Characteristic of the equivalent impedance for an *m×n* RLC network with an arbitrary boundary[*]

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Abstract: Considerable progress has been made recently in the development of techniques to determine exactly two-point resistances in networks of various topologies. In particular, a general resistance formula of a non-regular *m×n* resistor network with an arbitrary boundary is determined by the recursion-transform (RT) method. However, research on the complex impedance network is more difficult than that on the resistor network, and it is a problem worthy of study since the equivalent impedance has many different properties from equivalent resistance. In this study, the equivalent impedance of a non-regular *m×n* RLC network with an arbitrary boundary is studied based on the resistance formula, and the oscillation characteristics and resonance properties of the equivalent impedance are discovered. In the RLC network, it is found that our formula leads to the occurrence of resonances at the boundary condition holding a series of specific values with an external alternating current source. This curious result suggests the possibility of practical applications of our formula to resonant circuits.

Key words: RLC network; Resonance properties; Oscillation characteristics; Amplitude-frequency
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1 Introduction

A classic problem in electric circuit theory studied by numerous researchers for more than 170 years is the computation of the resistance between two arbitrary nodes in a resistor network. This problem has existed since the German scientist Kirchhoff described the node current law and the circuit voltage law in 1845 (Kirchhoff, 1847). Resistor networks have been widely studied and simulated as models for many scientific and engineering problems (Kirkpatrick, 1973; Klein and Randić, 1993; Xiao and Gutman, 2003). In physics and probability theory, mean field theory investigates the behavior of large and complex stochastic models by studying a simpler model such as a finite disordered network. A large number of small interacting individual components which interact with each other are considered in such models. The effect of all the other individuals on any given individual is approximated by a single averaged effect. Therefore, a many-body problem has been reduced to a one-body problem. Mean field theory has been widely used in statistical mechanics, condensed system research of complex systems, magnetism, structure transition, and other fields (Georges *et al.*, 1996; Chamberlin, 2000; Chitra and Kotliar, 2000; Haule, 2007; Baule *et al.*, 2013; Bao *et al.*, 2014). Obviously, the construction of the models of resistor networks and the research on them make sense both for theory and application.

As is clear from previous work on two-point resistance, it is usually very difficult to obtain the exact resistance in a complex $m \times n$ resistor network (Cserti, 2000; Bianco and Giordano, 2003; Wu, 2004; Giordano, 2005; Tzeng and Wu, 2006; Izmailian and Huang, 2010; Asad, 2013a; 2013b; Asad et al., 2013; Izmailian et al., 2014; Izmailian and Kenna, 2014). For the explicit computation of two-point resistances in a resistor network, Cserti (2000) evaluated the two-point resistance using the lattice Green's function. His study was confined to mainly regular lattices of an infinite size, and was applied to many regular infinite resistor and capacitance networks (Bianco and Giordano, 2003; Giordano, 2005; Asad et al., 2013). Wu (2004) proposed a different approach and derived an expression for two-point resistance in arbitrarily finite or infinite lattices in terms of the eigenvalues and eigenvectors of the Laplacian matrix. Laplacian analysis has also been extended to impedance networks after a slight modification of the formulas (Tzeng and Wu, 2006). The Laplacian method has been used to solve various types of resistance networks with all kinds of geometry. As well as the previous cases (Wu, 2004; Tzeng and Wu, 2006), there are the recent Tan-Zhou-Yang conjecture (Tan et al., 2013; Izmailian et al., 2014), a globe network (Izmailian and Kenna, 2014), and a hammock network (Essam et al., 2015). For applications, Chair (2012; 2014a; 2014b) researched several resistor networks using the Laplacian method, and several new resistance results were obtained. In recent years, a new method, Tan's recursion-transform (RT) method, was proposed (Tan, 2011; 2015a; 2015b; 2015c), which makes it easy to resolve the problem of non-regular lattices with an arbitrary boundary (Tan, 2011; 2017; Tan et al., 2013; 2014; Essam et al., 2014; Tan and Fang, 2015; Tan and Zhang, 2015; 2017; Zhou et al., 2017). Using the RT method to compute the equivalent resistance relies on only one matrix along one direction, and the resistance is expressed by single summation. In particular, the RT method can be used to solve a resistor network with one or two arbitrary boundaries (Tan, 2015a; 2015b; 2015c).

Research on a complex impedance network is more difficult than that on a resistor network, because the equivalent impedance has many different properties from equivalent resistance. From previous work on the equivalent impedance, the results of the equivalent complex impedance are always curious and nonlinear (Whan and Lobb, 1996; Gabelli et al., 2006; Tzeng and Wu, 2006; Tan and Zhang, 2015; 2017). Thus, complex impedance is a problem worthy of study. In this study, we investigate the equivalent impedance of a non-regular $m \times n$ rectangular RLC network with an arbitrary boundary based on the resistance formula obtained but not resolved before (Tan, 2015a). The non-regular $m \times n$ rectangular resistor network with an arbitrary boundary is shown in Fig. 1. Here m and n are the grid numbers in the vertical and horizontal directions, respectively; r_0 and r are resistors in the vertical and horizontal directions, respectively. Considering grid elements r and r_0 as the impedances, we focus on researching the equivalent impedance $Z_{m \times n}(d_1, d_2)$ in the complex impedance network.
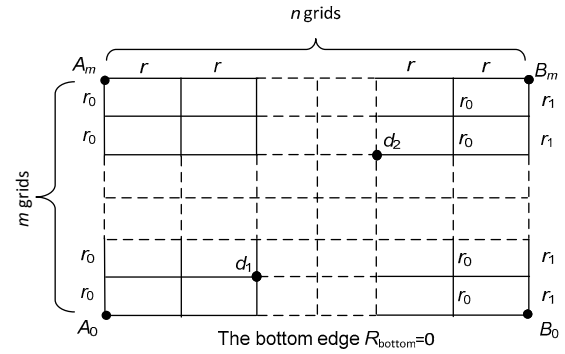

Fig. 1 An $m \times n$ resistor network with an arbitrary right boundary and null resistors on the bottom
Intersections in the horizontal and vertical directions are resistors r and r_0 except for the bottom and the right boundaries

