Frontiers of Information Technology & Electronic Engineering
www.jzus.zju.edu.cn; engineering.cae.cn; www.springerlink.com
ISSN 2095-9184 (print); ISSN 2095-9230 (online)
E-mail: jzus@zju.edu.cn

Complete synchronization of coupled Boolean networks with arbitrary finite delays*

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Received Aug. 26, 2019; Revision accepted Oct. 9, 2019; Crosschecked Dec. 13, 2019

Abstract: In this study, the complete synchronization problem of coupled delayed Boolean networks (CDBNs) is investigated. The state delays and output delays may not be equal, and the state delay in each Boolean network may be different in the proposed CDBN model. Based on the semi-tensor product of matrices, a necessary and sufficient condition for the complete synchronization of CDBNs is obtained. Then, an efficient algorithm for solving the synchronization of CDBNs is provided. Finally, numerical examples are presented to demonstrate the effectiveness of our algorithm.

Key words: Boolean networks; Synchronization; Time delay
https://doi.org/10.1631/FITEE.1900438 **CLC number:** O233

1 Introduction

Since Kauffman (1969) first introduced Boolean networks (BNs), they have been used extensively in modeling nonlinear and complex biological systems (Shmulevich et al., 2003; Richardson, 2005; Cheng and Qi, 2010). In a BN, each node's state is described by a binary variable, i.e., 1 (ON) or 0 (OFF). Each node updates its state based on the states of other nodes and the Boolean function (Heidel et al., 2003). BNs are quite significant in that they can not only be used in biological systems, but also provide a detailed description of the behavior in many other systems (Heidel et al., 2003; Zou and Zhu, 2014; Kobayashi and Hiraishi, 2017; Meng et al., 2018).

Recently, Cheng and Qi (2010) proposed the semi-tensor product (STP) of matrices, which has been applied to BNs successfully. Using the STP, the unique algebraic framework of BNs can be constructed and the BNs can be transformed into an equivalent discrete dynamical system (Cheng et al., 2011). Based on the STP technique, many basic problems concerning BNs have been studied, such as stabilization (Guo et al., 2015; Li YY et al., 2018a; Liu RJ et al., 2018; Lu et al., 2018; Zhu SY et al., 2018, 2019; Li BW et al., 2019a; Huang et al., 2020; Sun et al., 2020; Zhong et al., 2020), synchronization (Heidel et al., 2003; Li R and Chu, 2012; Liu Y et al., 2016; Li YY, 2017; Zhong et al., 2017; Li YY et al., 2018b; Yang et al., 2019, 2020), optimal control (Wu and Shen, 2018; Zhu QX et al., 2018; Zhong et al., 2019; Zhu SY et al., 2019), controllability and observability (Laschov et al., 2013; Zhu QX et al., 2019), output tracking (Liu Y et al., 2017; Li YY et al., 2019), fault detection (Fornasini and Valcher, 2015), and robust invariant set of BNs (Tong et al.,

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* Project supported by the National Natural Science Foundation of China (No. 61503115), the Fundamental Research Funds for the Central Universities, China (No. JZ2017HGTB0188), and the National Innovative Program for College Students, China (No. 201810359056)
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2018; Li BW et al., 2019b, 2019c).

One of the goals of studying nonlinear systems is to understand how collective behavior, such as synchronization and consensus, emerges in networked systems. Recently, synchronization of BNs has been studied extensively because of its wide applications in chemistry, economy, biology, and so on. Many interesting synchronization problems of BNs have been studied (Li R and Chu, 2012; Liu Y et al., 2016; Li YY, 2017; Yang et al., 2019). In Li R and Chu (2012), the complete synchronization of drive-response BNs without time delay has been studied.

In the real world, time delays are unavoidable and should be considered in BNs. In Zhong et al. (2014), the complete synchronization of coupled delayed Boolean networks (CDBNs) was investigated. In Liu RJ et al. (2018), a feedback controller was designed to realize the stabilization of BNs with state delay. However, it can be found that the models in Zhong et al. (2014) require a condition that the time delays between different nodes in different BNs need to be the same. In fact, time delays in message transmission between different nodes and BNs can be distinctive in many real-world systems. Moreover, the different output delays should be considered in CDBNs. Motivated by the above discussion, a complete synchronization of the coupled BNs with arbitrary finite delays is investigated. Based on the theory of STP, a necessary and sufficient criterion for the complete synchronization of the CDBNs is obtained. The contributions of this study can be listed as follows:

1. In contrast to previous works, the restriction that the output and state delays are equal is removed in our model. Furthermore, we do not require the assumption that the state delays in each BNs are equal.

2. A necessary and sufficient criterion is given for the complete synchronization of the proposed general BN model.

3. An efficient algorithm is proposed to verify the synchronization condition of CDBNs.

2 Preliminaries

2.1 Concepts and basic notations

To analyze the synchronization of CDBNs, we first introduce the STP of matrices. Let $\otimes$ denote the Kronecker product of matrices and $\boldsymbol{I}$ denote the identity matrix.

Definition 1 (Cheng et al., 2011) The STP of two matrices $\boldsymbol{C}_{pq}$ (p rows and q columns) and $\boldsymbol{D}_{mn}$ (m rows and n columns) is

$$\boldsymbol{C} \ltimes \boldsymbol{D} = (\boldsymbol{C} \otimes \boldsymbol{I}_{e/q})(\boldsymbol{D} \otimes \boldsymbol{I}_{e/m}),$$

where e is the least common multiple of q and m.

Based on the definition of STP, we can know that the general matrix product is just a special case of the STP when $q = m$. Hence, we will omit "$\ltimes$" for convenience in the following if no confusion is caused. Some symbols are presented as follows:

1. For a matrix $\boldsymbol{F}$ with p rows and q columns, we define that $\mathbf{Col}_k(\boldsymbol{F})$ is the k^{th} column of $\boldsymbol{F}$ and $\mathbf{Col}(\boldsymbol{F}) := \{\mathbf{Col}_k(\boldsymbol{F}) : 1 \leq k \leq q\}$.

2. $\boldsymbol{I}_k$ is an identity matrix with k rows and k columns. We define that $\boldsymbol{\delta}_k^i = \mathbf{Col}_i(\boldsymbol{I}_k)$ and $\boldsymbol{\Delta}_k := \{\boldsymbol{\delta}_k^i : 1 \leq i \leq k\}$.

3. For convenience, we express $\boldsymbol{F} = [\boldsymbol{\delta}_p^{i_1}, \boldsymbol{\delta}_p^{i_2}, \ldots, \boldsymbol{\delta}_p^{i_q}] \in \boldsymbol{\mathcal{L}}_{p\times q}$ as $\boldsymbol{F} = \boldsymbol{\delta}_p[i_1, i_2, \ldots, i_q]$.

4. We define $\boldsymbol{\Phi}_n$ as $\boldsymbol{\delta}_{2^{2n}}\{1, 2^n+2, \ldots, (2^n-2)\cdot 2^n + 2^n - 1, 2^{2n}\}$.

5. $\boldsymbol{W}_{[a,b]}$ is used to represent the swap matrix. Label $\boldsymbol{W}_{[a,b]}$'s columns by $(11, 12, \ldots, 1b, \ldots, a1, a2, \ldots, ab)$ and its rows by $(11, 21, \ldots, a1, \ldots, 1b, 2b, \ldots, ab)$. Then its element in position $((M,N),(m,n))$ is assigned as

$$w_{(M,N),(m,n)} = \boldsymbol{\delta}_{m,n}^{M,N} = \begin{cases} 1, & M = m \text{ and } N = n, \\ 0, & \text{otherwise.} \end{cases}$$

6. We use vectors $\boldsymbol{\delta}_2^1$ and $\boldsymbol{\delta}_2^2$ to denote the Boolean variables 1 and 0. The Boolean function $f : \{1,0\}^n \to \{1,0\}$ can be considered as a mapping from $\boldsymbol{\Delta}_2^n$ to $\boldsymbol{\Delta}_2$.

The STP of matrices has many good properties (Cheng et al., 2011) and we list some of them, which will be used in the following:

1. If $\boldsymbol{\sigma} \in \boldsymbol{\Delta}_n$, for any matrix $\boldsymbol{A}$, $\boldsymbol{\sigma} \ltimes \boldsymbol{A} = (\boldsymbol{I}_n \otimes \boldsymbol{A}) \ltimes \boldsymbol{\sigma}$.

2. If $\boldsymbol{\sigma} \in \boldsymbol{\Delta}_{2^n}$, then we can obtain $\boldsymbol{\sigma} \ltimes \boldsymbol{\sigma} = \boldsymbol{\Phi}_n \boldsymbol{\sigma}$.

3. We call $\mathbf{Ed} = \boldsymbol{\delta}_2[1,2,1,2]$ the dummy matrix. Then, we have $\mathbf{Ed}\boldsymbol{x}\boldsymbol{y} = \boldsymbol{y}$, $\forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{\Delta}_2$.

4. If $\boldsymbol{\phi}_1 \in \boldsymbol{\Delta}_m$ and $\boldsymbol{\phi}_2 \in \boldsymbol{\Delta}_n$, then $\boldsymbol{W}_{[m,n]}(\boldsymbol{\phi}_2 \ltimes \boldsymbol{\phi}_1) = \boldsymbol{\phi}_1 \ltimes \boldsymbol{\phi}_2$. For convenience, we use $\boldsymbol{W}_{[n]}$ to denote $\boldsymbol{W}_{[n,n]}$.

Lemma 1 (Cheng et al., 2011) Let $h : \boldsymbol{\Delta}_2^n \to \boldsymbol{\Delta}_2$ be a Boolean function and $\boldsymbol{H} \subseteq \boldsymbol{\mathcal{L}}_{2\times 2^n}$ the structure matrix of h. Then, for every $(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \ldots, \boldsymbol{\sigma}_n) \in \boldsymbol{\Delta}_2^n$, $h(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \ldots, \boldsymbol{\sigma}_n) = \boldsymbol{H} \ltimes \boldsymbol{\sigma}_1 \ltimes \boldsymbol{\sigma}_2 \ltimes \ldots \ltimes \boldsymbol{\sigma}_n$.

2.2 Problem formulation

The objective of this study is to give the necessary and sufficient condition of synchronization of the following system:

$$\begin{cases} \boldsymbol{X}_j^i(t+1) = f_j^i\left(\boldsymbol{X}_j^1(t-\tau_j^1), \boldsymbol{X}_j^2(t-\tau_j^2), \ldots,\right. \\ \qquad \boldsymbol{X}_j^N(t-\tau_j^N), \boldsymbol{Y}_1(t-\alpha_1), \\ \qquad \left.\boldsymbol{Y}_2(t-\alpha_2), \ldots, \boldsymbol{Y}_M(t-\alpha_M)\right), \\ \boldsymbol{Y}_j(t) = h_j\left(\boldsymbol{X}_j^1(t), \boldsymbol{X}_j^2(t), \ldots, \boldsymbol{X}_j^N(t)\right), \end{cases} \tag{1}$$

where $\boldsymbol{X}_j^i \in \boldsymbol{\Delta}_2$ means the state of the i^{th} node of the j^{th} BN and $\boldsymbol{Y}_j$ means the output of the j^{th} BN. Both f_j^i and h_j are Boolean functions ($j = 1, 2, \ldots, M$ and $i = 1, 2, \ldots, N$); α_l ($l = 1, 2, \ldots, M$) and τ_j^k ($k = 1, 2, \ldots, N$) are arbitrary integers, denoting output delays and state delays, respectively. Denote $\boldsymbol{X}_j(t) = \ltimes_{k=1}^N \boldsymbol{X}_j^k(t)$, which represents the state of the j^{th} BN.

