bar{\boldsymbol{V}}_2(t) \in \mathbb{R}^{\gamma_2}$, $\boldsymbol{B}_1 \in \mathbb{R}^{\gamma_1\times s}$, and $\boldsymbol{B}_2 \in \mathbb{R}^{\gamma_2\times s}$. Thus, we have

$$\begin{cases} \bar{\boldsymbol{V}}_1(t) = \mathrm{e}^{\bar{\boldsymbol{J}}t}\bar{\boldsymbol{V}}_1(0) + \displaystyle\int_0^t \mathrm{e}^{\bar{\boldsymbol{J}}(t-\tau)}\boldsymbol{B}_1\boldsymbol{f}_{[r_s]}(\tau)\mathrm{d}\tau, \\ \bar{\boldsymbol{V}}_2(t) = \mathrm{e}^{\hat{\boldsymbol{J}}t}\bar{\boldsymbol{V}}_2(0) + \displaystyle\int_0^t \mathrm{e}^{\hat{\boldsymbol{J}}(t-\tau)}\boldsymbol{B}_2\boldsymbol{f}_{[r_s]}(\tau)\mathrm{d}\tau. \end{cases}$$

It follows from Proposition 3 that

$$\|\bar{\boldsymbol{V}}_1(t)\| \to 0 \text{ if and only if } \boldsymbol{B}_1 = \boldsymbol{0}_{\gamma_1\times s}$$

and that

$$\|\bar{\boldsymbol{V}}_2(t)\| \to 0 \text{ if and only if } \bar{\boldsymbol{V}}_2(0) = \boldsymbol{0}_{\gamma_2} \text{ and } \boldsymbol{B}_2 = \boldsymbol{0}_{\gamma_2\times s}.$$

Therefore, $\|\boldsymbol{V}(t)\| \to 0$ if and only if $\boldsymbol{B}_1 = \boldsymbol{0}_{\gamma_1\times s}$, $\boldsymbol{B}_2 = \boldsymbol{0}_{\gamma_2\times s}$, and $\bar{\boldsymbol{V}}_2(0) = \boldsymbol{0}_{\gamma_2}$; i.e., $\boldsymbol{B} = \boldsymbol{0}_{r_s\times s}$ and $[\boldsymbol{0}_{\gamma_2\times\gamma_1}, \boldsymbol{I}_{\gamma_2}]\boldsymbol{N}\boldsymbol{v} = \boldsymbol{0}_{\gamma_2}$.

(b) $\gamma_2 = 0$. From Proposition 3, it can be seen that $\|\boldsymbol{V}(t)\| \to 0$ if and only if $\boldsymbol{B} = \boldsymbol{0}_{s\times s}$ and $\boldsymbol{0}_{1\times r_s}\boldsymbol{N}\boldsymbol{v} = 0$.

Hence, $\|\boldsymbol{V}(t)\| \to 0$ if and only if conditions (1) and (2) in Proposition 4 are established.

The equivalent condition given by Proposition 4 is actually a generalization of Proposition 3. Therefore, it is applicable to a general case.

Next, the definition of asymptotic stability and some corresponding results are given.

Definition 4 Given an initial state $\boldsymbol{x}_0 \in \mathcal{V}_{r_0}$, assume that $\boldsymbol{x}(t)$ is the solution to CCDLS (1). CCDLS (1) is said to be asymptotically stable with $\boldsymbol{x}_0$ if $\|\boldsymbol{x}(t)\| \to 0$.

Denote $\mathcal{M}_{r_0}$ as the domain of attraction for CCDLS (1) in $\mathcal{V}_{r_0}$; that is,

$$\mathcal{M}_{r_0} = \{\boldsymbol{x}_0 \in \mathcal{V}_{r_0} \mid \text{CCDLS (1) is asymptotically stable with } \boldsymbol{x}_0\}.$$

Then, there are some issues to discuss:

(1) What conditions does the initial state $\boldsymbol{x}_0 \in \mathcal{V}_{r_0}$ satisfy for CCDLS (1) to be asymptotically stable with $\boldsymbol{x}_0$?

(2) How to calculate $\mathcal{M}_{r_0}$?

(3) What is the relationship between $\mathcal{M}_{r_j}$ and $\mathcal{M}_{r_{j+1}}$ for any $j \in \mathbb{Z}_0^{s-1}$?

If $\mathcal{V}_{r_0}$ is $\boldsymbol{A}$-invariant, then we have $\boldsymbol{\Phi}_{[r_0,1]} = \boldsymbol{A}_{r_0} = \boldsymbol{A}|_{r_0}$. Hence, in this case, the solution to CCDLS (1) is $\boldsymbol{x}(t) = \mathrm{e}^{\boldsymbol{A}_{r_0}t}\boldsymbol{x}_0$. In addition, as can be seen from the proof of Theorem 1, CCDLS (1) can be transformed into the following linear system:

$$\begin{cases} \boldsymbol{x}'(t) = \boldsymbol{A}_{r_0}\boldsymbol{x}(t), \\ \boldsymbol{x}(0) \in \mathcal{V}_{r_0}. \end{cases}$$

Therefore, we always assume that $\mathcal{V}_{r_0}$ is not $\boldsymbol{A}$-invariant.

By combining Corollary 1 with Proposition 4, we give an equivalent condition for CCDLS (1) to be asymptotically stable with $\boldsymbol{x}_0$.