First, we consider the equivalent resistance formula of a non-regular $m \times n$ rectangular resistor network. Assume that A_0 is the origin of the coordinate system, and the left edge acts as the y axis and the bottom edge acts as the x axis. Denote nodes of the network by coordinates (x, y). The equivalent resistance between any two nodes $d_1(x_1, y_1)$ and $d_2(x_2, y_2)$ in the $m \times n$ resistor network can be expressed as (Tan, 2015a)

$$R_{m \times n}(d_1, d_2) = \frac{2r_0}{2m+1}$$
$$\cdot \sum_{i=1}^{m} \frac{\beta_{x_1,x_1}^{(i)} S_{1,i}^2 - 2\beta_{x_1,x_2}^{(i)} S_{1,i} S_{2,i} + \beta_{x_2,x_2}^{(i)} S_{2,i}^2}{(1-\cos\theta_i)[F_{n+1}^{(i)} + (h_1-1)F_n^{(i)}]}, \quad (1)$$

where

$$\theta_i = (2i-1)\pi/(2m+1), \quad (2)$$

$$\beta_{s,x}^{(i)} = \Delta F_s^{(i)}[\Delta F_{n-x}^{(i)} + (h_1 - 1)\Delta F_{n-x-1}^{(i)}], \qquad (3)$$

$$F_k^{(i)} = (\lambda_i^k - \overline{\lambda}_i^k)/(\lambda_i - \overline{\lambda}_i), \qquad (4)$$

$$\Delta F_k^{(i)} = F_{k+1}^{(i)} - F_k^{(i)}, \qquad (5)$$

$$h_1 = r_1/r_0, \qquad (6)$$

$$h = r/r_0, \qquad (7)$$

$$S_{j,i} = \sin(y_j\theta_i), \quad j=1, 2, \qquad (8)$$

$$\lambda_i = 1 + h - h\cos\theta_i + \sqrt{(1 + h - h\cos\theta_i)^2 - 1}, \qquad (9)$$

$$\overline{\lambda}_i = 1 + h - h\cos\theta_i - \sqrt{(1 + h - h\cos\theta_i)^2 - 1}. \qquad (10)$$

In particular, when $d_1=(0, y_1)$ and $d_2=(0, y_2)$, two nodes defined as $A_p(0, p)$ and $A_q(0, q)$ are at the left edge,

$$R_{m\times n}(A_p, A_q) = \frac{2r_0}{2m+1}$$
$$\cdot\sum_{i=1}^{m}\left[\left(1 - \frac{F_n^{(i)} + (h_1-1)F_{n-1}^{(i)}}{F_{n+1}^{(i)} + (h_1-1)F_n^{(i)}}\right)\frac{(S_{q,i} - S_{p,i})^2}{1 - \cos\theta_i}\right]. \qquad (11)$$

Because $\dfrac{2}{2m+1}\sum\limits_{i=1}^{m}\dfrac{(S_{1,i} - S_{2,i})^2}{1 - \cos\theta_i} = |y_2 - y_1|$ (Tan, 2015a), Eq. (11) can be rewritten as

$$R_{m\times n}(A_p, A_q) = |q - p|r_0 - \frac{2r_0}{2m+1}$$
$$\cdot\sum_{i=1}^{m}\left[\left(\frac{F_n^{(i)} + (h_1-1)F_{n-1}^{(i)}}{F_{n+1}^{(i)} + (h_1-1)F_n^{(i)}}\right)\frac{(S_{q,i} - S_{p,i})^2}{1 - \cos\theta_i}\right]. \qquad (12)$$

As is well known, if the resistors are replaced by the arbitrary complex impedances, the impedance characteristics will be very complex. In this study, the peculiarity of the equivalent impedance in an $m\times n$ RLC network with an arbitrary boundary is researched from Eq. (12), and novel results are found.

2 Impedance of the *m×n* RLC network

2.1 Impedance formula of the RLC network

In Fig. 1, the grid elements are arbitrary and can be either a resistor or an impedance. If the grid elements are complex impedances, the model is called an $m\times n$ complex impedance network, and its segment grid of the RLC network is shown in Fig. 2. Assuming

that the alternating current (AC) frequency is ω, we have the following mapping relations:

$$r_k = i\omega L_k + \frac{R_k}{1 + i\omega C_k R_k}, \; k = 0, 1, \qquad (13)$$

$$r = i\omega L + \frac{R}{1 + i\omega CR}, \qquad (14)$$

where $i^2=-1$. Using Eq. (12), the equivalent impedance of an $m\times n$ complex RLC network is obtained. Substituting Eq. (13) into Eq. (12), the general formula of the equivalent impedance between any two nodes A_p and A_q on the left boundary of an $m\times n$ RLC network can be written as

$$Z_{m\times n}(A_p, A_q) = \left(i\omega L_0 + \frac{R_0}{1 + i\omega C_0 R_0}\right)$$
$$\cdot\left[|q - p| - \frac{2}{2m+1}\sum_{i=1}^{m}\left(\frac{(S_{q,i} - S_{p,i})^2}{1 - \cos\theta_i}f_n(\lambda_i, h_1)\right)\right], \qquad (15)$$

where function $f_n(\lambda_i, h_1)$ is defined as

$$f_n(\lambda_i, h_1) = \frac{F_n^{(i)} + (h_1 - 1)F_{n-1}^{(i)}}{F_{n+1}^{(i)} + (h_1 - 1)F_n^{(i)}}$$
$$= \frac{\lambda_i^n - \overline{\lambda}_i^n + (h_1 - 1)(\lambda_i^{n-1} - \overline{\lambda}_i^{n-1})}{\lambda_i^{n+1} - \overline{\lambda}_i^{n+1} + (h_1 - 1)(\lambda_i^n - \overline{\lambda}_i^n)}, \qquad (16)$$

$$h = r/r_0 = \frac{i\omega L + \dfrac{R}{1 + i\omega CR}}{i\omega L_0 + \dfrac{R_0}{1 + i\omega C_0 R_0}}, \qquad (17)$$

$$h_1 = r_1/r_0 = \frac{i\omega L_1 + \dfrac{R_1}{1 + i\omega C_1 R_1}}{i\omega L_0 + \dfrac{R_0}{1 + i\omega C_0 R_0}}. \qquad (18)$$

The characteristics of the equivalent impedance $Z_{m\times n}(A_p, A_q)$ are determined by the function $f_n(\lambda_i, h_1)$. Thus, we will first study the characteristics of the function $f_n(\lambda_i, h_1)$.

As Eq. (15) is the plural, it is more valuable and can be applied more readily. Although the impedance sub-network in Fig. 2 is quite complicated, Eqs. (15) and (16) are simple. Many properties of the equivalent impedance can be derived from Eq. (16).
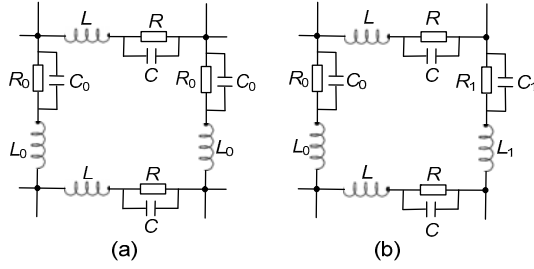
Fig. 2 Segment grids of the RLC network
(a) The segment grid except for the bottom and the right edges;
(b) The segment grid adjacent to the right boundary