Definition 2 The array of BNs (1) is said to realize complete synchronization if for any initial states $\boldsymbol{X}_j(-\tau)$, $\boldsymbol{X}_j(-\tau+1)$, ..., $\boldsymbol{X}_j(-1)$, $\boldsymbol{X}_j(0) \in \boldsymbol{\Delta}_{2^N}$, $j = 1, 2, \ldots, M$, there exists a positive integer T such that $\forall t \geq T$, $\boldsymbol{X}_j(t) = \boldsymbol{X}_i(t)$ holds for any $1 \leq i, j \leq M$.

3 Main results

3.1 A simplified model

Before we discuss the complete synchronization criterion for system (1), an easier model

$$\begin{cases} \boldsymbol{X}_j^i(t+1) = f_j^i\left(\boldsymbol{X}_j^1(t-\tau_1), \boldsymbol{X}_j^2(t-\tau_2), \ldots,\right. \\ \qquad \left.\boldsymbol{X}_j^N(t-\tau_N), \boldsymbol{Y}_1(t), \boldsymbol{Y}_2(t), \ldots, \boldsymbol{Y}_M(t)\right), \\ \boldsymbol{Y}_j(t) = h_j\left(\boldsymbol{X}_j^1(t), \boldsymbol{X}_j^2(t), \ldots, \boldsymbol{X}_j^N(t)\right), \end{cases} \tag{2}$$

will be studied first, in which the time delay between the i^{th} ($i = 1, 2, \ldots, N$) and j^{th} ($j = 1, 2, \ldots, M$) nodes in each Boolean network is the same and satisfies the condition $\tau_1 < \tau_2 < \ldots < \tau_N$. Moreover, the output delays do not exist in system (2).

Let $\boldsymbol{F}_j^i$ and $\boldsymbol{H}_j$ be the structure matrices of f_j^i and h_j, respectively. Define

$$\boldsymbol{Y}(t) = \ltimes_{k=1}^M \boldsymbol{Y}_k(t).$$

In the following, some useful lemmas for system (2) are given.

Lemma 2 Let $\boldsymbol{u}_k = \mathbf{Ed}^{N-k} \boldsymbol{W}_{[2,2^{N-k}]} \mathbf{Ed}^{k-1}$. Then

$$\boldsymbol{X}_j^k(t-\tau_k) = \boldsymbol{u}_k \boldsymbol{X}_j(t-\tau_k).$$

The proof is given in Appendix A.

Lemma 3 Let $\boldsymbol{M}_j^i = \boldsymbol{F}_j^i \boldsymbol{u}_1 \left\{\ltimes_{k=2}^N (\boldsymbol{I}_{2^{(k-1)N}} \otimes \boldsymbol{u}_k)\right\}$ and $\widehat{\boldsymbol{X}}_j(t) = \ltimes_{k=1}^N \boldsymbol{X}_j(t-\tau_k)$. It holds that

$$\boldsymbol{X}_j^i(t+1) = \boldsymbol{M}_j^i \widehat{\boldsymbol{X}}_j(t) \boldsymbol{Y}(t).$$

The proof is given in Appendix B.

Let

$$\boldsymbol{M}_j = \boldsymbol{M}_j^1 \left\{\ltimes_{i=2}^N \left[(\boldsymbol{I}_{2^{N^2+M}} \otimes \boldsymbol{M}_j^i) \ltimes \boldsymbol{\Phi}_{N^2+M}\right]\right\}.$$

Based on Lemmas 2 and 3, we have

$$\begin{aligned} \boldsymbol{X}_j(t+1) = &\ltimes_{k=1}^N \left(\boldsymbol{M}_j^k \widehat{\boldsymbol{X}}_j(t) \boldsymbol{Y}(t)\right) \\ =&\boldsymbol{M}_j^1 \{\ltimes_{i=2}^N [(\boldsymbol{I}_{2^{N^2+M}} \otimes \boldsymbol{M}_j^i) \ltimes \boldsymbol{\Phi}_{N^2+M}]\} \\ &\ltimes \widehat{\boldsymbol{X}}_j(t) \boldsymbol{Y}(t) \\ =&\boldsymbol{M}_j \widehat{\boldsymbol{X}}_j(t) \boldsymbol{Y}(t). \end{aligned} \tag{3}$$

Letting $\boldsymbol{H} = \otimes_{j=1}^M \boldsymbol{H}_j$, we can obtain

$$\begin{aligned} \boldsymbol{Y}(t) =& \left(\boldsymbol{H}_1 \ltimes_{k=1}^N \boldsymbol{X}_1^k(t)\right) \otimes \left(\boldsymbol{H}_2 \ltimes_{k=1}^N \boldsymbol{X}_2^k(t)\right) \otimes \ldots \\ &\otimes \left(\boldsymbol{H}_M \ltimes_{k=1}^N \boldsymbol{X}_M^k(t)\right) \\ =& (\boldsymbol{H}_1 \boldsymbol{X}_1(t)) \otimes (\boldsymbol{H}_2 \boldsymbol{X}_2(t)) \otimes \ldots \\ &\otimes (\boldsymbol{H}_M \boldsymbol{X}_M(t)) \\ =&(\otimes_{j=1}^M \boldsymbol{H}_j) \left(\ltimes_{k=1}^M \boldsymbol{X}_k(t)\right) \\ =&\boldsymbol{H} \ltimes_{j=1}^M \boldsymbol{X}_j(t). \end{aligned} \tag{4}$$

Hence, BN (2) can be described as algebraic representations (3) and (4). Denote

$$\begin{cases} \boldsymbol{W} = \boldsymbol{W}_{[2^M, 2^{N^2}]} \ltimes \left\{\ltimes_{i=2}^M \left[(\boldsymbol{I}_{2^M} \otimes \boldsymbol{W}_{[2^M, 2^{iN^2}]}) \boldsymbol{\Phi}_M\right]\right\}, \\ \boldsymbol{P} = \boldsymbol{P}_1 \ltimes_{j=1}^{N-2} \{\boldsymbol{I}_{2^{jMN}} \otimes \boldsymbol{P}_{j+1}\}, \\ \boldsymbol{I} = \ltimes_{j=0}^{N-1} \left(\boldsymbol{I}_{2^{(\tau_j+1)MN}} \otimes \mathbf{Ed}^{MN(\tau_{j+1}-\tau_j-1)}\right), \end{cases}$$

where $\boldsymbol{P}_q = \ltimes_{j=1}^{M-1} \left(\boldsymbol{I}_{2^{jN}} \otimes \boldsymbol{W}_{[2^N, 2^{j(N-q)N}]}\right)$, $q = 1, 2, \ldots, N-1$, and $\tau_0 = 0$.

Lemma 4 For system (2), the following equality holds:

$$\begin{aligned} \ltimes_{j=1}^M \boldsymbol{X}_j(t+1) =& (\otimes_{j=1}^M \boldsymbol{M}_j) \boldsymbol{W} \boldsymbol{H} (\boldsymbol{I}_{2^{MN}} \otimes \boldsymbol{P}) \\ &\ltimes \boldsymbol{I} \ltimes_{k=0}^{\tau_N} \left(\ltimes_{j=1}^M \boldsymbol{X}_j(t-k)\right). \end{aligned}$$

The proof is given in Appendix C.

Theorem 1 System (2) realizes synchronization if and only if there exists a positive integer d satisfying

$$\mathbf{Col}(\boldsymbol{Q}\boldsymbol{\Omega}^{d-1}) \subseteq \left\{\boldsymbol{\delta}_{2^{MN}}^{\rho_i} : \rho_i = 1 + \frac{(i-1)(2^{MN}-1)}{2^N - 1}\right\}, \tag{5}$$

where $d_0=\min\{m: m\geq 1, \boldsymbol{Q\Omega}^{m-1}=\boldsymbol{Q\Omega}^{n-1}, n>m\}$, $1\leq d\leq d_0$, $\boldsymbol{Q}=(\otimes_{j=1}^{M}\boldsymbol{M}_j)\boldsymbol{WH}(\boldsymbol{I}_{2^{MN}}\otimes\boldsymbol{P})\boldsymbol{I}$, $\boldsymbol{\Omega}=\boldsymbol{QW}_{[2^{MN\tau_N},2^{MN\tau_N+1}]}\boldsymbol{\Phi}_{MN\tau_N}$, and $i=1,2,\ldots,2^N$.

Proof First, let us show the necessity.

Let

$$\begin{cases}\boldsymbol{V}(t)=\ltimes_{k=0}^{\tau_N}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right),\\ \boldsymbol{\Omega}=\boldsymbol{QW}_{[2^{MN\tau_N},2^{MN\tau_N+1}]}\boldsymbol{\Phi}_{MN\tau_N},\\ \boldsymbol{Q}=(\otimes_{j=1}^{M}\boldsymbol{M}_j)\boldsymbol{WH}(\boldsymbol{I}_{2^{MN}}\otimes\boldsymbol{P})\boldsymbol{I}.\end{cases}$$

Then, Lemma 4 implies

$$\ltimes_{j=1}^{M}\boldsymbol{X}_j(t+1)=\boldsymbol{QV}(t). \tag{6}$$

Furthermore, it holds that

$$\begin{aligned}\boldsymbol{V}(t+1)=&\ltimes_{k=0}^{\tau_N}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t+1-k)\right)\\ =&\boldsymbol{QV}(t)\ltimes_{k=1}^{\tau_N}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t+1-k)\right)\\ =&\boldsymbol{QW}_{[2^{MN\tau_N},2^{MN(\tau_N+1)}]}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t)\right)\\ &\ltimes\ldots\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-\tau_N+1)\right)\boldsymbol{V}(t)\\ =&\boldsymbol{QW}_{[2^{MN\tau_N},2^{MN\tau_N+1}]}\boldsymbol{\Phi}_{MN\tau_N}\boldsymbol{V}(t)\\ =&\boldsymbol{\Omega V}(t).\end{aligned}$$

Hence, we have

$$\ltimes_{j=1}^{M}\boldsymbol{X}_j(t)=\boldsymbol{Q\Omega}^{t-1}\ltimes_{k=0}^{\tau_N}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(-k)\right).$$

If system (2) realizes complete synchronization, then for any initial values $\boldsymbol{X}_j(0)$, $\boldsymbol{X}_j(-1)$, ..., $\boldsymbol{X}_j(-\tau_N)$, $j=1,2,\ldots,M$, there exists a positive integer d such that $\boldsymbol{X}_j(d)=\boldsymbol{X}_i(d)$, $\forall 1\leq i,j\leq M$. Assume $\boldsymbol{X}_j(d)=\boldsymbol{\delta}_{2^N}^{r}$, where $1\leq r\leq 2^N$, $1\leq j\leq M$. Then, we have

$$\begin{aligned}&\boldsymbol{Q\Omega}^{d-1}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(0)\right)\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(-1)\right)\\ &\quad\ltimes\ldots\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(-\tau_N)\right)=\ltimes_{j=1}^{M}\boldsymbol{X}_j(d)\\ &\quad=\boldsymbol{\delta}_{2^N}^{r}\ltimes\boldsymbol{\delta}_{2^N}^{r}\ltimes\boldsymbol{\delta}_{2^N}^{r}\ltimes\ldots\ltimes\boldsymbol{\delta}_{2^N}^{r}\\ &\quad=\boldsymbol{\delta}_{2^{MN}}^{\rho_r},\end{aligned}\tag{7}$$

where $\rho_r=1+\dfrac{(r-1)(2^{MN}-1)}{2^N-1}$.

Since $\ltimes_{j=1}^{M}\boldsymbol{X}_j(0)$, $\ltimes_{j=1}^{M}\boldsymbol{X}_j(-1)$, ..., $\ltimes_{j=1}^{M}\boldsymbol{X}_j(-\tau_N)$ are arbitrarily given, Eq. (7) implies

$$\mathbf{Col}(\boldsymbol{Q\Omega}^{d-1})\subseteq\left\{\boldsymbol{\delta}_{2^{MN}}^{\rho_i}:\rho_i=1+\frac{(i-1)(2^{MN}-1)}{2^N-1}\right\},\tag{8}$$

where $i=1,2,\ldots,2^N$.