Theorem 2 If the initial state of CCDLS (1) is $\boldsymbol{x}_0 \in \mathcal{V}_{r_0}$, then CCDLS (1) is asymptotically stable with $\boldsymbol{x}_0$ if and only if the following conditions hold: $\boldsymbol{\varXi} = \boldsymbol{0}_{\alpha\times s}$, $\boldsymbol{\varGamma}\boldsymbol{x}_0 = \boldsymbol{0}_{\alpha-r_s}$, and $\boldsymbol{\varUpsilon}\boldsymbol{x}_0 = \boldsymbol{0}_{\max(\gamma_2,1)}$, where

$$\begin{cases} \boldsymbol{\varGamma} = [\boldsymbol{0}_{(\alpha-r_s)\times r_s}, \boldsymbol{I}_{\alpha-r_s}]\boldsymbol{P}_{[r_s,\frac{\alpha}{r_s}]}(\boldsymbol{I}_{r_0}\otimes \boldsymbol{1}_{\frac{\alpha}{r_0}}), \\ \boldsymbol{\varUpsilon} = [\boldsymbol{0}_{\max(\gamma_2,1)\times\gamma_1}, \boldsymbol{I}_{\gamma_2}]\boldsymbol{N}[\boldsymbol{I}_{r_s}, \boldsymbol{0}_{r_s\times(\alpha-r_s)}] \\ \quad \times\boldsymbol{P}_{[r_s,\frac{\alpha}{r_s}]}(\boldsymbol{I}_{r_0}\otimes \boldsymbol{1}_{\frac{\alpha}{r_0}}). \end{cases}$$

Proof To simplify $\boldsymbol{x}(t)$, let $\bar{\boldsymbol{x}}(t) = \boldsymbol{P}_{[r_s,\frac{\alpha}{r_s}]}\boldsymbol{x}(t) = [\bar{\boldsymbol{x}}_1^{\mathrm{T}}(t), \bar{\boldsymbol{x}}_2^{\mathrm{T}}(t)]^{\mathrm{T}}$, where $\bar{\boldsymbol{x}}_1(t) \in \mathbb{R}^{r_s}$ and $\bar{\boldsymbol{x}}_2(t) \in \mathbb{R}^{\alpha-r_s}$. By Eq. (11), we have

$$\begin{cases} \bar{\boldsymbol{x}}_1(t) = \mathrm{e}^{\boldsymbol{A}_{r_s}t}\bar{\boldsymbol{x}}_1(0) + \displaystyle\int_0^t \mathrm{e}^{\boldsymbol{A}_{r_s}(t-\tau)}\boldsymbol{\varXi}_1\boldsymbol{f}_{[s]}(\tau)\mathrm{d}\tau, \\ \bar{\boldsymbol{x}}_2(t) = \bar{\boldsymbol{x}}_2(0) + \boldsymbol{\varXi}_2\displaystyle\int_0^t \boldsymbol{f}_{[s]}(\tau)\mathrm{d}\tau. \end{cases}$$

From Proposition 4, it follows that $\|\bar{\boldsymbol{x}}_1(t)\| \to 0$ if and only if $\boldsymbol{\varXi}_1 = \boldsymbol{0}_{r_s\times s}$ and $\boldsymbol{\varUpsilon}\boldsymbol{x}_0 = \boldsymbol{0}_{\max(\gamma_2,1)}$. It is obvious that $\|\bar{\boldsymbol{x}}_2(t)\| \to 0$ is equivalent to $\boldsymbol{\varXi}_2 = \boldsymbol{0}_{(\alpha-r_s)\times s}$ and $\bar{\boldsymbol{x}}_2(0) = \boldsymbol{\varGamma}\boldsymbol{x}_0 = \boldsymbol{0}_{\alpha-r_s}$. Hence, $\|\boldsymbol{x}(t)\| \to 0$ if and only if the conditions in Theorem 2 are established.

Now, we can calculate $\mathcal{M}_{r_0}$ based on Theorem 2.

Corollary 2 If the initial space of CCDLS (1) is $\mathcal{V}_{r_0}$, then

$$\mathcal{M}_{r_0} = \{\boldsymbol{\zeta} \in \mathcal{V}_{r_0} | \boldsymbol{\varPsi}\boldsymbol{\zeta} = \boldsymbol{0}_{(s+1)\alpha+\max(-\gamma_1,1-r_s)}\},$$

where $\boldsymbol{\varPsi} = [\boldsymbol{\varGamma}^{\mathrm{T}}, \boldsymbol{\varUpsilon}^{\mathrm{T}}, \boldsymbol{\varPi}_1^{\mathrm{T}}, \boldsymbol{\varPi}_2^{\mathrm{T}}, \ldots, \boldsymbol{\varPi}_s^{\mathrm{T}}]^{\mathrm{T}}$ with $\boldsymbol{\varPi}_j = (\boldsymbol{A}^j \vec{\ltimes} \boldsymbol{I}_{r_0})\otimes \boldsymbol{1}_{\frac{\alpha}{r_j}} - \boldsymbol{\varPhi}^j_{[r_s,\frac{\alpha}{r_s}]}\vec{\ltimes}\boldsymbol{I}_{r_0},\ j = 1,2,\ldots,s$.