2.2 Results of special conditions

If $1+h-h\cos\theta_i=-1$, then $h\sin^2(\theta_i/2)=-1$, and $\lambda_i=-1$ is obtained from Eq. (9). Substituting it into Eq. (17), there is

$$i\omega L_0 + \frac{R_0}{1+i\omega C_0 R_0} = -\left(i\omega L + \frac{R}{1+i\omega CR}\right)\sin^2\left(\frac{\theta_i}{2}\right). \tag{19}$$

In this case, from Eq. (16) we derive

$$\lim_{\lambda\to-1} f_n(\lambda_i, h_1) = \lim_{\lambda\to-1} \frac{\lambda_i^n - \bar{\lambda}_i^n + (h_1-1)(\lambda_i^{n-1} - \bar{\lambda}_i^{n-1})}{\lambda_i^{n+1} - \bar{\lambda}_i^{n+1} + (h_1-1)(\lambda_i^n - \bar{\lambda}_i^n)}$$
$$= -\frac{2n-1-(n-1)h_1}{2n+1-nh_1}. \tag{20}$$

Substituting Eq. (18) into Eq. (20), we obtain

$$\lim_{\lambda\to-1} f_n(\lambda_i, h_1) = -\frac{(2n-1)r_0 - (n-1)r_1}{(2n+1)r_0 - nr_1}, \tag{21}$$

where r_k (k=0,1) is defined in Eq. (13).

If h_1=1, from Eq. (20) we obtain

$$\lim_{\lambda\to-1} f_n(\lambda_i, h_1) = -\frac{2n-1-(n-1)h_1}{2n+1-nh_1} = -\frac{n}{n+1}. \tag{22}$$

Namely, when Eq. (19) is established, some special results can be obtained from Eq. (16).

In particular, if $R=R_0=\infty$ and $L=L_0=0$, $h=C_0/C$ is obtained. Then Fig. 1 is called an $m\times n$ capacitance network. The equivalent capacitance is known as $C_n\leftrightarrow 1/(i\omega C_n)$, and then the equivalent capacitance can be obtained easily from Eq. (15).

Eq. (15) is concise and significant but complicated for understanding the essential meaning of the

RLC network, for which one needs to consider the special conditions of the parameters R, L, and C to find their application values. In special applications, one can obtain the readable results if the impedance parameters are special. The usefulness of Eq. (15) is best illustrated by applications. One application is given below.

3 Impedance of the *m×n* LC network

Considering an $m\times n$ impedance LC network, the equivalent impedance of a non-regular $m\times n$ LC network can be obtained from Eq. (15).

3.1 Impedance formula of the LC network

Assume that the impedance elements in the horizontal and vertical directions in the lattice (Fig. 3) are $r=i\omega L$, $r_0=1/(i\omega C)$, respectively, and the impedance elements on the right boundary are $r_1=i\omega L+1/(i\omega C)$. The equivalent impedances between any two nodes A_p and A_q on the left boundary of a non-regular $m\times n$ LC network are given by

$$Z_{m\times n}(A_p, A_q) = \frac{1}{i\omega C}$$
$$\cdot\left[|q-p| - \frac{2}{2m+1}\sum_{i=1}^{m}\left(\frac{(S_{q,i} - S_{p,i})^2}{1-\cos\theta_i} f_n(\lambda_i, h_1)\right)\right], \tag{23}$$

where $h=r/r_0=-\omega^2 LC$, $h_1=-\omega^2 LC+C/C_1$, and λ_i is worthy of discussion as h is negative. If $1+h-h\cos\theta_i=-1$, there is $\lambda_i=-1$. By using $h=r/r_0=-\omega^2 LC$, we obtain

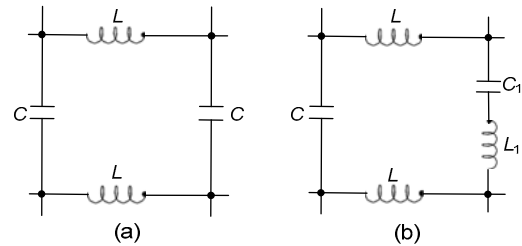$$\omega = \left(\sqrt{LC}\sin\frac{(2i-1)\pi}{2(2m+1)}\right)^{-1}. \tag{24}$$

Fig. 3 Segment grids of the LC network
(a) The segment grid except for the right edge; (b) The segment grid adjacent to right boundary except for the bottom edge

In this case, we obtain

$$\lim_{\lambda \to -1} f_n(\lambda_i, h_1) = -\frac{(2n-1)-(n-1)h_1}{(2n+1)-nh_1}. \quad (25)$$

Therefore, when Eq. (24) is established, many special results can be obtained from Eq. (23).

3.2 Analysis of three cases

In the case of $\omega \leq [\sqrt{LC}\sin(\theta_i/2)]^{-1}$,

$$\sqrt{(1+h-h\cos\theta_i)^2-1} \\ =i2\omega\sin(\theta_i/2)\sqrt{LC[1-\omega^2LC\sin^2(\theta_i/2)]}. \quad (26)$$

Therefore, when $\omega \leq [\sqrt{LC}\sin(\theta_i/2)]^{-1}$, by using Eqs. (9) and (24), $\lambda_i = \cos\phi_i + i\sin\phi_i$ is obtained with

$$\phi_i = \arccos(1-2\omega^2LC\sin^2(\theta_i/2)). \quad (27)$$

As a result,

$$f_n(\lambda_i, h_1) = \frac{\sin(n\phi_i)+(h_1-1)\sin[(n-1)\phi_i]}{\sin[(n+1)\phi_i]+(h_1-1)\sin(n\phi_i)}. \quad (28)$$

In the case of $[\sqrt{LC}\sin(\theta_k/2)]^{-1} \leq \omega \leq [\sqrt{LC}\sin(\theta_{k-1}/2)]^{-1}$, substituting Eq. (28) into Eq. (23), we have

$$Z_{m\times n}(A_p, A_q) = \frac{1}{i\omega C}\Bigg\{|q-p|-\frac{2}{2m+1} \\ \cdot\Bigg[\sum_{i=1}^{k-1}\Bigg(\frac{(S_{q,i}-S_{p,i})^2}{1-\cos\theta_i}\cdot\frac{\sin(n\phi_i)+(h_1-1)\sin[(n-1)\phi_i]}{\sin[(n+1)\phi_i]+(h_1-1)\sin(n\phi_i)}\Bigg) \\ +\sum_{i=k}^{m}\Bigg(\frac{(S_{q,i}-S_{p,i})^2}{1-\cos\theta_i}f_n(\lambda_i, h_1)\Bigg)\Bigg]\Bigg\}. \quad (29)$$

In the case of $\omega \leq \left(\sqrt{LC}\sin\dfrac{(2m-1)\pi}{2(2m+1)}\right)^{-1}$, substituting Eq. (28) into Eq. (23), we have

$$Z_{m\times n}(A_p, A_q) = \frac{1}{i\omega C}\Bigg[|q-p|-\frac{2}{2m+1} \\ \cdot\sum_{i=1}^{m}\Bigg(\frac{(S_{q,i}-S_{p,i})^2}{1-\cos\theta_i}\cdot\frac{\sin(n\phi_i)+(h_1-1)\sin[(n-1)\phi_i]}{\sin[(n+1)\phi_i]+(h_1-1)\sin(n\phi_i)}\Bigg)\Bigg]. \quad (30)$$