Assume d is the smallest positive integer satisfying Eq. (8). Next, we shall prove $1\leq d\leq d_0$ by the contradiction method. Suppose $d>d_0$. Denote $s_0=\min\{i\geq 0: \boldsymbol{Q\Omega}^{d_0+i}=\boldsymbol{Q\Omega}^{d_0-1}\}$ and there exists a positive constant l satisfying $\boldsymbol{Q\Omega}^{l-1}=\boldsymbol{Q\Omega}^{d-1}$, $d_0-1\leq l-1\leq d_0+s_0-1$. That is to say,

$$\begin{aligned}&\mathbf{Col}\{\boldsymbol{Q\Omega}^{d_0-1}\}=\mathbf{Col}\{\boldsymbol{Q\Omega}^{d_0+s_0}\}\\ &\quad\subseteq\mathbf{Col}\{\boldsymbol{Q\Omega}^{l-1}\}\subseteq\mathbf{Col}\{\boldsymbol{Q\Omega}^{d_0-1}\}.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\mathbf{Col}\{\boldsymbol{Q\Omega}^{d_0-1}\}=\mathbf{Col}\{\boldsymbol{Q\Omega}^{d_0+s_0}\}\\ &\quad=\mathbf{Col}\{\boldsymbol{Q\Omega}^{l-1}\}=\mathbf{Col}\{\boldsymbol{Q\Omega}^{d-1}\},\end{aligned}$$

which contradicts with the minimum of d. Therefore, we can conclude that $1\leq d\leq d_0$.

Next, let us show the sufficiency.

Assume there is a positive integer d satisfying Eq. (5). It follows from $\mathbf{Col}(\boldsymbol{Q\Omega}^{d-1})\subseteq\left\{\boldsymbol{\delta}_{2^{MN}}^{\rho_i}:\rho_i=1+\dfrac{(i-1)(2^{MN}-1)}{2^N-1}, i=1,2,\ldots,2^N\right\}$ and $\mathbf{Col}\{\boldsymbol{Q\Omega}^{t-1}\}\subseteq\mathbf{Col}\{\boldsymbol{Q\Omega}^{d-1}\}$ $(t\geq d)$ that

$$\mathbf{Col}(\boldsymbol{Q\Omega}^{d-1})\subseteq\left\{\boldsymbol{\delta}_{2^{MN}}^{\rho_i}:\rho_i=1+\frac{(i-1)(2^{MN}-1)}{2^N-1}\right\},\tag{9}$$

where $i=1,2,\ldots,2^N$.

For any initial values $\boldsymbol{X}_j(0),\boldsymbol{X}_j(-1),\ldots,\boldsymbol{X}_j(-\tau_N)$, $j=1,2,\ldots,M$, there exists a positive integer $\overline{r}$ satisfying $1\leq\overline{r}\leq 2^N$ and

$$\ltimes_{j=1}^{M}\boldsymbol{X}_j(t)=\boldsymbol{\delta}_{2^{MN}}^{\rho_{\overline{r}}}=\boldsymbol{\delta}_{2^N}^{\overline{r}}\boldsymbol{\delta}_{2^N}^{\overline{r}}\ldots\boldsymbol{\delta}_{2^N}^{\overline{r}}.$$

Notice that $\rho_{\overline{r}}=1+\dfrac{(\overline{r}-1)(2^{MN}-1)}{2^N-1}$, which implies

$$\boldsymbol{X}_j(t)=\boldsymbol{\delta}_{2^N}^{\overline{r}}, \forall t\geq d, 1\leq j\leq M.$$

Hence, we can obtain that for any initial values $\boldsymbol{X}_j(0)$, $\boldsymbol{X}_j(-1)$, ..., $\boldsymbol{X}_j(-\tau_N)$, $j=1,2,\ldots,M$, and $\forall t\geq d$,

$$\boldsymbol{X}_1(t)=\boldsymbol{X}_2(t)=\ldots=\boldsymbol{X}_M(t)=\boldsymbol{\delta}_{2^N}^{\overline{r}}.$$

3.2 Algorithm for solving system (1)

In this subsection, we consider the synchronization of system (1). According to Theorem 1, we know that the synchronization of system (2) is decided by matrices $\boldsymbol{Q}$ and $\boldsymbol{\Omega}$. Due to the complexity of system (1), it is hard to give the explicit expression of matrices $\boldsymbol{Q}$ and $\boldsymbol{\Omega}$ corresponding to system (1). In the following, specific steps will be taken to construct matrices $\boldsymbol{Q}$ and $\boldsymbol{\Omega}$ of system (1).

Step 1: Obtain the algebraic representation of system (1).

Let

$$\widehat{\boldsymbol{Y}}(t) = \ltimes_{k=1}^{M} \boldsymbol{Y}_k(t-\alpha_k)$$

and

$$\begin{cases} \tau_{(j1)} = \min\{\tau_j^1, \tau_j^2, \ldots, \tau_j^N\}, \\ \tau_{(j2)} = \min\limits_{1\le i_1<i_2\le N} \max\{\tau_j^{i_1}, \tau_j^{i_2}\}, \\ \tau_{(j3)} = \min\limits_{1\le i_1<i_2<i_3\le N} \max\{\tau_j^{i_1}, \tau_j^{i_2}, \tau_j^{i_3}\}, \\ \quad\vdots \\ \tau_{(jN)} = \max\{\tau_j^1, \tau_j^2, \ldots, \tau_j^N\}. \end{cases} \tag{10}$$

Denote $\boldsymbol{M}_j^i = \boldsymbol{F}_j^i \boldsymbol{u}_1 \left\{\ltimes_{k=2}^{N} (\boldsymbol{I}_{2^{(k-1)N}} \otimes \boldsymbol{u}_k)\right\}$, where $\boldsymbol{u}_k = \mathbf{Ed}^{N-k} \boldsymbol{W}_{[2,2^{N-k}]} \mathbf{Ed}^{k-1}$. It follows from Lemma 2 that

$$\begin{aligned} \boldsymbol{X}_j^i(t+1) &= \boldsymbol{F}_j^i \left\{\ltimes_{s=1}^{N} \boldsymbol{X}_j^s(t-\tau_j^s)\right\} \widehat{\boldsymbol{Y}}(t) \\ &= \boldsymbol{F}_j^i \boldsymbol{u}_1 \left\{\ltimes_{k=2}^{N} (\boldsymbol{I}_{2^{(k-1)N}} \otimes \boldsymbol{u}_k)\right\} \\ &\quad \ltimes \left\{\ltimes_{s=1}^{N} \boldsymbol{X}_j(t-\tau_j^s)\right\} \widehat{\boldsymbol{Y}}(t) \\ &= \boldsymbol{M}_j^i \left\{\ltimes_{s=1}^{N} \boldsymbol{X}_j(t-\tau_j^s)\right\} \widehat{\boldsymbol{Y}}(t). \end{aligned}$$

According to properties 1–4 of STP, we can change the form of $\boldsymbol{X}_j^i(t+1)$. The expression of $\boldsymbol{X}_j^i(t+1)$ can be rewritten as

$$\bar{\boldsymbol{M}}_j^i \boldsymbol{X}_j(t-\tau_{(j1)}) \boldsymbol{X}_j(t-\tau_{(j2)}) \ldots \boldsymbol{X}_j(t-\tau_{(jN)}) \widehat{\boldsymbol{Y}}(t), \tag{11}$$

where $\bar{\boldsymbol{M}}_j^i$ is determined by $\tau_j^1, \tau_j^2, \ldots, \tau_j^{(N-1)}, \tau_j^N$. Let $\bar{\boldsymbol{M}}_j = \bar{\boldsymbol{M}}_j^1 \left\{\ltimes_{i=2}^{N} \left[(\boldsymbol{I}_{2^{N^2+M}} \otimes \bar{\boldsymbol{M}}_j^i) \ltimes \boldsymbol{\Phi}_{N^2+M}\right]\right\}$ and $\widehat{\boldsymbol{X}}_j(t) = \ltimes_{k=1}^{N} \boldsymbol{X}_j(t-\tau_{(jk)})$. Hence,

$$\begin{aligned} \boldsymbol{X}_j(t+1) =& \left\{\bar{\boldsymbol{M}}_j^1 \ltimes_{k=1}^{N} \boldsymbol{X}_j(t-\tau_{(jk)}) \widehat{\boldsymbol{Y}}(t)\right\} \\ & \ltimes \left\{\bar{\boldsymbol{M}}_j^2 \ltimes_{k=1}^{N} \boldsymbol{X}_j(t-\tau_{(jk)}) \widehat{\boldsymbol{Y}}(t)\right\} \ldots \\ & \ltimes \left\{\bar{\boldsymbol{M}}_j^N \ltimes_{k=1}^{N} \boldsymbol{X}_j(t-\tau_{(jk)}) \widehat{\boldsymbol{Y}}(t)\right\} \\ =& \bar{\boldsymbol{M}}_j^1 \left\{ \ltimes_{i=2}^{N} \left[(\boldsymbol{I}_{2^{N^2+M}} \otimes \bar{\boldsymbol{M}}_j^i) \right.\right. \\ & \left.\left. \ltimes \boldsymbol{\Phi}_{N^2+M}\right]\right\} \ltimes_{k=1}^{N} \boldsymbol{X}_j(t-\tau_{(jk)}) \widehat{\boldsymbol{Y}}(t) \\ =& \bar{\boldsymbol{M}}_j \widehat{\boldsymbol{X}}_j(t) \widehat{\boldsymbol{Y}}(t). \end{aligned} \tag{12}$$

For the output of the BNs, let $\boldsymbol{H} = \otimes_{j=1}^{M} \boldsymbol{H}_j$. We can derive

$$\boldsymbol{Y}(t) = \boldsymbol{H} \ltimes_{j=1}^{M} \boldsymbol{X}_j(t). \tag{13}$$

Therefore, the algebraic representation of system (1) can be described by Eqs. (12) and (13).

Step 2: Obtain the expression of $\ltimes_{j=1}^{M} \left(\widehat{\boldsymbol{X}}_j(t) \widehat{\boldsymbol{Y}}(t)\right)$ using $\ltimes_{k=0}^{\tau_M} \left(\ltimes_{j=1}^{M} \boldsymbol{X}_j(t-k)\right)$ and $\widehat{\boldsymbol{Y}}(t)$.