Proof $\boldsymbol{\varXi} = \boldsymbol{0}_{\alpha\times s}$ if and only if, for any $j \in \mathbb{Z}_1^s$,

$$\begin{aligned} \mathrm{Col}_j(\boldsymbol{\varXi}) =& (\boldsymbol{A}^j \vec{\ltimes}\boldsymbol{x}_0)\otimes \boldsymbol{1}_{\frac{\alpha}{r_j}} \\ & - \boldsymbol{\varPhi}_{[r_s,\frac{\alpha}{r_s}]}[(\boldsymbol{A}^{j-1}\vec{\ltimes}\boldsymbol{x}_0)\otimes \boldsymbol{1}_{\frac{\alpha}{r_{j-1}}}] \\ =& \boldsymbol{0}_\alpha. \end{aligned} \tag{18}$$

Then for any $j \in \mathbb{Z}_1^s$, $(\boldsymbol{A}^j \vec{\ltimes}\boldsymbol{x}_0)\otimes \boldsymbol{1}_{\frac{\alpha}{r_j}} = \boldsymbol{\varPhi}^j_{[r_s,\frac{\alpha}{r_s}]}(\boldsymbol{x}_0 \otimes \boldsymbol{1}_{\frac{\alpha}{r_0}})$ holds. Thus, Eq. (18) is true if and only if, for any $j \in \mathbb{Z}_1^s$,

$$\begin{aligned} & (\boldsymbol{A}^j \vec{\ltimes}\boldsymbol{x}_0)\otimes \boldsymbol{1}_{\frac{\alpha}{r_j}} - \boldsymbol{\varPhi}^j_{[r_s,\frac{\alpha}{r_s}]}\vec{\ltimes}\boldsymbol{x}_0 \\ =& \left[(\boldsymbol{A}^j \vec{\ltimes}\boldsymbol{I}_{r_0})\otimes \boldsymbol{1}_{\frac{\alpha}{r_j}} - \boldsymbol{\varPhi}^j_{[r_s,\frac{\alpha}{r_s}]}\vec{\ltimes}\boldsymbol{I}_{r_0}\right]\boldsymbol{x}_0 \\ =& \boldsymbol{\varPi}_j\boldsymbol{x}_0 \\ =& \boldsymbol{0}_\alpha. \end{aligned}$$

By Theorem 4, CCDLS (1) is asymptotically stable with $\boldsymbol{x}_0$ if and only if $\boldsymbol{\varPsi}\boldsymbol{x}_0 = \boldsymbol{0}_{(s+1)\alpha+\max(-\gamma_1,1-r_s)}$.

Remark 2 Given the initial space $\mathcal{V}_{r_0}$, we can derive from the composition of $\boldsymbol{\Gamma}$ that for any $\boldsymbol{A} \in \mathbb{R}^{m\times km}$, $\boldsymbol{\Gamma}$ does not change. According to Corollary 2, it can be found that for any $\boldsymbol{A} \in \mathbb{R}^{m\times km}$, $\mathcal{M}_{r_0} \subseteq \mathcal{F}_{r_0}$ holds, where $\mathcal{F}_{r_0}$ is the solution space of $\boldsymbol{\Gamma x} = \boldsymbol{0}_{\alpha - r_s}$, which is critical for further discussion about stabilization.

Remark 3 If $\mathcal{V}_{r_0}$ is not $\boldsymbol{A}$-invariant, then $\mathcal{M}_{r_0} \subset \mathcal{V}_{r_0}$. This means that for any dimension-bounded operator $\boldsymbol{A}$, the domain of attraction for CCDLS (1) in $\mathcal{V}_{r_0}$ is not equal to the entire initial space $\mathcal{V}_{r_0}$.

Theorem 3 Let $\mathcal{H} = \{\boldsymbol{\zeta} \in \mathcal{V}_{r_0} | [\boldsymbol{\Gamma}^{\mathrm{T}}, \boldsymbol{\Upsilon}^{\mathrm{T}}, \boldsymbol{\Pi}_1^{\mathrm{T}}]^{\mathrm{T}}\boldsymbol{\zeta} = \boldsymbol{0}_{2\alpha+\max(-\gamma_1, 1-r_s)}\}$. If $\boldsymbol{x}_0 \in \mathcal{H}$, then $\boldsymbol{x}_0 \in \mathcal{M}_{r_0}$ if and only if $\boldsymbol{A} \vec{\ltimes} \boldsymbol{x}_0 \in \mathcal{M}_{r_1}$.

Proof (Necessity) Let $\bar{\boldsymbol{x}}_0 = \boldsymbol{A} \vec{\ltimes} \boldsymbol{x}_0 \in \mathcal{V}_{r_1}$. Then, the solution to CCDLS (1) starting from the initial state $\bar{\boldsymbol{x}}_0$ is

$$\bar{\boldsymbol{x}}(t) = \boldsymbol{A} \vec{\ltimes} \boldsymbol{x}_0 \vec{\boxplus} t\boldsymbol{A}^2 \vec{\ltimes} \boldsymbol{x}_0 \vec{\boxplus} \ldots \vec{\boxplus} \frac{t^s}{s!}\boldsymbol{A}^{s+1} \vec{\ltimes} \boldsymbol{x}_0 \vec{\boxplus} \ldots.$$

By Theorem 1 and Eq. (10), we conclude that

$$\boldsymbol{x}'(t) = \bar{\boldsymbol{x}}(t) \otimes \boldsymbol{1}_{\frac{\alpha}{\beta}} = \boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]}\boldsymbol{x}(t).$$

Then from $\|\boldsymbol{x}(t)\| \to 0$, it follows that $\|\bar{\boldsymbol{x}}(t)\| \to 0$.

