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Designing a novel consensus protocol for multiagent systems with general dynamics under directed networks*

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Abstract: The consensus problem for general linear multi-agent systems (MASs) under directed topology is investigated. First, a novel consensus protocol based on proportional-integral-derivative (PID) control is proposed. Second, the consensus problem is converted into an asymptotic stability problem through transformations. Third, through a state projection method the consensus condition is proved and the explicit expression of the consensus function is given. Then, a Lyapunov function is constructed and the gain matrices of the protocol are given based on the linear matrix inequality. Finally, two experiments are conducted to explain the advantages of the method. Simulation results show the effectiveness of the proposed algorithm.

Key words: Multi-agent; Consensus; PID control; Linear matrix inequality
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1 Introduction

Over the last several years, the consensus problem for multi-agent systems (MASs) has been a primary topic in the field of cooperative control. Autonomous vehicles working in a coordinated team have greater efficiency and operational capability than those perform solo missions (Ren *et al.*, 2007). The term 'consensus' means that all the dynamic agents eventually reach an agreement regarding a certain quantity of interests that depend on their states (Olfati-Saber *et al.*, 2007). The goal of most related literature is to design an effective control protocol or algorithm for MAS to achieve consensus, and the consensus algorithms have been widely applied in fields such as rendezvous (Lin *et al.*, 2015), formation control Cookson85 (Fax and Murray, 2004; Liu and Geng, 2015), flocking (Olfati-Saber, 2006; Zhang HT *et al.*, 2015), attitude alignment (Cai and Huang, 2014), and sensor networks (Sam and Khan, 2015).

In the condition where the dynamics of the agents are described by a first-order integrator, the most common continuous consensus algorithm was given by Fax and Murray (2004) and Ren and Beard (2005). It has been proved that if the interaction graph has a spanning tree, the consensus can be achieved. To reduce the frequency of the controller updates and the communication load, two distributed event-triggered control schemes were developed by Zhang H *et al.* (2015). Sun (2012) proposed a novel second-order consensus algorithm with bounded control inputs. The communication delay was considered by Su and Zhang (2008), who investigated two kinds of second-order consensus algorithms under fixed directed topology. Chen *et al.* (2015) studied the consensus problem of nonlinear second-order MAS in a more general framework. Yang *et al.* (2015) considered the general linear time-varying MAS with both multiple time delays and time-varying topologies. Zhang H *et al.* (2015)

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proposed an event-triggered control for general linear MAS with heterogeneous Markov switching topology to achieve H_∞ consensus. Obviously, the general linear dynamic is more universal than other forms; therefore, it is chosen as the object in this study. In addition to the dynamics of MAS, the topologies of the network have a crucial impact on the consensus of MAS. Li *et al.* (2013) considered undirected communication topology and proposed two types of distributed adaptive dynamic consensus protocols based on the relative output information of neighboring agents. Ren and Beard (2005) showed that a consensus could be achieved asymptotically if the union of the directed graphs had a spanning tree as the system evolved. When the switch of the topology is stochastic, Mu *et al.* (2015) relaxed the graph condition and assumed only that the union graph has a directed spanning tree, under which the conditions for containment tracking were given.

The proportional-integral-derivative (PID) controller has mature algorithms and is very robust; therefore, it has been widely used in the design of coordinated control (Cheng *et al.*, 2015; Wang *et al.*, 2015), in which mainly the PI^n type algorithm was adopted. Daniel and Mario (2014) adopted a distributed PID protocol which is a notable extension of the results in Ruggero *et al.* (2011) and Andreasson *et al.* (2012). A fast decentralized consensus tracking algorithm based on a PID controller was introduced by Yang *et al.* (2014) to improve the convergence speed of the MAS. Ji and Liao (2013) investigated the consensus problem with a proportional-derivative (PD) controller. There are some drawbacks to these approaches. For example, the dynamics of MAS was described by a simple first-order integrator and the topology was undirected in Ji and Liao (2013) and Lombana and di Bernardo (2015). Yang *et al.* (2014) considered only the consensus of MAS in the discrete domain. Thus, their control algorithms are not sufficiently representative.

In this paper, we propose a novel consensus protocol based on a distributed PID control, and the dynamics of MAS are assumed to be continuously linear time invariant and can be of any order. Thus, a theoretical framework for the MAS with a common mode is proposed. Our purpose is to verify the effectiveness of the proposed consensus protocol and make comparisons with other consensus algorithms. The control system is implemented with a dual-loop structure, including an outer-loop controller to guarantee consensus and an inner-loop one to adjust the dynamic property of the whole system.