First, the algebraic representation of system (1) implies

$$\ltimes_{j=1}^{M} \boldsymbol{X}_j(t+1) = (\otimes_{j=1}^{M} \bar{\boldsymbol{M}}_j) \left[\ltimes_{j=1}^{M} \left(\widehat{\boldsymbol{X}}_j(t) \widehat{\boldsymbol{Y}}(t)\right)\right]. \tag{14}$$

Note that

$$\begin{cases} \widehat{\boldsymbol{X}}_1(t) = \boldsymbol{X}_1(t-\tau_{(11)}) \boldsymbol{X}_1(t-\tau_{(12)}) \\ \qquad \ltimes \ldots \boldsymbol{X}_1(t-\tau_{(1N)}), \\ \widehat{\boldsymbol{X}}_2(t) = \boldsymbol{X}_2(t-\tau_{(21)}) \boldsymbol{X}_2(t-\tau_{(22)}) \\ \qquad \ltimes \ldots \boldsymbol{X}_2(t-\tau_{(2N)}), \\ \quad\vdots \\ \widehat{\boldsymbol{X}}_M(t) = \boldsymbol{X}_M(t-\tau_{(M1)}) \boldsymbol{X}_M(t-\tau_{(M2)}) \\ \qquad \ltimes \ldots \boldsymbol{X}_M(t-\tau_{(MN)}). \end{cases} \tag{15}$$

Since $\tau_{j1}, \tau_{j2}, \ldots, \tau_{jN}$ are arbitrary integers, it is possible that

$$\begin{aligned} \boldsymbol{X}_i(t-\tau_{(iq)}) &= \boldsymbol{X}_i(t-\tau_{(i(q+1))}) \\ &= \ldots = \boldsymbol{X}_i(t-\tau_{(i(q+k))}). \end{aligned}$$

In fact, if there are two terms $\boldsymbol{X}_i(t-\tau_{(ij_1)}) = \boldsymbol{X}_i(t-\tau_{(ij_2)})$, we need to make sure that there is no identical item on the right-hand side of each equation in Eq. (15). Then, based on properties 1–4 of STP, Eq. (15) can be transformed into

$$\begin{cases} \widehat{\boldsymbol{X}}_1(t) = \boldsymbol{A}_1 \boldsymbol{X}_1(t-\phi_{(11)}) \boldsymbol{X}_1(t-\phi_{(12)}) \\ \qquad \ltimes \ldots \boldsymbol{X}_1(t-\phi_{(1N_1)}), \\ \widehat{\boldsymbol{X}}_2(t) = \boldsymbol{A}_2 \boldsymbol{X}_2(t-\phi_{(21)}) \boldsymbol{X}_2(t-\phi_{(22)}) \\ \qquad \ltimes \ldots \boldsymbol{X}_2(t-\phi_{(2N_2)}), \\ \quad\vdots \\ \widehat{\boldsymbol{X}}_M(t) = \boldsymbol{A}_M \boldsymbol{X}_M(t-\phi_{(M1)}) \boldsymbol{X}_M(t-\phi_{(M2)}) \\ \qquad \ltimes \ldots \boldsymbol{X}_M(t-\phi_{(MN_M)}), \end{cases} \tag{16}$$

where $\phi_{(ij)} \in \{\tau_{i1}, \tau_{i2}, \ldots, \tau_{iN}\}$, $\phi_{(i1)} < \phi_{(i2)} < \ldots < \phi_{(iN_i)}$, and $i = 1, 2, \ldots, M$. In addition, $\boldsymbol{A}_j$ can be derived from $\tau_{j1}, \tau_{j2}, \ldots, \tau_{jN}$.

For example, equation

$$\widehat{\boldsymbol{X}}_1(t) = \boldsymbol{X}_1(t-1) \boldsymbol{X}_1(t-3) \boldsymbol{X}_1(t-1) \tag{17}$$

is transformed into

$$\begin{aligned}\widehat{\boldsymbol{X}}_1(t) =&\boldsymbol{X}_1(t-1)\boldsymbol{W}_{[2^N]}\boldsymbol{X}_1(t-1)\boldsymbol{X}_1(t-3)\\ =&(\boldsymbol{I}_{2^N}\otimes \boldsymbol{W}_{[2^N]})\boldsymbol{\Phi}_N\boldsymbol{X}_1(t-1)\boldsymbol{X}_1(t-3)\\ =&\boldsymbol{A}_1\boldsymbol{X}_1(t-1)\boldsymbol{X}_1(t-3),\end{aligned}$$

where $\boldsymbol{A}_1=(\boldsymbol{I}_{2^N}\otimes \boldsymbol{W}_{[2^N]})\boldsymbol{\Phi}_N$.

For the convenience of subsequent calculations, based on properties 1–4 of STP, we can transform Eq. (16) to the following equivalent equations:

$$\begin{cases}\widehat{\boldsymbol{X}}_1(t)=\boldsymbol{D}_1\ltimes_{k=0}^{\tau_m}\boldsymbol{X}_1(t-k),\\ \widehat{\boldsymbol{X}}_2(t)=\boldsymbol{D}_2\ltimes_{k=0}^{\tau_m}\boldsymbol{X}_2(t-k),\\ \qquad\vdots\\ \widehat{\boldsymbol{X}}_M(t)=\boldsymbol{D}_M\ltimes_{k=0}^{\tau_m}\boldsymbol{X}_M(t-k),\end{cases}\tag{18}$$

where $\tau_m=\max\{\phi_{(1N_1)},\phi_{(2N_2)},\ldots,\phi_{(MN_M)},\alpha_1,\alpha_2,\ldots,\alpha_M\}$ and $\boldsymbol{D}_j$ $(j=1,2,\ldots,M)$ are obtained based on $\phi_{(j1)},\phi_{(j2)},\ldots,\phi_{(jN_j)}$, and τ_m. Let $\widetilde{\boldsymbol{X}}_k(t)=\ltimes_{s=0}^{\tau_m}\boldsymbol{X}_k(t-s)$, $\boldsymbol{D}=\boldsymbol{D}_1\ltimes_{k=2}^{M}\{(\boldsymbol{I}_{2^{(k-1)N(\tau_m+1)}}\otimes\boldsymbol{D}_k)\}$. According to Eq. (18), we can obtain

$$\widehat{\boldsymbol{X}}_s(t)=\boldsymbol{D}_s\widetilde{\boldsymbol{X}}_s(t),\ s=1,\,2,\,\ldots,\,M,$$

and

$$\begin{aligned}\ltimes_{k=1}^{M}\widehat{\boldsymbol{X}}_k(t)=&\boldsymbol{D}_1\widetilde{\boldsymbol{X}}_1(t)\boldsymbol{D}_2\widetilde{\boldsymbol{X}}_2(t)\ldots\boldsymbol{D}_M\widetilde{\boldsymbol{X}}_M(t)\\ =&\boldsymbol{D}_1(\boldsymbol{I}_{2^{N(\tau_m+1)}}\otimes\boldsymbol{D}_2)\widetilde{\boldsymbol{X}}_1(t)\widetilde{\boldsymbol{X}}_2(t)\\ &\ltimes\boldsymbol{D}_3\widetilde{\boldsymbol{X}}_3(t)\ldots\boldsymbol{D}_M\widetilde{\boldsymbol{X}}_M(t)\\ &\ldots\\ =&\boldsymbol{D}_1\ltimes_{k=2}^{M}\{(\boldsymbol{I}_{2^{(k-1)N(\tau_m+1)}}\otimes\boldsymbol{D}_k)\}\\ &\ltimes\widetilde{\boldsymbol{X}}_1(t)\widetilde{\boldsymbol{X}}_2(t)\ldots\widetilde{\boldsymbol{X}}_M(t)\\ =&\boldsymbol{D}\ltimes_{k=1}^{M}\widetilde{\boldsymbol{X}}_k(t).\end{aligned}\tag{19}$$

Denote

$$\begin{cases}\boldsymbol{W}=\boldsymbol{W}_{[2^M,2^{N^2}]}\ltimes\left\{\ltimes_{i=2}^{M}\left[(\boldsymbol{I}_{2^M}\otimes\boldsymbol{W}_{2^M,2^{iN^2}})\boldsymbol{\Phi}_M\right]\right\},\\ \bar{\boldsymbol{P}}=\bar{\boldsymbol{P}}_1(\boldsymbol{I}_{2^{MN}}\otimes\bar{\boldsymbol{P}}_2)\ldots(\boldsymbol{I}_{2^{(\tau_m-1)MN}}\otimes\bar{\boldsymbol{P}}_{\tau_m}),\end{cases}$$

where $\bar{\boldsymbol{P}}_q=\ltimes_{j=1}^{M-1}(\boldsymbol{I}_{2^{jN}}\otimes\boldsymbol{W}_{[2^N,2^{j(\tau_m-q+1)N}]}),q=1,2,\ldots,\tau_m$.

Lemma 5 For system (1), the following equality holds:

$$\ltimes_{j=1}^{M}\{\widehat{\boldsymbol{X}}_j(t)\widehat{\boldsymbol{Y}}(t)\}=\boldsymbol{W}(\boldsymbol{I}_{2^M}\otimes\boldsymbol{D}\bar{\boldsymbol{P}})\widehat{\boldsymbol{Y}}(t)\ltimes_{k=0}^{\tau_M}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right).$$

The proof is given in Appendix D.

Step 3: Obtain matrices $\boldsymbol{Q}$ and $\boldsymbol{\Omega}$.

Note that $\widehat{\boldsymbol{Y}}(t)=\boldsymbol{Y}_1(t-\alpha_1)\boldsymbol{Y}_2(t-\alpha_2)\ldots\ltimes\boldsymbol{Y}_M(t-\alpha_M)$, and it can be obtained that

$$\begin{aligned}\widehat{\boldsymbol{Y}}(t)=&\{\boldsymbol{H}_1\boldsymbol{X}_1(t-\alpha_1)\}\otimes\{\boldsymbol{H}_2\boldsymbol{X}_2(t-\alpha_2)\}\\ &\otimes\ldots\otimes\{\boldsymbol{H}_M\boldsymbol{X}_M(t-\alpha_M)\}\\ =&\{\otimes_{j=1}^{M}\boldsymbol{H}_j\}\ltimes_{k=1}^{M}\boldsymbol{X}_k(t-\alpha_k)\\ =&\boldsymbol{H}\ltimes_{k=1}^{M}\boldsymbol{X}_k(t-\alpha_k),\end{aligned}\tag{20}$$

where $\boldsymbol{H}=\otimes_{j=1}^{M}\boldsymbol{H}_j$.

Lemma 6 Let

$$\begin{aligned}\boldsymbol{R}=\ltimes_{k=1}^{M}\Big(&\boldsymbol{I}_{2^{N(M-k)}}\otimes\Big\{\boldsymbol{W}_{[2^{MN(\alpha_{M+1-k}+1)-N},2^N]}\\ &\ltimes(\boldsymbol{I}_{2^{MN(\alpha_{M+1-k}+1)-N}}\otimes\boldsymbol{\Phi}_N)\Big\}\Big).\end{aligned}$$

Then

$$\begin{aligned}&\widehat{\boldsymbol{Y}}(t)\ltimes_{k=0}^{\tau_M}\left\{\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right\}\\ &=\boldsymbol{H}\boldsymbol{R}\ltimes_{k=0}^{\tau_M}\left\{\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right\}.\end{aligned}$$

The proof is given in Appendix E.

Based on Lemmas 5 and 6, we have

$$\ltimes_{j=1}^{M}\widehat{\boldsymbol{X}}_j(t)\widehat{\boldsymbol{Y}}(t)=\boldsymbol{W}(\boldsymbol{I}_{2^N}\otimes\boldsymbol{D}\boldsymbol{P})\boldsymbol{H}\boldsymbol{R}\ltimes_{k=0}^{\tau_M}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right).\tag{21}$$

Eqs. (14) and (21) imply

$$\ltimes_{j=1}^{M}\boldsymbol{X}_j(t+1)=(\otimes_{j=1}^{M}\bar{\boldsymbol{M}}_j)\boldsymbol{W}(\boldsymbol{I}_{2^N}\otimes\boldsymbol{D}\boldsymbol{P})\boldsymbol{H}\boldsymbol{R}\ltimes_{k=0}^{\tau_M}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right).\tag{22}$$

Let $\boldsymbol{V}(t)=\ltimes_{s=0}^{\tau_M}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-s)\right)$. Eq. (22) implies

$$\ltimes_{j=1}^{M}\boldsymbol{X}_j(t+1)=\boldsymbol{Q}\boldsymbol{V}(t),$$

where

$$\boldsymbol{Q}=(\otimes_{j=1}^{M}\bar{\boldsymbol{M}}_j)\boldsymbol{W}(\boldsymbol{I}_{2^N}\otimes\boldsymbol{D}\boldsymbol{P})\boldsymbol{H}\boldsymbol{R}.\tag{23}$$

Hence, the relationship between $\boldsymbol{V}(t+1)$ and $\boldsymbol{V}(t)$ is as follows:

$$\begin{aligned}\boldsymbol{V}(t+1)=&\ltimes_{s=0}^{\tau_M}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t+1-s)\right)\\ =&\boldsymbol{Q}\boldsymbol{V}(t)\ltimes_{j=1}^{M}\boldsymbol{X}_j(t)\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-1)\\ &\ldots\ltimes_{j=1}^{M}\boldsymbol{X}_j(t+1-\tau_M)\\ =&\boldsymbol{Q}\boldsymbol{W}_{[2^{MN(\tau_M)},2^{MN(\tau_M+1)}]}\ltimes_{j=1}^{M}\boldsymbol{X}_j(t)\\ &\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-1)\ldots\ltimes_{j=1}^{M}\boldsymbol{X}_j(t+1-\tau_M)\\ &\cdot\boldsymbol{V}(t)\\ =&\boldsymbol{Q}\boldsymbol{W}_{[2^{MN(\tau_M)},2^{MN(\tau_M+1)}]}\boldsymbol{\Phi}_{MN(\tau_M)}\boldsymbol{V}(t).\end{aligned}$$

Therefore, we can obtain matrix $\boldsymbol{\Omega}$ as

$$\boldsymbol{\Omega} = \boldsymbol{Q}\boldsymbol{W}_{[2^{MN(\tau_M)},2^{MN(\tau_M+1)}]}\boldsymbol{\Phi}_{MN(\tau_M)}. \tag{24}$$

This completes step 3 and matrices $\boldsymbol{Q}$ and $\boldsymbol{\Omega}$ are obtained as Eqs. (23) and (24), respectively.