3.3 Circuit resonance conditions

An important feature of the complex impedance network is that its circuit can resonate under certain conditions. This is completely different from the resistor network. For example, when $\sin[(n+1)\phi_i]+(h_1-1)\sin(n\phi_i)=0$, we have

$$h_1 = 1-\frac{\sin[(n+1)\phi_i]}{\sin(n\phi_i)} = 1-\cos\phi_i-\sin\phi_i\cot(n\phi_i). \quad (31)$$

Substituting $h_1=-\omega^2LC+C/C_1$ into Eq. (31), there is

$$h_1 = -\omega^2L_1C+C/C_1 = 1-\cos\phi_i-\sin\phi_i\cot(n\phi_i). \quad (32)$$

If Eq. (32) holds, Eqs. (29) and (30) are in resonance. In the case of $\omega^2LC=1$, the 3D graphics are shown (Fig. 4) to clarify the specific relationship among h_1, i, and n.
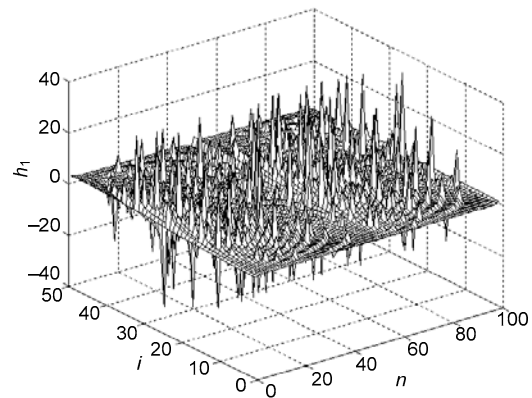
Fig. 4 The 3D graphics of h_1 with the changes of i and n under the condition of circuit resonance ($1\leq i\leq m=50$, $n\leq 100$)

Fig. 4 shows that there are many h_1 that can meet Eq. (32). That is to say, a situation, in which the effective impedance (Eq. (30)) between two nodes A_p, A_q diverges and the network is in resonance, can occur at specific frequencies ω in an AC circuit. Since L_1 and C_1 are the right boundary elements which can be controlled artificially, L_1 and C_1 can be adjusted to lead to the result of a resonance circuit. In other words, the external boundary conditions will affect the circuit resonance conditions. In these cases, the boundary condition of h_1 with the frequency ω of an external AC source can bring in resonance in the LC circuit, which can be calculated from Eq. (32). The result

suggests the possibility of practical applications to resonant circuits.

3.4 Semi-infinite LC network

When $\omega > [\sqrt{LC}\sin(\theta_i/2)]^{-1}$, λ_i and $\bar{\lambda}_i$ are real numbers, and $\bar{\lambda}_i \le -1 \le \lambda_i < 0$ (Eq. (9)). Therefore, if $n \to \infty$, Eq. (16) yields

$$\lim_{n\to\infty} f_n(\lambda_i, h_1) = \frac{1+(h_1-1)\lambda_i}{\bar{\lambda}_i+(h_1-1)}. \tag{33}$$

From Eq. (23), we obtain

$$Z_{m\times\infty}(A_p, A_q) = \frac{1}{\mathrm{i}\omega C}\left[|q-p| - \frac{2}{2m+1}\right.$$
$$\left. \cdot\sum_{i=1}^{m}\left(\frac{(S_{q,i}-S_{p,i})^2}{1-\cos\theta_i}\cdot\frac{1+(h_1-1)\lambda_i}{\bar{\lambda}_i+(h_1-1)}\right)\right]. \tag{34}$$

Eq. (34) is the equivalent impedance of the semi-infinite $m\times n$ LC network.

4 Impedance of the 2×n LC network

4.1 Impedance formula of a 2×n LC network

Studies of the complex impedance networks are usually very complicated. To reveal more characteristics of the impedance network, a simple model (Fig. 5) is studied first.

In Fig. 5, assuming the AC frequency is ω, the general equivalent impedance of a 2×n LC network with an arbitrary capacitance can be deduced from Eq. (23) to

$$Z_{2\times n}(A, D) = \frac{2}{\mathrm{i}\omega C}\left[1 - \frac{1}{5}\right.$$
$$\left. \cdot\left(\frac{\sin^2(2\theta_1)}{1-\cos\theta_1}f_n(\lambda_1, h_1) + \frac{\sin^2(2\theta_2)}{1-\cos\theta_2}f_n(\lambda_2, h_1)\right)\right], \tag{35}$$
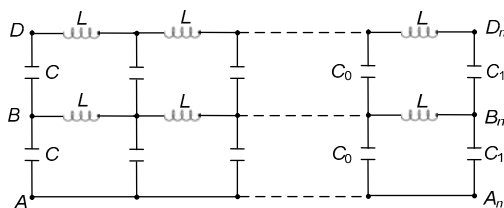

Fig. 5 The 2×n LC network with an arbitrary right boundary

$$Z_{2\times n}(A, B) = \frac{1}{\mathrm{i}\omega C}\left[1 - \frac{2}{5}\right.$$
$$\left. \cdot\left(\frac{\sin^2\theta_1}{1-\cos\theta_1}f_n(\lambda_1, h_1) + \frac{\sin^2\theta_2}{1-\cos\theta_2}f_n(\lambda_2, h_1)\right)\right], \tag{36}$$

$$Z_{2\times n}(B, D) = \frac{1}{\mathrm{i}\omega C}\left[1 - \frac{2}{5}\left(\frac{(\sin(2\theta_1)-\sin\theta_1)^2}{1-\cos\theta_1}\right.\right.$$
$$\left.\left. \cdot f_n(\lambda_1, h_1) + \frac{(\sin(2\theta_2)-\sin\theta_2)^2}{1-\cos\theta_2}f_n(\lambda_2, h_1)\right)\right], \tag{37}$$

where $\theta_i=(2i-1)\pi/5$ ($i=1, 2$), and $f_n(\lambda_i, h_1)$ is defined in Eq. (16) with $\lambda_i=1+h-h\cos\theta_i+\sqrt{(1+h-h\cos\theta_i)^2-1}$ and $h=-\omega^2 LC$, $h_1=C/C_1$.