(Sufficiency) Since $\boldsymbol{A} \vec{\ltimes} \boldsymbol{x}_0 \in \mathcal{M}_{r_1}$, we know that for any $j \in \mathbb{Z}_2^s$,

$$(\boldsymbol{A}^{j-1} \vec{\ltimes} \bar{\boldsymbol{x}}_0) \otimes \boldsymbol{1}_{\frac{\beta}{r_j}} = \boldsymbol{\Phi}_{[r_s, \frac{\beta}{r_s}]}[(\boldsymbol{A}^{j-2} \vec{\ltimes} \bar{\boldsymbol{x}}_0) \otimes \boldsymbol{1}_{\frac{\beta}{r_{j-1}}}],$$

where $\beta = \vee_{j=1}^s r_j$. Then for any $j \in \mathbb{Z}_2^s$, we have

$$\begin{aligned}
&(\boldsymbol{A}^j \vec{\ltimes} \boldsymbol{x}_0) \otimes \boldsymbol{1}_{\frac{\alpha}{r_j}} \\
=&\left(\boldsymbol{\Phi}_{[r_s, \frac{\beta}{r_s}]} \otimes \boldsymbol{1}_{\frac{\alpha}{\beta}}\right)\left[(\boldsymbol{A}^{j-1} \vec{\ltimes} \boldsymbol{x}_0) \otimes \boldsymbol{1}_{\frac{\beta}{r_{j-1}}}\right] \\
=&\frac{\beta}{\alpha}\left[(\boldsymbol{\Phi}_{[r_s, \frac{\beta}{r_s}]} \otimes \boldsymbol{1}_{\frac{\alpha}{\beta}}) \otimes \boldsymbol{1}_{\frac{\alpha}{\beta}}^{\mathrm{T}}\right]\left[(\boldsymbol{A}^{j-1} \vec{\ltimes} \boldsymbol{x}_0) \otimes \boldsymbol{1}_{\frac{\alpha}{r_{j-1}}}\right] \\
=&\boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]}\left[(\boldsymbol{A}^{j-1} \vec{\ltimes} \boldsymbol{x}_0) \otimes \boldsymbol{1}_{\frac{\alpha}{r_{j-1}}}\right].
\end{aligned}$$

It follows that for any $j \in \mathbb{Z}_2^s$, $\boldsymbol{\Pi}_j \boldsymbol{x}_0 = \boldsymbol{0}_\alpha$. Since $\boldsymbol{x}_0 \in \mathcal{H}$, we know that $\boldsymbol{x}_0 \in \mathcal{M}_{r_0}$.

From Theorem 3, we can obtain some more general results which reveal the relationship between $\mathcal{M}_{r_j}$ and $\mathcal{M}_{r_{j+1}}$ for any $j \in \mathbb{Z}_0^{s-1}$.

Corollary 3 For CCDLS (1), the following statements are true:

(1) If $\boldsymbol{x}_0 \in \mathcal{M}_{r_0}$, then for any $j \in \mathbb{Z}_1^s$, $\boldsymbol{A}^j \vec{\ltimes} \boldsymbol{x}_0 \in \mathcal{M}_{r_j}$ holds.

(2) If $\mathcal{M}_{r_0} \neq \varnothing$, then for any $j \in \mathbb{Z}_0^{s-1}$, $\varnothing \neq \boldsymbol{A} \vec{\ltimes} \mathcal{M}_{r_j} \subseteq \mathcal{M}_{r_{j+1}}$, where $\boldsymbol{A} \vec{\ltimes} \mathcal{M}_{r_j} = \{\boldsymbol{A} \vec{\ltimes} \boldsymbol{\zeta} \mid \boldsymbol{\zeta} \in \mathcal{M}_{r_j}\}$.

5 Stabilization

Consider CCDLCS (3). Given the initial space $\mathcal{V}_{r_0}$, the dimension of its solution is invariant. Thus, under the feedback $\boldsymbol{u}(t) = \boldsymbol{K x}(t)$, the following closed-loop system can be obtained:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \hat{\boldsymbol{A}} \vec{\ltimes} \boldsymbol{x}(t),\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathcal{V}_{r_0}, \\ \boldsymbol{y}(t) = \boldsymbol{C} \vec{\ltimes} \boldsymbol{x}(t), \end{cases} \tag{19}$$

where $\hat{\boldsymbol{A}} = \boldsymbol{A} \vec{\ltimes} \boldsymbol{I}_\alpha + (\boldsymbol{B} \otimes \boldsymbol{1}_{\beta/m})\boldsymbol{K} \in \mathbb{R}^{\beta\times\alpha}$ and $\beta = \vee_{j=1}^s r_j$.

$\hat{\boldsymbol{A}}$ is a dimension-bounded operator. Given the initial state $\boldsymbol{x}_0 \in \mathcal{V}_{r_0}$, it can be easily verified that for any $j \in \mathbb{Z}^+$, $\hat{\boldsymbol{A}}^j \vec{\ltimes} \boldsymbol{x}_0 \in \mathcal{V}_\beta$ holds. Thus, $\mathcal{V}_{r_1} = \mathcal{V}_\beta$ is $\hat{\boldsymbol{A}}$-invariant, and in this case, we have $\boldsymbol{A}|_{\mathcal{V}_\beta} = \hat{\boldsymbol{A}}_\beta = \hat{\boldsymbol{A}}(\boldsymbol{I}_\beta \otimes \boldsymbol{1}_{\alpha/\beta})$. Similar to CCDLS (1), the closed-loop system (19) can be transformed into the following inhomogeneous linear differential equation:

$$\begin{cases} \boldsymbol{x}'(t) = \hat{\boldsymbol{\Phi}}_{[\beta, \alpha/\beta]}\boldsymbol{x}(t) + t\hat{\boldsymbol{\Xi}}\boldsymbol{x}_0, \\ \boldsymbol{x}(0) = \hat{\boldsymbol{x}}_0, \end{cases}$$

where $\hat{\boldsymbol{\Phi}}_{[\beta,\ (\alpha/\beta)]} = (\beta/\alpha)(\hat{\boldsymbol{A}}_\beta \otimes \boldsymbol{1}_{\alpha/\beta}) \otimes \boldsymbol{1}_{\alpha/\beta}^{\mathrm{T}}$ and $\hat{\boldsymbol{\Xi}} = (\hat{\boldsymbol{A}} \vec{\ltimes} \boldsymbol{I}_{r_0}) \otimes \boldsymbol{1}_{\alpha/\beta} - \hat{\boldsymbol{\Phi}}_{[\beta,\ (\alpha/\beta)]} \vec{\ltimes} \boldsymbol{I}_{r_0}$.

Then, the definition of stabilization and some related results are given.

Definition 5 Given an initial state $\boldsymbol{x}_0 \in \mathcal{V}_{r_0}$, assume that $\boldsymbol{x}(t)$ is the solution to the closed-loop system (19). CCDLCS (3) is said to be asymptotically stabilizable with $\boldsymbol{x}_0$ by feedback $\boldsymbol{u}(t) = \boldsymbol{K x}(t)$ if closed-loop system (19) is asymptotically stable with $\boldsymbol{x}_0$.

Denote $\mathcal{M}_{[\boldsymbol{K}, r_0]}$ as the domain of attraction for closed-loop system (19) in $\mathcal{V}_{r_0}$; that is,

$$\begin{aligned}\mathcal{M}_{[\boldsymbol{K}, r_0]} = \{&\boldsymbol{x}_0 \in \mathcal{V}_{r_0} \mid \text{CCDLCS(3) is} \\ &\text{asymptotically stabilizable with } \boldsymbol{x}_0 \text{ by} \\ &\text{feedback } \boldsymbol{u}(t) = \boldsymbol{K x}(t)\}.\end{aligned}$$

Inspired by the Jordan decomposition of $\boldsymbol{A}_{r_s}$, we can assume that

$$\hat{\boldsymbol{N}}\hat{\boldsymbol{A}}_\beta\hat{\boldsymbol{N}}^{-1} = \begin{bmatrix} \hat{\boldsymbol{J}}_{\hat{\gamma}_1\times\hat{\gamma}_1} & \boldsymbol{0}_{\hat{\gamma}_1\times\hat{\gamma}_2} \\ \boldsymbol{0}_{\hat{\gamma}_2\times\hat{\gamma}_1} & \bar{\boldsymbol{J}}_{\hat{\gamma}_2\times\hat{\gamma}_2} \end{bmatrix},$$

where $\hat{\gamma}_1$ is the sum of the geometrical multiplicities of eigenvalues with a negative real part, $\hat{\gamma}_2 = \beta - \hat{\gamma}_1$, and $\hat{\boldsymbol{J}}$ represents the Jordan matrix generated by the eigenvalues of $\hat{\boldsymbol{A}}_\beta$ with a negative real part and

$\bar{\boldsymbol{J}}$ generated by eigenvalues with a non-negative real part.

Given the initial space and the corresponding controller, the closed-loop system (19) is determined. In this case, the computing method of $\mathcal{M}_{[\boldsymbol{K},r_0]}$ is obtained based on the Jordan canonical form of $\hat{\boldsymbol{A}}_\beta$.

Proposition 5 If the initial space $\mathcal{V}_{r_0}$ of CCDLCS (3) and the corresponding feedback controller $\boldsymbol{K}\in\mathbb{R}^{p\times\alpha}$ are given, then

$$\mathcal{M}_{[\boldsymbol{K},r_0]}=\left\{\boldsymbol{\zeta}\in\mathcal{V}_{r_0}\middle|\left[\hat{\boldsymbol{\varGamma}}^{\mathrm{T}},\hat{\boldsymbol{\varUpsilon}}^{\mathrm{T}}\right]^{\mathrm{T}}\boldsymbol{\zeta}=\mathbf{0}_{\alpha+\pi}\right\},$$

where

$$\begin{cases}\pi=\max(-\hat{\gamma}_1,1-\beta),\\ \hat{\boldsymbol{\varGamma}}=[\mathbf{0}_{(\alpha-\beta)\times\beta},\boldsymbol{I}_{\alpha-\beta}]\boldsymbol{P}_{[\beta,\frac{\alpha}{\beta}]}(\boldsymbol{I}_{r_0}\otimes\mathbf{1}_{\frac{\alpha}{r_0}}),\\ \hat{\boldsymbol{\varUpsilon}}=[\mathbf{0}_{\max(\hat{\gamma}_2,1)\times\hat{\gamma}_1},\boldsymbol{I}_{\hat{\gamma}_2}]\hat{N}[\boldsymbol{I}_\beta,\mathbf{0}_{\beta\times(\alpha-\beta)}]\\ \qquad\times\boldsymbol{P}_{[\beta,\frac{\alpha}{\beta}]}(\boldsymbol{I}_\beta\otimes\mathbf{1}_{\frac{\alpha}{\beta}}).\end{cases}$$