Step 4: Obtain the expression of $\ltimes_{j=1}^{M}\boldsymbol{X}_j(t+1)$ with $\ltimes_{s=0}^{\tau_M}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(-s)\right)$.

According to step 3, we have

$$\begin{cases} \ltimes_{j=1}^{M}\boldsymbol{X}_j(t+1) = \boldsymbol{Q}\boldsymbol{V}(t), \\ \boldsymbol{V}(t+1) = \boldsymbol{\Omega}\boldsymbol{V}(t). \end{cases}$$

Based on the results above, we have

$$\ltimes_{j=1}^{M}\boldsymbol{X}_j(t+1) = \boldsymbol{Q}\boldsymbol{\Omega}^t \ltimes_{s=0}^{\tau_M}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(-s)\right).$$

Based on the above discussion, an efficient algorithm for constructing matrices $\boldsymbol{Q}$ and $\boldsymbol{\Omega}$ of system (1) and the expression of $\ltimes_{j=1}^{M}\boldsymbol{X}_j(t+1)$ can be summarized (Algorithm 1).

From Algorithm 1, matrices $\boldsymbol{Q}$ and $\boldsymbol{\Omega}$ can be obtained. Furthermore, we can judge the synchronization of system (1) based on these two matrices.

Algorithm 1 Construction of $\boldsymbol{Q}$ and $\boldsymbol{\Omega}$

1: Obtain the algebraic representation of the system
2: Obtain the expression of $\ltimes_{j=1}^{M}\left(\widehat{\boldsymbol{X}}_j(t)\widehat{\boldsymbol{Y}}(t)\right)$ with $\ltimes_{k=0}^{\tau_M}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right)$ and $\widehat{\boldsymbol{Y}}(t)$
3: Obtain matrices $\boldsymbol{Q}$ and $\boldsymbol{\Omega}$
4: Obtain the expression $\ltimes_{j=1}^{M}\boldsymbol{X}_j(t+1) = \boldsymbol{Q}\boldsymbol{\Omega}^t \ltimes_{s=0}^{\tau_M}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(-s)\right)$

Theorem 2 System synchronization occurs if and only if there exists a positive integer d satisfying

$$\mathbf{Col}(\boldsymbol{Q}\boldsymbol{\Omega}^{d-1}) \subseteq \left\{\delta_{2^{MN}}^{\rho_i} : \rho_i = 1 + \frac{(i-1)(2^{MN}-1)}{2^N-1}\right\},$$

where $d_0 = \min\{m : m \geq 1, \boldsymbol{Q}\boldsymbol{\Omega}^{m-1} = \boldsymbol{Q}\boldsymbol{\Omega}^{n-1}, n > m\}$, $1 \leq d \leq d_0$, and $i = 1, 2, \ldots, 2^N$.

The proofs of Theorems 1 and 2 are similar. So, we omit that of Theorem 2 here.

Remark 1 The difference between Theorems 1 and 2 is just the matrices $\boldsymbol{Q}$ and $\boldsymbol{\Omega}$. According to the proof of Theorem 1, it can be found that we may obtain different d_0 for different matrices $\boldsymbol{Q}$ and $\boldsymbol{\Omega}$.

Remark 2 In many real coupled BNs, time delays are ubiquitous at the moment of information exchange. It should be pointed out that in most previous works it is assumed that there are no time delays in the BNs or that time delays between different nodes in different BNs need to be the same. This requirement is removed in our model, meaning that the results of this study are more general compared with previous results (Zhong et al., 2014; Lu et al., 2016; Liu Y et al., 2017; Liu RJ et al., 2018). Furthermore, the unified framework is constructed to analyze the delayed coupled BNs, which can be seen as the extension of the method in Zhong et al. (2014), Lu et al. (2016), Liu Y et al. (2017), and Liu RJ et al. (2018). Using the framework proposed here to analyze the delayed coupled Boolean networks, we can further extend many previous results to more general cases.

4 Numerical examples

In this section, we illustrate the effectiveness of the proposed algorithm through two numerical examples.

Example 1 Consider the CDBNs as follows:

$$\begin{cases} \boldsymbol{X}_1^1(t+1) = \left(\boldsymbol{X}_1^1(t-1) \wedge (\boldsymbol{Y}_1(t) \vee \boldsymbol{Y}_2(t))\right) \\ \qquad \vee \left(\neg\boldsymbol{X}_1^1(t-1) \wedge \neg\left(\boldsymbol{Y}_1(t) \rightarrow \boldsymbol{Y}_2(t)\right)\right), \\ \boldsymbol{X}_2^1(t+1) = \left(\boldsymbol{X}_2^1(t-2) \wedge \neg\left(\boldsymbol{Y}_1(t) \rightarrow \boldsymbol{Y}_2(t)\right)\right) \\ \qquad \vee \left(\neg\boldsymbol{X}_2^1(t-1) \wedge (\boldsymbol{Y}_1(t) \vee \boldsymbol{Y}_2(t))\right), \\ \boldsymbol{Y}_1(t) = \neg\boldsymbol{X}_1^1(t), \\ \boldsymbol{Y}_2(t) = \neg\boldsymbol{X}_2^1(t). \end{cases} \tag{25}$$

By some computations, the algebraic representation of Eq. (25) is

$$\begin{cases} \boldsymbol{X}_1^1(t+1) = \boldsymbol{F}_1^1\boldsymbol{X}_1^1(t-1)\boldsymbol{Y}_1(t)\boldsymbol{Y}_2(t), \\ \boldsymbol{X}_2^1(t+1) = \boldsymbol{F}_2^1\boldsymbol{X}_2^1(t-2)\boldsymbol{Y}_1(t)\boldsymbol{Y}_2(t), \\ \boldsymbol{Y}_1(t) = \boldsymbol{H}_1\boldsymbol{X}_1^1(t), \\ \boldsymbol{Y}_2(t) = \boldsymbol{H}_2\boldsymbol{X}_2^1(t), \end{cases}$$

where

$$\begin{cases} \boldsymbol{F}_1^1 = \boldsymbol{\delta}_2[1,1,1,2,2,1,2,2], \\ \boldsymbol{F}_2^1 = \boldsymbol{\delta}_2[2,1,2,2,1,1,1,2], \\ \boldsymbol{H}_1 = \boldsymbol{\delta}_2[2,1], \\ \boldsymbol{H}_2 = \boldsymbol{\delta}_2[2,1]. \end{cases}$$

Matrices $\boldsymbol{Q}$ and $\boldsymbol{\Omega}$ can be computed based on Algorithm 1 and we can obtain

$$\boldsymbol{Q}\boldsymbol{\Omega}^6 = \boldsymbol{Q}\boldsymbol{\Omega}^8$$

and

$$d_0 = \min\{m : m \geq 1, \boldsymbol{Q}\boldsymbol{\Omega}^{m-1} = \boldsymbol{Q}\boldsymbol{\Omega}^{n-1}, n > m\} = 7.$$

Then, we check whether there exists a positive integer d satisfying $1 \leq d \leq 7$ and $\mathbf{Col}(\boldsymbol{Q}\boldsymbol{\Omega}^{d-1}) \subseteq \{\boldsymbol{\delta}_4^i : i=1, i=4\}$. When $d=7$, we can obtain by computation that

$$\begin{aligned}\boldsymbol{Q}\boldsymbol{\Omega}^6 =&\boldsymbol{\delta}_4[4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,\\&1,1,1,1,4,1,4,1,1,1,1,1,1,1,1,1,\\&1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,\\&1,1,1,1,1,1,1,1,4,4,4,4,1,4,1,4].\end{aligned}$$

Hence,

$$\mathbf{Col}\{\boldsymbol{Q}\boldsymbol{\Omega}^6\} \subseteq \{\boldsymbol{\delta}_4^i : i=1, i=4\}.$$

Thus, the system can realize synchronization according to Theorem 2. The initial values are chosen as $\boldsymbol{X}_1(-2)=\boldsymbol{\delta}_2^1$, $\boldsymbol{X}_2(-2)=\boldsymbol{\delta}_2^2$, $\boldsymbol{X}_1(-1)=\boldsymbol{\delta}_2^2$, $\boldsymbol{X}_2(-1)=\boldsymbol{\delta}_2^1$, $\boldsymbol{X}_1(0)=\boldsymbol{\delta}_2^1$, and $\boldsymbol{X}_2(0)=\boldsymbol{\delta}_2^2$. Fig. 1 shows the state evolution of CDBNs (25), where $\boldsymbol{X}_1(t)=\boldsymbol{X}_2(t)$ is represented by 1 and $\boldsymbol{X}_1(t) \neq \boldsymbol{X}_2(t)$ is represented by -1.

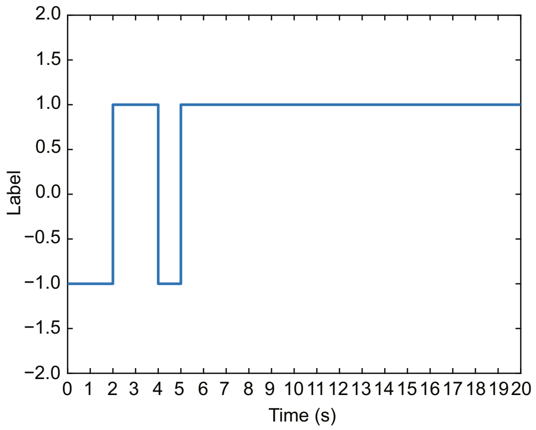

Fig. 1 States of the system in Example 1

Example 2 Consider the CDBNs as follows:

$$\begin{cases}\boldsymbol{X}_1^1(t+1)=\left(\boldsymbol{X}_1^1(t-1)\vee \boldsymbol{Y}_1(t)\right)\rightarrow(\neg \boldsymbol{Y}_2(t)),\\ \boldsymbol{X}_2^1(t+1)=\boldsymbol{X}_2^1(t-2)\leftrightarrow(\boldsymbol{Y}_1(t)\uparrow \boldsymbol{Y}_2(t)),\\ \boldsymbol{Y}_1(t)=\neg \boldsymbol{X}_1^1(t),\\ \boldsymbol{Y}_2(t)=\boldsymbol{X}_2^1(t).\end{cases} \tag{26}$$

The algebraic representation of Eq. (26) can be expressed as follows:

$$\begin{cases}\boldsymbol{X}_1^1(t+1)=\boldsymbol{F}_1^1\boldsymbol{X}_1^1(t-1)\boldsymbol{Y}_1(t)\boldsymbol{Y}_2(t),\\ \boldsymbol{X}_2^1(t+1)=\boldsymbol{F}_2^1\boldsymbol{X}_2^1(t-2)\boldsymbol{Y}_1(t)\boldsymbol{Y}_2(t),\\ \boldsymbol{Y}_1(t)=\boldsymbol{H}_1\boldsymbol{X}_1^1(t),\\ \boldsymbol{Y}_2(t)=\boldsymbol{H}_2\boldsymbol{X}_2^1(t),\end{cases}$$

where

$$\begin{cases}\boldsymbol{F}_1^1=\boldsymbol{\delta}_2[2,1,2,1,2,1,1,1],\\ \boldsymbol{F}_2^1=\boldsymbol{\delta}_2[2,1,1,1,1,2,2,2],\\ \boldsymbol{H}_1=\boldsymbol{\delta}_2[2,1],\\ \boldsymbol{H}_2=\boldsymbol{\delta}_2[1,2].\end{cases}$$

Matrices $\boldsymbol{Q}$ and $\boldsymbol{\Omega}$ can be computed based on Algorithm 1. By computation, we can obtain

$$\boldsymbol{Q}\boldsymbol{\Omega}^4=\boldsymbol{Q}\boldsymbol{\Omega}^7$$

and

$$d_0=\min\{m: m\geq 1, \boldsymbol{Q}\boldsymbol{\Omega}^{m-1}=\boldsymbol{Q}\boldsymbol{\Omega}^{n-1}, n>m\}=5.$$

Then, we need to check whether there exists a positive integer d satisfying $1 \leq d \leq 5$ and $\mathbf{Col}(\boldsymbol{Q}\boldsymbol{\Omega}^{d-1}) \subseteq \{\boldsymbol{\delta}_4^i : i=1, i=4\}$.