Since $\theta_1=\pi/5$, $\theta_2=3\pi/5$, from Eqs. (35)–(37) we obtain

$$Z_{2\times n}(A, D) = \frac{2}{\mathrm{i}\omega C}\left[1 - \left(\frac{5+2\sqrt{5}}{10}f_n(\lambda_1, h_1)\right.\right.$$
$$\left.\left. + \frac{5-2\sqrt{5}}{10}f_n(\lambda_2, h_1)\right)\right], \tag{38}$$

$$Z_{2\times n}(A, B) = \frac{1}{\mathrm{i}\omega C}\left[1 - \left(\frac{5+\sqrt{5}}{10}f_n(\lambda_1, h_1)\right.\right.$$
$$\left.\left. + \frac{5-\sqrt{5}}{10}f_n(\lambda_2, h_1)\right)\right], \tag{39}$$

$$Z_{2\times n}(B, D) = \frac{1}{\mathrm{i}\omega C}\left[1 - \left(\frac{5-\sqrt{5}}{10}f_n(\lambda_1, h_1)\right.\right.$$
$$\left.\left. + \frac{5+\sqrt{5}}{10}f_n(\lambda_2, h_1)\right)\right]. \tag{40}$$

Eqs. (38)–(40) are the general formulas of the 2×n impedance LC network and can be applied in a variety of situations. However, Eqs. (38)–(40) are not enough to understand the meaning and characteristics of networks. So, the amplitude-frequency characteristics and the resonance characteristics of $f_n(\lambda_i, h_1)$ should be considered and researched.

4.2 Amplitude-frequency characteristics

Setting $n=10$, the 3D graphics of the complex impedance function $f_n(\lambda_1, h_1)$ are shown in Figs. 6 and 7. Fig. 6 shows oscillation characteristics and resonance properties of $f_n(\lambda_1, h_1)$ when $\omega^2 LC \le 11$, while Fig. 7 shows that $f_n(\lambda_1, h_1)$ gradually increases with the increase of $\omega^2 LC$ when $\omega^2 LC \ge \csc^2(\pi/10)$.
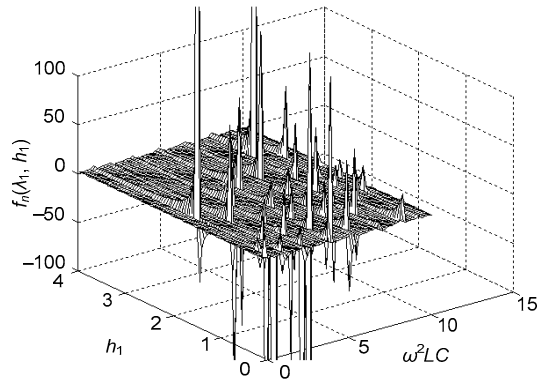
Fig. 6 The 3D graphics of the function $f_n(\lambda_1, h_1)$ with the changes of ω and h_1 ($\omega^2LC \leq 11$)
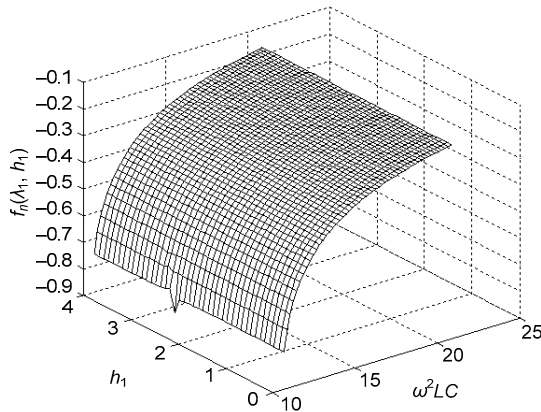

Fig. 7 The 3D graphics of the function $f_n(\lambda_1, h_1)$ with the changes of ω and h_1 ($\omega^2LC > 11$)

Figs. 6 and 7 express the overall situation, and 2D graphics are needed to obtain clear characteristics. Setting $n=10$ and $h_1=0.5, 4$, the complex impedance function $f_n(\lambda_1, h_1)$ changes in an oscillatory manner with the change of ω (Figs. 8 and 9).
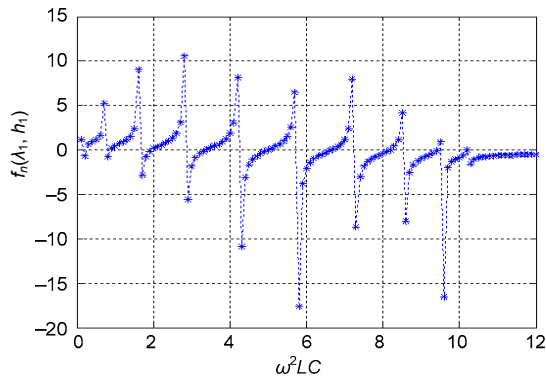

Fig. 8 The curve of the function $f_n(\lambda_1, h_1)$ with the change of ω^2LC ($h_1=0.5, n=10$)
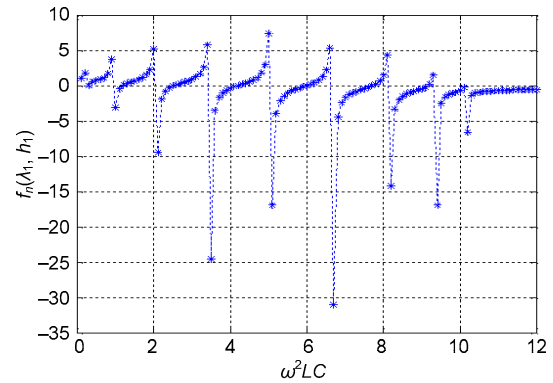

Fig. 9 The curve of the function $f_n(\lambda_1, h_1)$ with the change of ω^2LC ($h_1=4, n=10$)

Trends of curves in Figs. 8 and 9 are non-periodic but periodic-like.

In particular, following the same steps mentioned above, the 3D graphics of amplitude-frequency characteristics of $f_n(\lambda_2, h_1)$ can be obtained. To make the study concise, the 3D graphics of amplitude-frequency characteristics of $f_n(\lambda_2, h_1)$ are omitted.

4.3 Analysis of impedance under different conditions

Case 1: When $\omega\sqrt{LC} > \csc(\pi/10)$, there is $\bar{\lambda}_i \leq -1 \leq \lambda_i < 0$. When $n \to \infty$, substituting Eq. (33) into Eqs. (38)–(40), we have

$$Z_{2\times\infty}(A,D) = \frac{2}{i\omega C}\left[1 - \frac{5+2\sqrt{5}}{10}\left(\frac{1+(h_1-1)\lambda_1}{\bar{\lambda}_1+(h_1-1)}\right)\right.$$
$$\left. - \frac{5-2\sqrt{5}}{10}\left(\frac{1+(h_1-1)\lambda_2}{\bar{\lambda}_2+(h_1-1)}\right)\right], \quad (41)$$

$$Z_{2\times\infty}(A,B) = \frac{1}{i\omega C}\left[1 - \frac{5+\sqrt{5}}{10}\left(\frac{1+(h_1-1)\lambda_1}{\bar{\lambda}_1+(h_1-1)}\right)\right.$$
$$\left. - \frac{5-\sqrt{5}}{10}\left(\frac{1+(h_1-1)\lambda_2}{\bar{\lambda}_2+(h_1-1)}\right)\right], \quad (42)$$

$$Z_{2\times\infty}(B,D) = \frac{1}{i\omega C}\left[1 - \frac{5-\sqrt{5}}{10}\left(\frac{1+(h_1-1)\lambda_1}{\bar{\lambda}_1+(h_1-1)}\right)\right.$$
$$\left. - \frac{5+\sqrt{5}}{10}\left(\frac{1+(h_1-1)\lambda_2}{\bar{\lambda}_2+(h_1-1)}\right)\right]. \quad (43)$$