Proof By Corollary 2, we have

$$\mathcal{M}_{[\boldsymbol{K},r_0]}=\left\{\boldsymbol{\zeta}\in\mathcal{V}_{r_0}\middle|\left[\hat{\boldsymbol{\varXi}}^{\mathrm{T}},\hat{\boldsymbol{\varGamma}}^{\mathrm{T}},\hat{\boldsymbol{\varUpsilon}}^{\mathrm{T}}\right]^{\mathrm{T}}\boldsymbol{\zeta}=\mathbf{0}_{2\alpha+\pi}\right\}. \tag{20}$$

As can be seen from Eq. (20), we need only to prove $\hat{\mathcal{F}}_{r_0}=\mathcal{Q}$, where $\hat{\mathcal{F}}_{r_0}=\{\boldsymbol{\zeta}\in\mathcal{V}_{r_0}\mid\hat{\boldsymbol{\varGamma}}\boldsymbol{\zeta}=\mathbf{0}_{\alpha-\beta}\}$ and $\mathcal{Q}=\left\{\boldsymbol{\zeta}\in\mathcal{V}_{r_0}\middle|\left[\hat{\boldsymbol{\varXi}}^{\mathrm{T}},\ \hat{\boldsymbol{\varGamma}}^{\mathrm{T}}\right]^{\mathrm{T}}\boldsymbol{\zeta}=\mathbf{0}_{2\alpha-\beta}\right\}$. Clearly, $\mathcal{Q}\subseteq\hat{\mathcal{F}}_{r_0}$, so we need only to prove $\hat{\mathcal{F}}_{r_0}\subseteq\mathcal{Q}$.

Since $\hat{\boldsymbol{A}}_\beta=\hat{\boldsymbol{A}}(\boldsymbol{I}_\beta\otimes\mathbf{1}_{\frac{\alpha}{\beta}})$, we can obtain

$$\begin{aligned}\hat{\boldsymbol{\varPhi}}_{[\beta,\frac{\alpha}{\beta}]}&=\frac{\beta}{\alpha}(\hat{\boldsymbol{A}}_\beta\otimes\mathbf{1}_{\frac{\alpha}{\beta}})\otimes\mathbf{1}^{\mathrm{T}}_{\frac{\alpha}{\beta}}\\&=\frac{\beta}{\alpha}[(\hat{\boldsymbol{A}}\otimes\mathbf{1}_{\frac{\alpha}{\beta}})(\boldsymbol{I}_\beta\otimes\mathbf{1}_{\frac{\alpha}{\beta}})]\otimes\mathbf{1}^{\mathrm{T}}_{\frac{\alpha}{\beta}}\\&=\frac{\beta}{\alpha}(\hat{\boldsymbol{A}}\otimes\mathbf{1}_{\frac{\alpha}{\beta}})[(\boldsymbol{I}_\beta\otimes\mathbf{1}_{\frac{\alpha}{\beta}})\otimes\mathbf{1}^{\mathrm{T}}_{\frac{\alpha}{\beta}}].\end{aligned}$$

It follows from Proposition 2 that

$$\begin{aligned}\hat{\boldsymbol{\varXi}}&=(\hat{\boldsymbol{A}}\vec{\ltimes}\boldsymbol{I}_{r_0})\otimes\mathbf{1}_{\frac{\alpha}{\beta}}-\hat{\boldsymbol{\varPhi}}_{[\beta,\frac{\alpha}{\beta}]}\vec{\ltimes}\boldsymbol{I}_{r_0}\\&=(\hat{\boldsymbol{A}}\otimes\mathbf{1}_{\frac{\alpha}{\beta}}-\hat{\boldsymbol{\varPhi}}_{[\beta,\frac{\alpha}{\beta}]})(\boldsymbol{I}_{r_0}\otimes\mathbf{1}_{\frac{\alpha}{r_0}})\\&=(\hat{\boldsymbol{A}}\otimes\mathbf{1}_{\frac{\alpha}{\beta}})\left[\boldsymbol{I}_\alpha-\frac{\beta}{\alpha}(\boldsymbol{I}_\beta\otimes\mathbf{1}_{\frac{\alpha}{\beta}})\otimes\mathbf{1}^{\mathrm{T}}_{\frac{\alpha}{\beta}}\right](\boldsymbol{I}_{r_0}\otimes\mathbf{1}_{\frac{\alpha}{r_0}})\\&=(\hat{\boldsymbol{A}}\otimes\mathbf{1}_{\frac{\alpha}{\beta}})\boldsymbol{P}^{-1}_{[\beta,\frac{\alpha}{\beta}]}\begin{bmatrix}\mathbf{0}_{\beta\times\beta}&\mathbf{0}\\\mathbf{0}&\boldsymbol{I}_{\alpha-\beta}\end{bmatrix}\boldsymbol{P}_{[\beta,\frac{\alpha}{\beta}]}(\boldsymbol{I}_{r_0}\otimes\mathbf{1}_{\frac{\alpha}{r_0}})\\&=(\hat{\boldsymbol{A}}\otimes\mathbf{1}_{\frac{\alpha}{\beta}})\boldsymbol{P}^{-1}_{[\beta,\frac{\alpha}{\beta}]}\begin{bmatrix}\mathbf{0}_{\beta\times r_0}\\\hat{\boldsymbol{\varGamma}}\end{bmatrix}.\end{aligned}$$

For any $\boldsymbol{\zeta}\in\hat{\mathcal{F}}_{r_0}$, we have $\hat{\boldsymbol{\varXi}}\boldsymbol{\zeta}=\mathbf{0}_\alpha$. Therefore, $\hat{\mathcal{F}}_{r_0}\subseteq\mathcal{Q}$.