If $d=1$, then

$$\begin{aligned}\boldsymbol{Q}\boldsymbol{\Omega}^{d-1} =&\boldsymbol{\delta}_4[3,4,3,4,3,4,3,4,1,2,1,2,1,2,1,2,\\&1,2,1,2,1,2,1,2,1,2,1,2,1,2,1,2,\\&4,3,4,3,4,3,4,3,4,3,4,3,4,3,4,3,\\&1,2,1,2,1,2,1,2,1,2,1,2,1,2,1,2].\end{aligned}$$

However,

$$\mathbf{Col}\{\boldsymbol{Q}\} \nsubseteq \{\boldsymbol{\delta}_4^i : i=1, i=4\}.$$

If $d=2$, then

$$\begin{aligned}\boldsymbol{Q}\boldsymbol{\Omega}^{d-1} =&\boldsymbol{\delta}_4[4,1,4,1,3,2,3,2,3,1,3,1,4,2,4,2,\\&3,1,3,1,4,2,4,2,3,1,3,1,4,2,4,2,\\&1,4,1,4,2,3,2,3,1,4,1,4,2,3,2,3,\\&1,1,1,1,2,2,2,2,1,1,1,1,2,2,2,2].\end{aligned}$$

However,

$$\mathbf{Col}\{\boldsymbol{Q}\boldsymbol{\Omega}\} \nsubseteq \{\boldsymbol{\delta}_4^i : i=1, i=4\}.$$

If $d=3$, then

$$\begin{aligned}\boldsymbol{Q}\boldsymbol{\Omega}^{d-1} =&\boldsymbol{\delta}_4[1,1,1,1,1,4,1,4,4,3,4,3,1,1,1,1,\\&3,4,3,4,2,2,2,2,3,4,3,4,2,2,2,2,\\&1,1,1,1,1,4,1,4,1,1,1,1,1,4,1,4,\\&4,4,4,4,2,2,2,2,4,4,4,4,2,2,2,2].\end{aligned}$$

However,

$$\mathbf{Col}\{\boldsymbol{Q}\boldsymbol{\Omega}^2\} \nsubseteq \{\boldsymbol{\delta}_4^i : i=1, i=4\}.$$

If $d = 4$, then

$$\begin{aligned}\boldsymbol{Q\Omega}^{d-1} =&\boldsymbol{\delta}_4[1,4,1,4,1,4,1,4,1,3,1,3,1,4,1,4,\\&4,2,4,2,1,2,1,2,4,2,4,2,1,2,1,2,\\&4,1,4,1,4,1,4,1,4,1,4,1,4,1,4,1,\\&1,2,1,2,1,2,1,2,1,2,1,2,1,2,1,2].\end{aligned}$$

However,

$$\textbf{Col}\{\boldsymbol{Q\Omega}^3\} \nsubseteq \{\boldsymbol{\delta}_4^i : i=1, i=4\}.$$

If $d = 5$, then

$$\begin{aligned}\boldsymbol{Q\Omega}^{d-1} =&\boldsymbol{\delta}_4[4,1,4,1,1,2,1,2,1,4,1,4,4,2,4,2,\\&1,1,1,1,4,2,4,2,1,1,1,1,4,2,4,2,\\&1,4,1,4,2,1,2,1,1,4,1,4,2,1,2,1,\\&1,1,1,1,4,2,4,2,1,1,1,1,4,2,4,2].\end{aligned}$$

However,

$$\textbf{Col}\{\boldsymbol{Q\Omega}^4\} \nsubseteq \{\boldsymbol{\delta}_4^i : i=1, i=4\}.$$

Therefore, there does not exist any positive integer d satisfying $1 \leq d \leq 5$ and $\textbf{Col}(\boldsymbol{Q\Omega}^{d-1}) \subseteq \{\boldsymbol{\delta}_4^i : i=1, i=4\}$. Thus, the system cannot realize synchronization according to Theorem 2. The initial values of the system are chosen as $\boldsymbol{X}_1(-2) = \boldsymbol{\delta}_2^1$, $\boldsymbol{X}_2(-2) = \boldsymbol{\delta}_2^2$, $\boldsymbol{X}_1(-1) = \boldsymbol{\delta}_2^1$, $\boldsymbol{X}_2(-1) = \boldsymbol{\delta}_2^1$, $\boldsymbol{X}_1(0) = \boldsymbol{\delta}_2^2$, and $\boldsymbol{X}_2(0) = \boldsymbol{\delta}_2^2$. Fig. 2 shows the state evolution of CDBNs (26), where $\boldsymbol{X}_1(t) = \boldsymbol{X}_2(t)$ is represented by 1 and $\boldsymbol{X}_1(t) \neq \boldsymbol{X}_2(t)$ is represented by -1. It can be seen from Fig. 2 that the system is not synchronous.

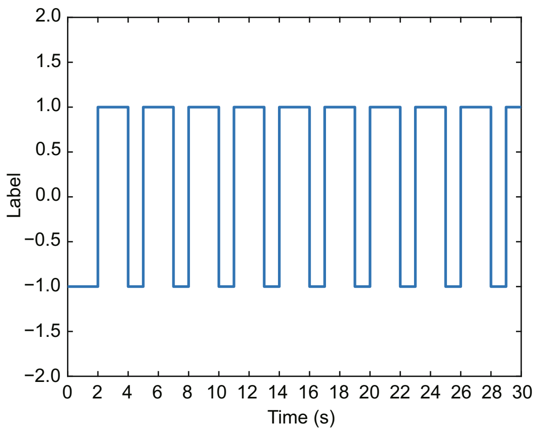

Fig. 2 States of the system in Example 2

5 Conclusions and discussion

We have studied mainly complete synchronization of CDBNs. We first converted the equations of CDBNs into an algebraic form. Then a necessary and sufficient criterion for the complete synchronization of CDBNs has been presented. Note that the model proposed in this study is more general than those in previous studies. Furthermore, a design algorithm has been provided to judge the synchronization of CDBNs. Finally, numerical examples have been presented to demonstrate the effectiveness of the algorithm.

Contributors

Jie LIU designed the research. Lulu LI and Jie LIU completed the mathematical proof. Jie LIU drafted the manuscript. Lulu LI helped organize the manuscript. Habib M. FARDOUN revised and finalized the manuscript.

Compliance with ethics guidelines

Jie LIU, Lulu LI, and Habib M. FARDOUN declare that they have no conflict of interest.

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Appendix A: Proof of Lemma 2

It can be derived that

$$\begin{aligned}\boldsymbol{X}_j^N(t-\tau_N) =&\mathbf{Ed}^{N-1}\boldsymbol{X}_j^1(t-\tau_N)\boldsymbol{X}_j^2(t-\tau_N)\\ &\ltimes\ldots\boldsymbol{X}_j^N(t-\tau_N)\\ =&\mathbf{Ed}^{N-1}\boldsymbol{X}_j(t-\tau_N).\end{aligned}\tag{A1}$$

Similarly, we can obtain

$$\begin{aligned}&\boldsymbol{X}_j^{N-1}(t-\tau_{N-1})\\ &=\mathbf{Ed}\boldsymbol{X}_j^N(t-\tau_{N-1})\boldsymbol{X}_j^{N-1}(t-\tau_{N-1})\\ &=\mathbf{Ed}\boldsymbol{W}_{[2,2]}\boldsymbol{X}_j^{N-1}(t-\tau_{N-1})\boldsymbol{X}_j^N(t-\tau_{N-1})\\ &=\mathbf{Ed}\boldsymbol{W}_{[2,2]}\mathbf{Ed}^{N-2}\boldsymbol{X}_j^1(t-\tau_{N-1})\\ &\quad\ltimes\boldsymbol{X}_j^2(t-\tau_{N-1})\ldots\boldsymbol{X}_j^N(t-\tau_{N-1})\\ &=\mathbf{Ed}\boldsymbol{W}_{[2,2]}\mathbf{Ed}^{N-2}\boldsymbol{X}_j(t-\tau_{N-1})\\ &\quad\ldots\end{aligned}\tag{A2}$$

and

$$\begin{aligned}&\boldsymbol{X}_j^1(t-\tau_1)\\ &=\mathbf{Ed}^{N-1}\boldsymbol{X}_j^2(t-\tau_1)\ldots\boldsymbol{X}_j^N(t-\tau_1)\boldsymbol{X}_j^1(t-\tau_1)\\ &=\mathbf{Ed}^{N-1}\boldsymbol{W}_{[2,2^{N-1}]}\ltimes_{k=1}^N\boldsymbol{X}_j^k(t-\tau_1)\\ &=\mathbf{Ed}^{N-1}\boldsymbol{W}_{[2,2^{N-1}]}\boldsymbol{X}_j(t-\tau_1).\end{aligned}\tag{A3}$$

Let $\boldsymbol{u}_k=\mathbf{Ed}^{N-k}\boldsymbol{W}_{[2,2^{N-k}]}\mathbf{Ed}^{k-1}$. According to Eqs. (A1)−(A3), it can be summarized that

$$\begin{aligned}\boldsymbol{X}_j^k(t-\tau_k)&=\mathbf{Ed}^{N-k}\boldsymbol{W}_{[2,2^{N-k}]}\mathbf{Ed}^{k-1}\boldsymbol{X}_j(t-\tau_k)\\ &=\boldsymbol{u}_k\boldsymbol{X}_j(t-\tau_k).\end{aligned}$$