Case 2: When $\omega\sqrt{LC} = \csc(\pi/10)$ with $\lambda_1 = -1$, Eq. (25) is obtained, and then from Eqs. (38)–(40) we obtain

$$Z_{2\times n}(A, D) = \frac{2}{\mathrm{i}\omega C}\left[1 + \frac{5+2\sqrt{5}}{10}\left(\frac{(2n-1)-(n-1)h_1}{(2n+1)-nh_1}\right)\right. \\ \left. -\frac{5-2\sqrt{5}}{10}f_n(\lambda_2, h_1)\right],$$

(44)

$$Z_{2\times n}(A, B) = \frac{1}{\mathrm{i}\omega C}\left[1 + \frac{5+\sqrt{5}}{10}\left(\frac{(2n-1)-(n-1)h_1}{(2n+1)-nh_1}\right)\right. \\ \left. -\frac{5-\sqrt{5}}{10}f_n(\lambda_2, h_1)\right],$$

(45)

$$Z_{2\times n}(B, D) = \frac{1}{\mathrm{i}\omega C}\left[1 + \frac{5-\sqrt{5}}{10}\left(\frac{(2n-1)-(n-1)h_1}{(2n+1)-nh_1}\right)\right. \\ \left. -\frac{5+\sqrt{5}}{10}f_n(\lambda_2, h_1)\right].$$

(46)

Case 3: When $\csc(3\pi/10) < \omega\sqrt{LC} < \csc(\pi/10)$, $\lambda_1 = \cos\phi_1 + \mathrm{i}\sin\phi_1$ is obtained with

$$\phi_1 = \arccos(1 - 2\omega^2 LC \sin^2(\pi/10)). \qquad (47)$$

From Eq. (28) we obtain

$$f_n(\lambda_1, h_1) = \frac{\sin(n\phi_1) + (h_1-1)\sin[(n-1)\phi_1]}{\sin[(n+1)\phi_1] + (h_1-1)\sin(n\phi_1)}. \quad (48)$$

From Eqs. (38)–(40) we obtain

$$Z_{2\times n}(A, D) = \frac{2}{\mathrm{i}\omega C}\left(1 - \frac{5+2\sqrt{5}}{10}\right. \\ \left. \cdot\frac{\sin(n\phi_1) + (h_1-1)\sin[(n-1)\phi_1]}{\sin[(n+1)\phi_1] + (h_1-1)\sin(n\phi_1)} - \frac{5-2\sqrt{5}}{10}f_n(\lambda_2, h_1)\right),$$

(49)

$$Z_{2\times n}(A, B) = \frac{1}{\mathrm{i}\omega C}\left(1 - \frac{5+\sqrt{5}}{10}\right. \\ \left. \cdot\frac{\sin(n\phi_1) + (h_1-1)\sin[(n-1)\phi_1]}{\sin[(n+1)\phi_1] + (h_1-1)\sin(n\phi_1)} - \frac{5-\sqrt{5}}{10}f_n(\lambda_2, h_1)\right),$$

(50)

$$Z_{2\times n}(B, D) = \frac{1}{\mathrm{i}\omega C}\left[1 - \frac{5-\sqrt{5}}{10}\right. \\ \left. \cdot\frac{\sin(n\phi_1) + (h_1-1)\sin[(n-1)\phi_1]}{\sin[(n+1)\phi_1] + (h_1-1)\sin(n\phi_1)} - \frac{5+\sqrt{5}}{10}f_n(\lambda_2, h_1)\right).$$

(51)

When studying the characteristics of Eq. (48), it is found that the characteristics of $Z_{2\times n}(A, D)$, $Z_{2\times n}(A, B)$, and $Z_{2\times n}(B, D)$ change unsteadily with the change of n and h_1.

Case 4: When $\omega\sqrt{LC} = \csc(3\pi/10)$ with $\lambda_2 = -1$, Eq. (25) is obtained, and then from Eqs. (49)–(51) we obtain

$$Z_{2\times n}(A, D) = \frac{2}{\mathrm{i}\omega C}\left[1 - \frac{5+2\sqrt{5}}{10}\right. \\ \cdot\left(\frac{\sin(n\phi_1) + (h_1-1)\sin[(n-1)\phi_1]}{\sin[(n+1)\phi_1] + (h_1-1)\sin(n\phi_1)}\right) \\ \left. +\frac{5-2\sqrt{5}}{10}\left(\frac{(2n-1)-(n-1)h_1}{(2n+1)-nh_1}\right)\right],$$

(52)

$$Z_{2\times n}(A, B) = \frac{1}{\mathrm{i}\omega C}\left[1 - \frac{5+\sqrt{5}}{10}\right. \\ \cdot\left(\frac{\sin(n\phi_1) + (h_1-1)\sin[(n-1)\phi_1]}{\sin[(n+1)\phi_1] + (h_1-1)\sin(n\phi_1)}\right) \\ \left. +\frac{5-\sqrt{5}}{10}\left(\frac{(2n-1)-(n-1)h_1}{(2n+1)-nh_1}\right)\right],$$

(53)

$$Z_{2\times n}(B, D) = \frac{1}{\mathrm{i}\omega C}\left[1 - \frac{5-\sqrt{5}}{10}\right. \\ \cdot\left(\frac{\sin(n\phi_1) + (h_1-1)\sin[(n-1)\phi_1]}{\sin[(n+1)\phi_1] + (h_1-1)\sin(n\phi_1)}\right) \\ \left. +\frac{5+\sqrt{5}}{10}\left(\frac{(2n-1)-(n-1)h_1}{(2n+1)-nh_1}\right)\right],$$

(54)

where $\lambda_1 = \cos\phi_1 + \mathrm{i}\sin\phi_1$ with

$$\phi_1 = \arccos\left[1 - 2\left(\sin\frac{\pi}{10}\Big/\sin\frac{3\pi}{10}\right)^2\right]. \qquad (55)$$