For the given $\mathcal{V}_{r_0}$, $\mathcal{M}_{[\boldsymbol{K},r_0]}$ depends only on controller $\boldsymbol{K}$. If $\boldsymbol{K}$ varies, then $\mathcal{M}_{[\boldsymbol{K},r_0]}$ may also vary. It can be seen from Proposition 5 that $\mathcal{M}_{[\boldsymbol{K},r_0]}\subseteq\hat{\mathcal{F}}_{r_0}$ can be obtained regarding the choice of $\boldsymbol{K}\in\mathbb{R}^{p\times\alpha}$ after $\mathcal{V}_{r_0}$ is given. In this situation, there are two problems to discuss in the following:

(1) What conditions does $\boldsymbol{K}$ satisfy for $\mathcal{M}_{[\boldsymbol{K},r_0]}$ to be equal to $\hat{\mathcal{F}}_{r_0}$?

(2) How to design $\boldsymbol{K}$ such that $\mathcal{M}_{[\boldsymbol{K},r_0]}=\hat{\mathcal{F}}_{r_0}$?

According to Proposition 5, we give the following theorem:

Theorem 4 If there exists $\boldsymbol{K}$ such that all eigenvalues of $\hat{\boldsymbol{A}}_\beta$ have a negative real part, then $\mathcal{M}_{[\boldsymbol{K},r_0]}=\hat{\mathcal{F}}_{r_0}$.

Proof If $\boldsymbol{K}$ satisfies the condition in Theorem 4, then $\hat{\boldsymbol{\varUpsilon}}=\mathbf{0}_{1\times r_0}$. Under this circumstance, $\mathcal{M}_{[\boldsymbol{K},r_0]}=\hat{\mathcal{F}}_{r_0}$.

Note that Theorem 4 is necessary and sufficient if $r_0=\alpha$.

Using Theorem 4 and the Lyapunov theorem, an algorithm for designing the corresponding controller $\boldsymbol{K}$ is obtained (Algorithm 1).

Algorithm 1 Iterative algorithm for computing the controller $\boldsymbol{K}$ satisfying $\mathcal{M}_{[\boldsymbol{K},r_0]}=\hat{\mathcal{F}}_{r_0}$

1: Let $\bar{\boldsymbol{A}}=(\boldsymbol{A}\vec{\ltimes}\boldsymbol{I}_\alpha)(\boldsymbol{I}_\beta\otimes\mathbf{1}_{\frac{\alpha}{\beta}})$ and $\bar{\boldsymbol{B}}=\boldsymbol{B}\otimes\mathbf{1}_{\frac{\beta}{m}}$
2: Find $\hat{\boldsymbol{K}}$ and a positive definite matrix $\boldsymbol{W}$ to satisfy $\boldsymbol{W}\bar{\boldsymbol{A}}^{\mathrm{T}}+\bar{\boldsymbol{A}}\boldsymbol{W}+\hat{\boldsymbol{K}}^{\mathrm{T}}\bar{\boldsymbol{B}}^{\mathrm{T}}+\bar{\boldsymbol{B}}\hat{\boldsymbol{K}}<\mathbf{0}_{\beta\times\beta}$
3: **for** $i=1$ to β **do**
4: Find $\boldsymbol{K}_i\in\mathbb{R}^{p\times\frac{\alpha}{\beta}}$ satisfying $\boldsymbol{K}_i\mathbf{1}_{\frac{\alpha}{\beta}}=\mathrm{Col}_i(\hat{\boldsymbol{K}}\boldsymbol{W}^{-1})$
5: **end for**
6: Set $\boldsymbol{K}=[\boldsymbol{K}_1,\ \boldsymbol{K}_2,\ldots,\ \boldsymbol{K}_\beta]$

6 Examples

In this section, we give two examples to illustrate the validity of the theoretical results.

Example 1 Consider CCDLS (1) with

$$\boldsymbol{A}=\begin{bmatrix}-9&1&-6&4\\7&4&1&10\end{bmatrix}\text{ and }\boldsymbol{x}_0\in\mathbb{R}^8.$$

It follows that $\boldsymbol{A}\vec{\ltimes}\boldsymbol{x}_0\in\mathbb{R}^4$ and $\boldsymbol{A}^2\vec{\ltimes}\boldsymbol{x}_0\in\mathbb{R}^2$. We can verify that $\mathbb{R}^2$ is $\boldsymbol{A}$-invariant. Therefore, we can calculate that

$$\boldsymbol{A}_{r_2}=\boldsymbol{A}_{|\mathbb{R}^2}=\begin{bmatrix}-8&-2\\11&11\end{bmatrix},$$

and that its eigenvalues are $-9973/1475$ and $859/88$. Furthermore,

$$\boldsymbol{N}=\begin{bmatrix}-1322/1555&269/2404\\1469/2790&-633/637\end{bmatrix}$$

and

$$\boldsymbol{P}_{[2,4]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

By the definition of $\boldsymbol{\Psi}$, it is easy to calculate that $\operatorname{rank}(\boldsymbol{\Psi}) = 7 < 8$. To solve $\boldsymbol{\Psi x} = \boldsymbol{0}$, it can be seen that $\boldsymbol{\zeta} = [134/71, 134/71, 134/71, 134/71, 1, 1, 1, 1]^{\mathrm{T}}$ is the fundamental solution. By Theorem 1, the solution to CCDLS (1) with the initial state $\boldsymbol{\zeta}$ is $\boldsymbol{x}(t) = \mathrm{e}^{\boldsymbol{\Phi}_{[2,4]}t}\boldsymbol{\zeta}$. According to Corollary 2, we have $\mathcal{M}_8 = \{k\boldsymbol{\zeta} | k \in \mathbb{R}\}$.