Appendix B: Proof of Lemma 3

Let $\boldsymbol{M}_j^i=\boldsymbol{F}_j^i\boldsymbol{u}_1\left\{\ltimes_{k=2}^N(\boldsymbol{I}_{2^{(k-1)N}}\otimes\boldsymbol{u}_k)\right\}$ and $\widehat{\boldsymbol{X}}_j(t)=\ltimes_{k=1}^N\boldsymbol{X}_j(t-\tau_k)$. Then, we have

$$\begin{aligned}\boldsymbol{X}_j^i(t+1)&=\boldsymbol{F}_j^i\ltimes_{k=1}^N\boldsymbol{X}_j^k(t-\tau_k)\ltimes_{k=1}^N\boldsymbol{Y}_k(t)\\ &=\boldsymbol{F}_j^i\ltimes_{k=1}^N\{\boldsymbol{u}_k\boldsymbol{X}_j(t-\tau_k)\}\boldsymbol{Y}(t)\\ &=\boldsymbol{F}_j^i\boldsymbol{u}_1(\boldsymbol{I}_{2^N}\otimes\boldsymbol{u}_2)\boldsymbol{X}_j(t-\tau_1)\boldsymbol{X}_j(t-\tau_2)\\ &\quad\ltimes_{k=3}^N\{\boldsymbol{u}_k\boldsymbol{X}_j(t-\tau_k)\}\boldsymbol{Y}(t)\\ &=\boldsymbol{F}_j^i\boldsymbol{u}_1\left\{\ltimes_{k=2}^N(\boldsymbol{I}_{2^{(k-1)N}}\otimes\boldsymbol{u}_k)\right\}\\ &\quad\ltimes_{k=1}^N\boldsymbol{X}_j(t-\tau_k)\boldsymbol{Y}(t)\\ &=\boldsymbol{M}_j^i\widehat{\boldsymbol{X}}_j(t)\boldsymbol{Y}(t).\end{aligned}$$

Appendix C: Proof of Lemma 4

Let

$$\begin{cases}\boldsymbol{W}=\boldsymbol{W}_{[2^M,2^{N^2}]}\ltimes\left\{\ltimes_{i=2}^M\left[(\boldsymbol{I}_{2^M}\otimes\boldsymbol{W}_{[2^M,2^{iN^2}]})\boldsymbol{\Phi}_M\right]\right\},\\ \boldsymbol{P}=\boldsymbol{P}_1\ltimes_{j=1}^{N-2}\{\boldsymbol{I}_{2^{jMN}}\otimes\boldsymbol{P}_{j+1}\},\end{cases}$$

where $\boldsymbol{P}_q=\ltimes_{j=1}^{M-1}\left(\boldsymbol{I}_{2^{jN}}\otimes\boldsymbol{W}_{[2^N,2^{j(N-q)N}]}\right)$, $q=1,2,\ldots,N-1$. It holds that

$$\begin{aligned}&\ltimes_{j=1}^M\left(\widehat{\boldsymbol{X}}_j(t)\boldsymbol{Y}(t)\right)\\ &=\boldsymbol{W}_{[2^M,2^{N^2}]}\boldsymbol{Y}(t)\widehat{\boldsymbol{X}}_1(t)\ltimes_{k=2}^M\left\{\widehat{\boldsymbol{X}}_k(t)\boldsymbol{Y}(t)\right\}\\ &=\boldsymbol{W}_{[2^M,2^{N^2}]}\boldsymbol{Y}(t)\boldsymbol{W}_{[2^M,2^{2N^2}]}\boldsymbol{Y}(t)\\ &\quad\ltimes\widehat{\boldsymbol{X}}_1(t)\widehat{\boldsymbol{X}}_2(t)\ltimes_{k=3}^M\left\{\widehat{\boldsymbol{X}}_k(t)\boldsymbol{Y}(t)\right\}\\ &\quad\ldots\\ &=\boldsymbol{W}_{[2^M,2^{N^2}]}\ltimes_{i=2}^M\left[\left(\boldsymbol{I}_{2^M}\otimes\boldsymbol{W}_{[2^M,2^{iN^2}]}\right)\boldsymbol{\Phi}_M\right]\\ &\quad\ltimes\boldsymbol{Y}(t)\ltimes_{k=1}^M\widehat{\boldsymbol{X}}_k(t)\\ &=\boldsymbol{W}\boldsymbol{Y}(t)\ltimes_{k=1}^M\widehat{\boldsymbol{X}}_k(t).\end{aligned}\tag{C1}$$

One can obtain from Eq. (3) that

$$\ltimes_{j=1}^M\boldsymbol{X}_j(t+1)=\{\otimes_{j=1}^M\boldsymbol{M}_j\}\left\{\ltimes_{j=1}^M\left(\widehat{\boldsymbol{X}}_j(t)\boldsymbol{Y}(t)\right)\right\}.\tag{C2}$$

Hence, we have

$$\ltimes_{j=1}^M\boldsymbol{X}_j(t+1)=(\otimes_{j=1}^M\boldsymbol{M}_j)\boldsymbol{W}\boldsymbol{Y}(t)\ltimes_{s=1}^M\widehat{\boldsymbol{X}}_s(t).\tag{C3}$$

Let $\boldsymbol{P}_1=\ltimes_{j=1}^{M-1}(\boldsymbol{I}_{2^{jN}}\otimes\boldsymbol{W}_{[2^N,2^{j(N-1)N}]})$. It holds that

$$\begin{aligned}&\ltimes_{k=1}^M\widehat{\boldsymbol{X}}_k(t)\\ &=\boldsymbol{X}_1(t-\tau_1)\{\ltimes_{i=2}^N\boldsymbol{X}_1(t-\tau_i)\}\boldsymbol{X}_2(t-\tau_1)\\ &\quad\ltimes\{\ltimes_{i=2}^N\boldsymbol{X}_2(t-\tau_i)\}\ldots\boldsymbol{X}_M(t-\tau_1)\{\ltimes_{i=2}^N\boldsymbol{X}_M(t-\tau_i)\}\\ &=\boldsymbol{X}_1(t-\tau_1)\boldsymbol{W}_{[2^N,2^{(N-1)N}]}\boldsymbol{X}_2(t-\tau_1)\ltimes_{i=2}^N\boldsymbol{X}_1(t-\tau_i)\\ &\quad\ltimes_{i=2}^N\boldsymbol{X}_2(t-\tau_i)\boldsymbol{X}_3(t-\tau_1)\ltimes_{i=2}^N\boldsymbol{X}_3(t-\tau_i)\\ &\quad\ltimes\ldots\boldsymbol{X}_M(t-\tau_1)\ltimes_{i=2}^N\boldsymbol{X}_M(t-\tau_i)\\ &=(\boldsymbol{I}_{2^N}\otimes W_{[2^N,2^{(N-1)N}]})\boldsymbol{X}_1(t-\tau_1)\boldsymbol{X}_2(t-\tau_1)\\ &\quad\ltimes\boldsymbol{W}_{[2^N,2^{2(N-1)N}]}\boldsymbol{X}_3(t-\tau_1)\ltimes_{i=2}^N\boldsymbol{X}_1(t-\tau_i)\\ &\quad\ltimes_{i=2}^N\boldsymbol{X}_2(t-\tau_i)\ltimes_{i=2}^N\boldsymbol{X}_3(t-\tau_i)\boldsymbol{X}_4(t-\tau_1)\\ &\quad\ltimes_{i=2}^N\boldsymbol{X}_4(t-\tau_i)\ldots\boldsymbol{X}_M(t-\tau_1)\ltimes_{i=2}^N\boldsymbol{X}_M(t-\tau_i)\\ &=(\boldsymbol{I}_{2^N}\otimes\boldsymbol{W}_{[2^N,2^{(N-1)N}]})(\boldsymbol{I}_{2^{2N}}\otimes\boldsymbol{W}_{[2^N,2^{2(N-1)N}]})\\ &\quad\ltimes\ldots(\boldsymbol{I}_{2^{(M-1)N}}\otimes\boldsymbol{W}_{[2^N,2^{(M-1)(N-1)N}]})\\ &\quad\ltimes_{k=1}^M\boldsymbol{X}_k(t-\tau_1)\left\{\ltimes_{j=1}^M\left[\ltimes_{i=2}^N\boldsymbol{X}_j(t-\tau_i)\right]\right\}\\ &=\boldsymbol{P}_1\{\ltimes_{k=1}^M\boldsymbol{X}_k(t-\tau_1)\}\{\ltimes_{j=1}^M(\ltimes_{i=2}^N\boldsymbol{X}_j(t-\tau_i))\}.\end{aligned}\tag{C4}$$

Similar to Eq. (C4), let $\boldsymbol{P}_2 = \ltimes_{j=1}^{M-1}(\boldsymbol{I}_{2^{jN}} \otimes \boldsymbol{W}_{[2^N,2^{j(N-2)N}]})$. We have

$$\ltimes_{j=1}^{M}\left(\ltimes_{i=2}^{N}\boldsymbol{X}_j(t-\tau_i)\right) = \boldsymbol{P}_2\left\{\ltimes_{k=1}^{M}\boldsymbol{X}_k(t-\tau_2)\right\}\left\{\ltimes_{j=1}^{M}\left[\ltimes_{i=3}^{N}\boldsymbol{X}_j(t-\tau_i)\right]\right\}. \tag{C5}$$

Similarly, let $\boldsymbol{P}_q = \ltimes_{j=1}^{M-1}(\boldsymbol{I}_{2^{jN}} \otimes \boldsymbol{W}_{[2^N,2^{j(N-q)N}]})$, $q=1,2,\ldots,N-1$. We have

$$\left\{\ltimes_{j=1}^{M}\left[\ltimes_{i=3}^{N}\boldsymbol{X}_j(t-\tau_i)\right]\right\} = \boldsymbol{P}_3\left\{\ltimes_{k=1}^{M}\boldsymbol{X}_k(t-\tau_3)\right\}\boldsymbol{P}_4\left\{\ltimes_{k=1}^{M}\boldsymbol{X}_k(t-\tau_4)\right\}\ldots \ltimes \boldsymbol{P}_{N-1}\left\{\ltimes_{k=1}^{M}\boldsymbol{X}_k(t-\tau_{N-1})\right\}\left\{\ltimes_{k=1}^{M}\boldsymbol{X}_k(t-\tau_N)\right\}. \tag{C6}$$

Let $\boldsymbol{P} = \boldsymbol{P}_1(\boldsymbol{I}_{2^{MN}}\otimes\boldsymbol{P}_2)(\boldsymbol{I}_{2^{2MN}}\otimes\boldsymbol{P}_3)\ldots(\boldsymbol{I}_{2^{(N-2)MN}}\otimes\boldsymbol{P}_{N-1})$. According to Eqs. (C4) and (C6), it holds that

$$\begin{aligned}\ltimes_{k=1}^{M}\widehat{\boldsymbol{X}}_k(t) =&\boldsymbol{P}_1\left\{\ltimes_{k-1}^{M}\boldsymbol{X}_k(t-\tau_1)\right\}\ldots\boldsymbol{P}_{N-1}\\ &\ltimes_{k=1}^{M}\boldsymbol{X}_k(t-\tau_{N-1})\ltimes_{k=1}^{M}\boldsymbol{X}_k(t-\tau_N)\\ =&\boldsymbol{P}_1(\boldsymbol{I}_{2^{MN}}\otimes\boldsymbol{P}_2)(\boldsymbol{I}_{2^{2MN}}\otimes\boldsymbol{P}_3)\ldots\\ &\ltimes(\boldsymbol{I}_{2^{(N-2)MN}}\otimes\boldsymbol{P}_{N-1})\ltimes_{k=1}^{M}\boldsymbol{X}_k(t-\tau_1)\\ &\ltimes_{k=1}^{M}\boldsymbol{X}_k(t-\tau_2)\ldots\ltimes_{k=1}^{M}\boldsymbol{X}_k(t-\tau_N)\\ =&\boldsymbol{P}\ltimes_{j=1}^{N}\left\{\ltimes_{k=1}^{M}\boldsymbol{X}_k(t-\tau_j)\right\}.\end{aligned} \tag{C7}$$

Based on Eq. (C3) and (C7), let $\tau_0=0$. We have

$$\begin{aligned}\ltimes_{j=1}^{M}\boldsymbol{X}_j(t+1) =&(\otimes_{s=1}^{M}\boldsymbol{M}_s)\boldsymbol{W}\boldsymbol{Y}(t)\boldsymbol{P}\\ &\ltimes_{j=1}^{N}\left\{\ltimes_{k=1}^{M}\boldsymbol{X}_k(t-\tau_j)\right\}\\ =&\left(\otimes_{s=1}^{M}\boldsymbol{M}_s)\boldsymbol{W}\boldsymbol{H}(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t)\right)\boldsymbol{P}\\ &\ltimes_{j=1}^{N}\left\{\ltimes_{k=1}^{M}\boldsymbol{X}_k(t-\tau_j)\right\}\\ =&(\otimes_{s=1}^{M}\boldsymbol{M}_s)\boldsymbol{W}\boldsymbol{H}(\boldsymbol{I}_{2^{MN}}\otimes\boldsymbol{P})\\ &\ltimes_{j=0}^{N}\left\{\ltimes_{k=1}^{M}\boldsymbol{X}_k(t-\tau_j)\right\}.\end{aligned} \tag{C8}$$