Case 5: When $\omega\sqrt{LC} < \csc(3\pi/10)$, there are $\lambda_1 = \cos\phi_1 + \mathrm{i}\sin\phi_1$, $\phi_1 = \arccos(1 - 2\omega^2 LC \sin^2(\pi/10))$, $\lambda_2 = \cos\phi_2 + \mathrm{i}\sin\phi_2$, and $\phi_2 = \arccos(1 - 2\omega^2 LC \sin^2(3\pi/10))$. From Eqs. (49)–(51) we obtain

$$Z_{2\times n}(A, D) = \frac{2}{\mathrm{i}\omega C}\left[1 - \frac{5+2\sqrt{5}}{10}\right. \\ \cdot\left(\frac{\sin(n\phi_1) + (h_1-1)\sin[(n-1)\phi_1]}{\sin[(n+1)\phi_1] + (h_1-1)\sin(n\phi_1)}\right) \\ \left. -\frac{5-2\sqrt{5}}{10}\left(\frac{\sin(n\phi_2) + (h_1-1)\sin[(n-1)\phi_2]}{\sin[(n+1)\phi_2] + (h_1-1)\sin(n\phi_2)}\right)\right],$$

(56)

$$Z_{2\times n}(A,B) = \frac{1}{i\omega C}\left[1 - \frac{5+\sqrt{5}}{10}\right.$$
$$\cdot\left(\frac{\sin(n\phi_1)+(h_1-1)\sin[(n-1)\phi_1]}{\sin[(n+1)\phi_1]+(h_1-1)\sin(n\phi_1)}\right) \qquad (57)$$
$$\left.-\frac{5-\sqrt{5}}{10}\left(\frac{\sin(n\phi_2)+(h_1-1)\sin[(n-1)\phi_2]}{\sin[(n+1)\phi_2]+(h_1-1)\sin(n\phi_2)}\right)\right],$$

$$Z_{2\times n}(B,D) = \frac{1}{i\omega C}\left[1 - \frac{5-\sqrt{5}}{10}\right.$$
$$\cdot\left(\frac{\sin(n\phi_1)+(h_1-1)\sin[(n-1)\phi_1]}{\sin[(n+1)\phi_1]+(h_1-1)\sin(n\phi_1)}\right) \qquad (58)$$
$$\left.-\frac{5+\sqrt{5}}{10}\left(\frac{\sin(n\phi_2)+(h_1-1)\sin[(n-1)\phi_2]}{\sin[(n+1)\phi_2]+(h_1-1)\sin(n\phi_2)}\right)\right].$$

The above equations are the personalized formulas under different conditions. It is shown that the analysis formulas of the equivalent complex impedance in the LC network are indeed complicated.

4.4 Impedance features under different conditions

An important feature of the complex impedance network is that its circuit can produce oscillation and resonance phenomena, which are completely different from the resistor network. Eqs. (49)–(51), (52)–(54), and (56)–(58) are nonlinear because Eq. (48) is nonlinear. The nonlinearity of Eq. (48) is illustrated above. The oscillation and resonance phenomena will be revealed by the 3D graphics of $Z_{AD}(n)$. Setting $\omega^2 LC$=0.5, 1, 5, and 20, Figs. 10–13 are obtained.

Figs. 10–13 express the overall situation. To obtain clear characteristics, 2D graphics are needed. Setting $\omega^2 LC$=0.5, 1, and 5, and h_1=0.5, 1, and 2, the 2D graphics of the complex impedances $Z_{2\times n}(A, D)$ are shown in Figs. 14–17.
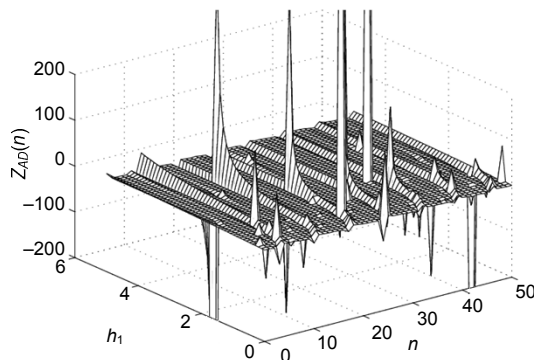

Fig. 10 The 3D graphics of the impedance $Z_{AD}(n)$ with the changes of h_1 and n ($\omega^2 LC$=0.5)
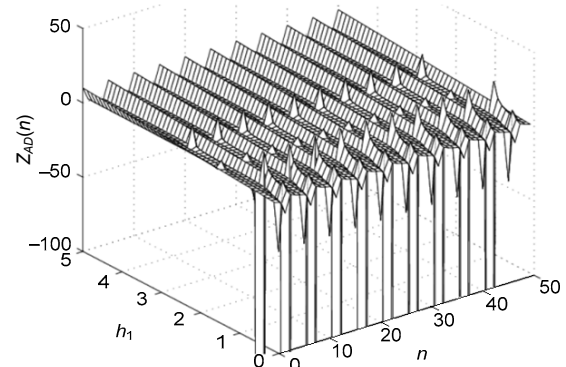

Fig. 11 The 3D graphics of the impedance $Z_{AD}(n)$ with the changes of h_1 and n ($\omega^2 LC$=1)
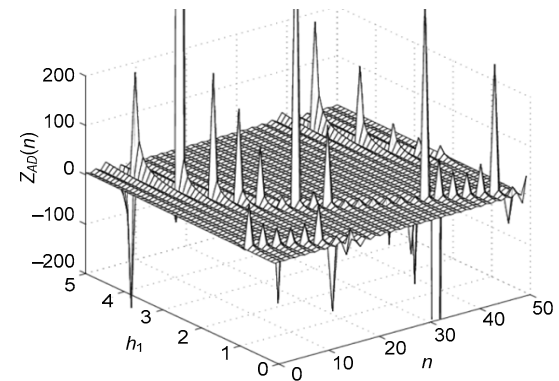

Fig. 12 The 3D graphics of the impedance $Z_{AD}(n)$ with the changes of h_1 and n ($\omega^2 LC$=5)
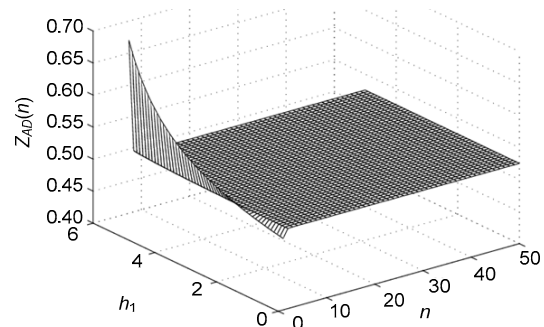

Fig. 13 The 3D graphics of the impedance $Z_{AD}(n)$ with the changes of h_1 and n ($\omega^2 LC$=20)

Case 1: When $\omega^2 LC$=0.5 and h_1=0.5, the complex impedance $Z_{2\times n}(A, D)$ changes unsteadily with the change of n (Fig. 14).

Case 2: When $\omega^2 LC$=1 and h_1=0.5, the complex impedance $Z_{2\times n}(A, D)$ changes periodically with the change of n (Fig. 15).