Example 2 Considering CCDLCS (3) with

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 5 & 3 \\ -1 & -5 & 3 & 3 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 2 & -5 \end{bmatrix},$$

and $\boldsymbol{x}_0 \in \mathbb{R}^4$, we have $\boldsymbol{A} \vec{\ltimes} x_0 \in \mathbb{R}^2$. Clearly, $\mathbb{R}^2$ is $\boldsymbol{A}$-invariant. Then we can obtain

$$\boldsymbol{A}_{r_1} = \boldsymbol{A}_{|\mathbb{R}^2} = \begin{bmatrix} 3 & 8 \\ -6 & 6 \end{bmatrix},$$

$$\hat{\boldsymbol{\Gamma}} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

and

$$\boldsymbol{P}_{[2,2]} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

To solve $\hat{\boldsymbol{\Gamma}}\boldsymbol{x} = \boldsymbol{0}$, we know that $\boldsymbol{\xi} = [1, 1, 0, 0]^{\mathrm{T}}$ and $\boldsymbol{\zeta} = [0, 0, 1, 1]^{\mathrm{T}}$ are the fundamental solutions. Then, $\hat{\mathcal{F}}_4 = \{k_1\boldsymbol{\xi} + k_2\boldsymbol{\zeta} | k_1, k_2 \in \mathbb{R}\}$. $\boldsymbol{X}_1(t) = \mathrm{e}^{\boldsymbol{\Phi}_{[2,2]}t}\boldsymbol{\xi}$ is the solution to CCDLCS (3) starting from $\boldsymbol{\xi}$, and $\boldsymbol{X}_2(t) = \mathrm{e}^{\boldsymbol{\Phi}_{[2,2]}t}\boldsymbol{\zeta}$ is the solution to CCDLCS (3) starting from $\boldsymbol{\zeta}$.

It is easy to calculate that $\hat{\boldsymbol{K}} = [3.2948,\ -7.2214]$. There are many corresponding controllers, such as $\boldsymbol{K}' = [1.1393,\ 2.1555, -5.5225, -1.6989]$, $\boldsymbol{K}'' = [-0.5,\ 3.2948,\ -3.5,\ 3.7214]$, and $\boldsymbol{K}''' = [1,\ 2.2948,\ 1,\ -8.2214]$. That is to say,

$$\mathcal{M}_{[\boldsymbol{K}',4]} = \mathcal{M}_{[\boldsymbol{K}'',4]} = \mathcal{M}_{[\boldsymbol{K}''',4]} = \hat{\mathcal{F}}_4.$$

Therefore, CCDLCS (3) is asymptotically stabilizable with $\hat{\mathcal{F}}_4$ by feedback $\boldsymbol{u}(t) = \boldsymbol{K}'\boldsymbol{x}(t)$ ($\boldsymbol{u}(t) = \boldsymbol{K}''\boldsymbol{x}(t)$, $\boldsymbol{u}(t) = \boldsymbol{K}'''\boldsymbol{x}(t)$).

Through the analysis of this study, we know that there are many corresponding controllers. However, here, only three kinds of controllers are given to verify the effectiveness of the method in this study.

Remark 4 Fig. 1a shows the variation of the norm of $\boldsymbol{X}_1(t)$ and $\boldsymbol{X}_2(t)$ under the case of $\boldsymbol{u}(t) \equiv 0$, and Fig. 1b shows them under the case of closed-loop control.

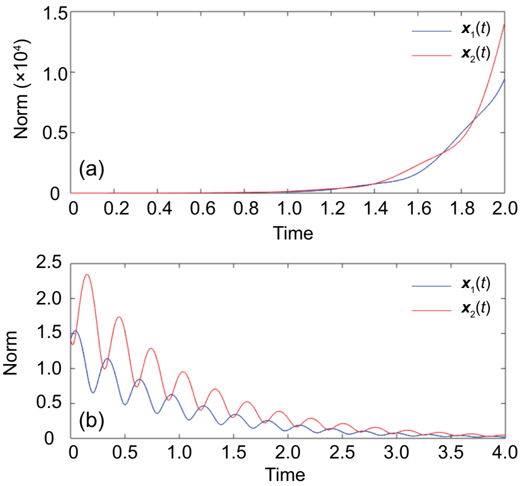

Fig. 1 Norm of the trajectory (Example 2): (a) open-loop system; (b) closed-loop system

7 Conclusions

In this study, the solution to continuous-time cross-dimensional linear systems has been proposed through integral iteration. This solution has a simpler algebraic form than that in Cheng et al. (2017). Based on this, an equivalent condition has been obtained when a CCDLS is asymptotically stable with a given initial value. Then we have analyzed which initial state can be stabilized, and provided a method to calculate the corresponding controller. In the future, the controllability and observability of CCDLCSs may be studied with the solutions obtained in this study.

Contributors

Qing-le ZHANG designed the research and drafted the manuscript. Qing-le ZHANG and Jun-e FENG processed the data. Biao WANG and Jun-e FENG helped organize the manuscript. Qing-le ZHANG and Jun-e FENG revised and finalized the paper.

Compliance with ethics guidelines

Qing-le ZHANG, Biao WANG, and Jun-e FENG declare that they have no conflict of interest.

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