Note that

$$\begin{aligned}&\ltimes_{k=0}^{N}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-\tau_k)\right) = \left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t)\right)\\ &\ltimes\mathbf{Ed}^{MN(\tau_1-1)}\ltimes_{k=1}^{\tau_1}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right)\\ &\ltimes\mathbf{Ed}^{MN(\tau_2-\tau_1-1)}\ltimes_{k=\tau_1+1}^{\tau_2}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right)\\ &\ltimes\mathbf{Ed}^{MN(\tau_3-\tau_2-1)}\ltimes_{k=\tau_2+1}^{\tau_3}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right)\ldots\\ &\ltimes\mathbf{Ed}^{MN(\tau_N-\tau_{N-1}-1)}\ltimes_{k=\tau_{N-1}+1}^{\tau_N}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right)\\ =&(\boldsymbol{I}_{2^{MN}}\otimes\mathbf{Ed}^{MN(\tau_1-1)})\\ &\ltimes(\boldsymbol{I}_{2^{(\tau_1+1)MN}}\otimes\mathbf{Ed}^{MN(\tau_2-\tau_1-1)})\\ &\ltimes\ldots(\boldsymbol{I}_{2^{(\tau_{N-1}+1)MN}}\otimes\mathbf{Ed}^{MN(\tau_N-\tau_{N-1}-1)})\\ &\ltimes_{k=0}^{\tau_n}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right)\\ =&\boldsymbol{I}\ltimes_{k=0}^{\tau_n}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right),\end{aligned} \tag{C9}$$

where $\boldsymbol{I} = \left\{\ltimes_{j=0}^{N-1}(\boldsymbol{I}_{2^{(\tau_j+1)MN}}\otimes\mathbf{Ed}^{MN(\tau_{j+1}-\tau_j-1)})\right\}$.

According to Eqs. (C8) and (C9), we have

$$\begin{aligned}\ltimes_{j=1}^{M}\boldsymbol{X}_j(t+1) =&(\otimes_{j=1}^{M}\boldsymbol{M}_j)\boldsymbol{W}\boldsymbol{H}(\boldsymbol{I}_{2^{MN}}\otimes\boldsymbol{P})\\ &\ltimes\boldsymbol{I}\ltimes_{k=0}^{\tau_n}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right).\end{aligned}$$

Appendix D: Proof of Lemma 5

Let

$$\boldsymbol{W}=\boldsymbol{W}_{[2^M,2^{N^2}]}\left\{\ltimes_{i=2}^{M}\left[(\boldsymbol{I}_{2^M}\otimes\boldsymbol{W}_{[2^M,2^{iN^2}]})\boldsymbol{\Phi}_M\right]\right\}.$$

Thus, we have

$$\begin{aligned}&\ltimes_{j=1}^{M}\left(\widehat{\boldsymbol{X}}_j(t)\widehat{\boldsymbol{Y}}(t)\right) = \boldsymbol{W}_{[2^M,2^{N^2}]}\widehat{\boldsymbol{Y}}(t)\widehat{\boldsymbol{X}}_1(t)\\ &\ltimes_{k=2}^{M}\left\{\widehat{\boldsymbol{X}}_k(t)\widehat{\boldsymbol{Y}}(t)\right\}\\ =&\boldsymbol{W}_{[2^M,2^{N^2}]}\widehat{\boldsymbol{Y}}(t)\boldsymbol{W}_{[2^M,2^{2N^2}]}\widehat{\boldsymbol{Y}}(t)\widehat{\boldsymbol{X}}_1(t)\widehat{\boldsymbol{X}}_2(t)\\ &\ltimes_{k=3}^{M}\{\widehat{\boldsymbol{X}}_k(t)\widehat{\boldsymbol{Y}}(t)\}\\ =&\boldsymbol{W}_{[2^M,2^{N^2}]}(\boldsymbol{I}_{2^M}\otimes\boldsymbol{W}_{[2^M,2^{2N^2}]})\boldsymbol{\Phi}_M\boldsymbol{Y}(t)\\ &\ltimes\widehat{\boldsymbol{X}}_1(t)\widehat{\boldsymbol{X}}_2(t)\ltimes_{k=3}^{M}\left\{\widehat{\boldsymbol{X}}_k(t)\widehat{\boldsymbol{Y}}(t)\right\}\\ &\ldots\\ =&\boldsymbol{W}_{[2^M,2^{N^2}]}\left\{\ltimes_{i=2}^{M}\left[(\boldsymbol{I}_{2^M}\otimes\boldsymbol{W}_{[2^M,2^{iN^2}]})\boldsymbol{\Phi}_M\right]\right\}\\ &\ltimes\widehat{\boldsymbol{Y}}(t)\ltimes_{k=1}^{M}\widehat{\boldsymbol{X}}_k(t)\\ =&\boldsymbol{W}\widehat{\boldsymbol{Y}}(t)\ltimes_{k=1}^{M}\widehat{\boldsymbol{X}}_k(t).\end{aligned} \tag{D1}$$

According to Eqs. (19) and (D1), we have

$$\ltimes_{j=1}^{M}\{\widehat{\boldsymbol{X}}_j(t)\widehat{\boldsymbol{Y}}(t)\} = \boldsymbol{W}\widehat{\boldsymbol{Y}}(t)\boldsymbol{D}\ltimes_{j=1}^{M}\widetilde{\boldsymbol{X}}_j(t). \tag{D2}$$

Similar to Eqs. (C4)−(C7), let $\bar{\boldsymbol{P}}_q = \ltimes_{j=1}^{M-1}(\boldsymbol{I}_{2^{jN}}\otimes\boldsymbol{W}_{[2^N,2^{j(\tau_m-q+1)N}]})$, $q=1,2,\ldots,\tau_m$, and $\bar{\boldsymbol{P}} = \bar{\boldsymbol{P}}_1\ltimes_{k=2}^{\tau_m}(\boldsymbol{I}_{2^{(k-1)MN}}\otimes\bar{\boldsymbol{P}}_k)$. We can obtain

$$\begin{aligned}&\left\{\ltimes_{j=1}^{M}\{\ltimes_{i=0}^{\tau_m}\boldsymbol{X}_j(t-i)\}\right\}\\ =&\bar{\boldsymbol{P}}_1\left\{\ltimes_{k=1}^{M}\boldsymbol{X}_k(t)\right\}\bar{\boldsymbol{P}}_2\left\{\ltimes_{k=1}^{M}\boldsymbol{X}_k(t-1)\right\}\\ &\ltimes\ldots\bar{\boldsymbol{P}}_{\tau_m}\left\{\ltimes_{k=1}^{M}\boldsymbol{X}_k(t-(\tau_m-1))\right\}\\ &\ltimes\left\{\ltimes_{k=1}^{M}\boldsymbol{X}_k(t-\tau_m)\right\},\end{aligned}$$

$$\ltimes_{k=1}^{M}\widetilde{\boldsymbol{X}}_k(t) = \bar{\boldsymbol{P}}\ltimes_{k=0}^{\tau_M}\left\{\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right\}. \tag{D3}$$

According to Eqs. (D2) and (D3), we have

$$\begin{aligned}\ltimes_{j=1}^{M}\left\{\widehat{\boldsymbol{X}}_j(t)\widehat{\boldsymbol{Y}}(t)\right\} =&\boldsymbol{W}\widehat{\boldsymbol{Y}}(t)\boldsymbol{D}\bar{\boldsymbol{P}}\\ &\ltimes_{k=0}^{\tau_M}\left\{\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right\}\\ =&\boldsymbol{W}(\boldsymbol{I}_{2^M}\otimes\boldsymbol{D}\bar{\boldsymbol{P}})\widehat{\boldsymbol{Y}}(t)\\ &\ltimes_{k=0}^{\tau_M}\left\{\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right\}.\end{aligned}$$

Appendix E: Proof of Lemma 6

Let

$$\boldsymbol{R} = \ltimes_{k=1}^{M} \left(\boldsymbol{I}_{2^{N(M-k)}} \otimes \left\{\boldsymbol{W}_{[2^{MN(\alpha_{M+1-k}+1)-N},2^N]} \ltimes \left(\boldsymbol{I}_{2^{MN(\alpha_{M+1-k}+1)-N}} \otimes \boldsymbol{\Phi}_N\right)\right\}\right).$$

Then, we have

$$\begin{aligned}
&\boldsymbol{X}_1(t-\alpha_1)\boldsymbol{X}_2(t-\alpha_2)\ldots\boldsymbol{X}_M(t-\alpha_M) \\
&\ltimes_{k=0}^{\tau_M}\left\{\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right\} \\
=&\boldsymbol{X}_1(t-\alpha_1)\boldsymbol{X}_2(t-\alpha_2)\ldots\boldsymbol{X}_M(t-\alpha_M) \\
&\ltimes\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t)\ldots\ltimes_{j=1}^{M-1}\boldsymbol{X}_j(t-\alpha_M)\right) \\
&\ltimes\boldsymbol{X}_M(t-\alpha_M)\ldots\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-\tau_M) \\
=&\boldsymbol{X}_1(t-\alpha_1)\boldsymbol{X}_2(t-\alpha_2)\ldots\boldsymbol{X}_{M-1}(t-\alpha_{M-1}) \\
&\ltimes\boldsymbol{W}_{[2^{MN(\alpha_M+1)-N},2^N]}\left(\ltimes_{j=1}^{M}\boldsymbol{X}_j(t)\ldots\right. \\
&\left.\ltimes_{j=1}^{M-1}\boldsymbol{X}_j(t-\alpha_M)\right)\boldsymbol{X}_M(t-\alpha_M) \\
&\ltimes\boldsymbol{X}_M(t-\alpha_M)\ldots\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-\tau_M) \\
&\ldots \\
=&\Big(\boldsymbol{I}_{2^{N(M-1)}}\otimes\Big\{\boldsymbol{W}_{[2^{MN(\alpha_M+1)-N},2^N]}(\boldsymbol{I}_{2^{MN(\alpha_M+1)-N}} \\
&\otimes\boldsymbol{\Phi}_N)\Big\}\Big)\ldots\Big(\boldsymbol{I}_{2^N}\otimes\Big\{\boldsymbol{W}_{[2^{MN(\alpha_2+1)-N},2^N]} \\
&\ltimes(\boldsymbol{I}_{2^{MN(\alpha_2+1)-N}}\otimes\boldsymbol{\Phi}_N)\Big\}\Big)\Big\{\boldsymbol{W}_{[2^{MN(\alpha_1+1)-N},2^N]} \\
&\ltimes\left(\boldsymbol{I}_{2^{MN(\alpha_1+1)-N}}\otimes\boldsymbol{\Phi}_N\right)\Big\}\ltimes_{k=0}^{\tau_M}\left\{\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right\} \\
=&\boldsymbol{R}\ltimes_{k=0}^{\tau_M}\left\{\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right\}.
\end{aligned} \tag{E1}$$

According to Eqs. (20) and (E1), we have

$$\begin{aligned}
&\widehat{\boldsymbol{Y}}(t)\ltimes_{k=0}^{\tau_M}\left\{\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right\} \\
=&\boldsymbol{H}\boldsymbol{R}\ltimes_{k=0}^{\tau_M}\left\{\ltimes_{j=1}^{M}\boldsymbol{X}_j(t-k)\right\}.
\end{aligned}$$