Case 3: When $\omega^2 LC$=1 and h_1=1, the complex

impedance $Z_{2\times n}(A, D)$ resonates periodically with the change of n (Fig. 16). From Fig. 16 it can be found that the complex impedances of $Z_{2\times n}(A, D)$ are in resonance.

Case 4: When $\omega^2 LC=5$ and $h_1=2$, the complex impedance $Z_{2\times n}(A, D)$ changes nonlinearly with the change of n (Fig. 17).
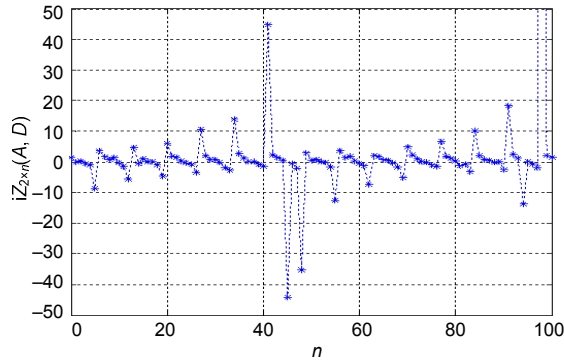

Fig. 14 The oscillation properties of the impedance $Z_{2\times n}(A, D)$ with the change of n (h_1=0.5, $\omega^2 LC$=0.5)
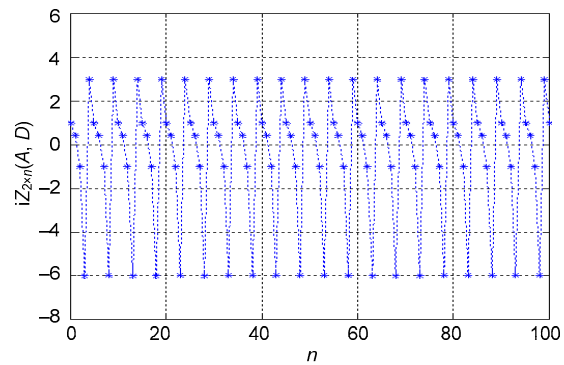

Fig. 15 The oscillation properties of the impedance $Z_{2\times n}(A, D)$ with the change of n (h_1=0.5, $\omega^2 LC$=1)
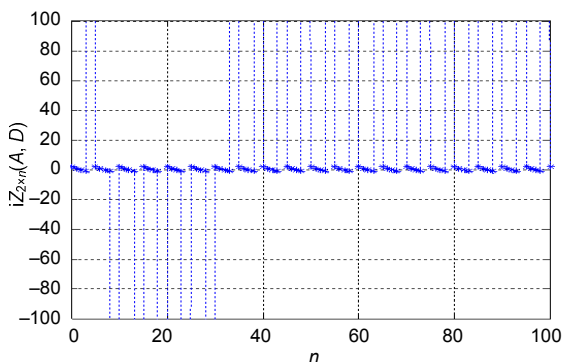

Fig. 16 The resonance properties of the impedance $Z_{2\times n}(A, D)$ with the change of n (h_1=1, $\omega^2 LC$=1)
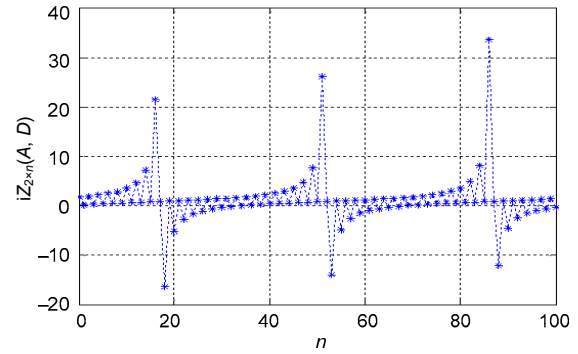

Fig. 17 The oscillation properties of the impedance $Z_{2\times n}(A, D)$ with the change of n (h_1=2, $\omega^2 LC$=5)

5 Summary and discussion

The study of a complex impedance network is more complex than that of a resistor network, because the equivalent complex impedance is related to the frequency of the input circuit. Tzeng and Wu (2006) researched the problem of complex impedance of the arbitrary LC network with a free boundary, and obtained the equivalent complex impedance between arbitrary two nodes in the two-dimensional $m\times n$ LC network, which is expressed by double summation using the Laplacian matrix method.

In this study, we have investigated the equivalent impedance between two nodes A_p and A_q of a non-regular rectangular $m\times n$ impedance network with an arbitrary boundary, and the results were expressed by a single summation. Besides, the arbitrary boundary is different from the free boundary and has never been resolved before. The advantage of the arbitrary boundary is that it can be artificially controlled to design the control circuit, but it is impossible for a free boundary. This study is based on the resistance formula obtained by Tan (2015a), and the analysis of oscillation characteristics and resonance properties is based mainly on the element r_1 on the right edge (Eq. (28)), which differs from the analysis of Tzeng and Wu (2006). Three cases have been presented: the general $m\times n$ RLC network, the general $m\times n$ LC network, and finally the specific $2\times n$ LC network. In the first two cases, the equivalent impedance has been expressed in a one summation formula, while in the third case the result was written in a compact form.

In the RLC network, the formula leads to the occurrence of resonances when the boundary

condition holding a series of special values with an external AC source. This result suggests the possibility of practical applications of the formula to resonant circuits.

The characteristics of the equivalent impedances are determined by the function $f_n(\lambda_i, h_1)$ which is defined as Eq. (16). When $R=R_0=\infty$ and $L=L_0=0$, we obtained an $m \times n$ capacitance network and its equivalent capacitance is $C_n \leftrightarrow 1/(i\omega C_n)$, which can be deduced from Eq. (15).

As a next step, an $m \times n$ impedance LC network has been considered. Its equivalent impedance was obtained by Eq. (23). An important property of the complex impedance network is the circuit resonance. This phenomenon was illustrated in the 3D graphics (Fig. 4).

Finally, the $2 \times n$ LC network has been studied and the features of the amplitude-frequency characteristics of the function $f_n(\lambda_i, h_1)$ were illustrated (Figs. 6 and 7). Two 2D graphics (Figs. 8 and 9) were obtained for specified values of n and h_1. The complex impedance function $f_n(\lambda_i, h_1)$ changes with the change of ω.

In the last part of this work, four 3D graphics and four 2D graphics of the impedance of $Z_{2 \times n}(A, D)$ have been shown.

In particular, this study has shown that the equivalent impedance $Z_{2 \times n}(A, D)$ increases gradually with the increase of ω when the value of $\omega^2 LC$ is sufficiently large ($\omega^2 LC > \csc^2(\pi/10)$) except for the condition of $n \leq 2$, where there is no oscillation (Fig. 13) (such as $\omega^2 LC=20$).

This study has also shown that the equivalent impedance has many different properties from equivalent resistance, and properties of oscillation and resonances suggest the possibility of practical applications of the formulas.

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