2 Preliminaries

2.1 Graph theory and matrix notation

Let $\boldsymbol{G}=(\boldsymbol{V},\boldsymbol{\mathcal{E}},\boldsymbol{W})$ stand for a directed graph which consists of a node set $\boldsymbol{V}=\{1,2,\cdots,N\}$ and an edge set $\boldsymbol{\mathcal{E}}\in\boldsymbol{V}\times\boldsymbol{V}$. For an edge $e_{ij}=(v_i,v_j)$, node v_i is the parent node and node v_j the child node, and v_j is a neighbor of v_i. A directed graph contains a directed spanning tree if there exists a node called the 'root node', which has no parent node such that the node has directed paths to all other nodes in the graph. The adjacency matrix $\boldsymbol{W}=[w_{ij}]$ associated with the directed graph $\boldsymbol{G}$ is defined by w_{ij}, which is a positive value if $(v_j,v_i)\in\boldsymbol{\mathcal{E}}$. w_{ij} denotes the weight for the edge $(v_j,v_i)\in\boldsymbol{\mathcal{E}}$. If the weights are not considered, then w_{ij} is set to 1 if $(v_j,v_i)\in\boldsymbol{\mathcal{E}}$. The in-degree of node i is defined as $\deg_{\mathrm{in}}(i)=\sum_{j\in N_i}w_{ij}$, where N_i denotes the neighbor set of node i. The Laplacian matrix of graph $\boldsymbol{G}$ is defined as $\boldsymbol{\mathcal{L}}=\boldsymbol{D}-\boldsymbol{W}$, where $\boldsymbol{D}=\mathrm{diag}\,(\deg_{\mathrm{in}}(1),\deg_{\mathrm{in}}(2),\cdots,\deg_{\mathrm{in}}(N))$ is a diagonal matrix. Note that in general, the Laplacian matrix for a diagraph is asymmetric (Olfati-Saber *et al.*, 2007).

Throughout this paper, the following notations and definitions will be used: $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$ denote the sets of $n\times m$ real and complex matrices, respectively. $\boldsymbol{I}_N$ represents the identity matrix of dimension N. $\mathbf{1}$ ($\mathbf{0}$) denotes a column vector with all entries equal to one (zero). $\mathrm{Re}(\lambda)$ is the real part of a complex number λ. Symbol '$\otimes$' represents the Kronecker product.

2.2 Problem description

Consider a group of N identical agents with general linear dynamics. Each agent in MAS is modeled as

$$\dot{\boldsymbol{x}}_i(t)=\boldsymbol{x}_i(t)+\boldsymbol{B}\boldsymbol{u}_i(t),\ i=1,2,\cdots,N, \tag{1}$$

where $\boldsymbol{A}\in\mathbb{R}^{n\times n}$ and $\boldsymbol{B}\in\mathbb{R}^{n\times p}$ are constant matrices, $\boldsymbol{x}_i(t)=(x_{i_1}(t),x_{i_2}(t),\cdots,x_{i_n}(t))^{\mathrm{T}}\in\mathbb{R}^n$ is the state vector of the ith node, and $\boldsymbol{u}_i\in\mathbb{R}^p$ is the control input.

In this study, we consider the following consensus protocol for the ith agent:

$$\boldsymbol{u}_i(t)=\boldsymbol{K}_1\boldsymbol{x}_i(t)+\boldsymbol{K}_2\sum_{j\in N_i}\omega_{ij}\left[T_{\rm p}(\boldsymbol{x}_j(t)-\boldsymbol{x}_i(t))\right.$$
$$\left.+T_{\rm I}\int_0^t(\boldsymbol{x}_j(\tau)-\boldsymbol{x}_i(\tau)){\rm d}t+T_{\rm D}\frac{{\rm d}(\boldsymbol{x}_j(t)-\boldsymbol{x}_i(t))}{{\rm d}t}\right], \tag{2}$$

where $i,j\in\{1,2,\cdots,N\}$, $\boldsymbol{K}_1$, $\boldsymbol{K}_2\in\mathbb{R}^{p\times n}$, $\boldsymbol{K}_1$ is the state-feedback matrix, $\boldsymbol{K}_2$ is the consensus gain matrix, N_i denotes the neighbor set of agent i, and $T_{\rm p}>0, T_{\rm I}>0, T_{\rm D}>0$ are the proportional, integral, and derivative gain, respectively. Then we can obtain the dynamics of each agent in the following form:

$$\dot{\boldsymbol{x}}_i(t)=(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}_1)\boldsymbol{x}_i(t)$$
$$+\boldsymbol{B}\boldsymbol{K}_2\sum_{j\in N_i}\omega_{ij}\left[T_{\rm p}(\boldsymbol{x}_j(t)-\boldsymbol{x}_i(t))\right. \tag{3}$$
$$\left.+T_{\rm I}\int_0^t(\boldsymbol{x}_j(\tau)-\boldsymbol{x}_i(\tau)){\rm d}t+T_{\rm D}\frac{{\rm d}(\boldsymbol{x}_j(t)-\boldsymbol{x}_i(t))}{{\rm d}t}\right].$$

Let $\boldsymbol{x}(t)=[\boldsymbol{x}_1^{\rm T}(t),\boldsymbol{x}_2^{\rm T}(t),\cdots,\boldsymbol{x}_N^{\rm T}(t)]^{\rm T}$. Then system (3) can be written in a compact matrix form as

$$\dot{\boldsymbol{x}}(t)=[\boldsymbol{I}_N\otimes(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}_1)-T_{\rm p}\boldsymbol{\mathcal{L}}\otimes\boldsymbol{B}\boldsymbol{K}_2]\boldsymbol{x}(t)$$
$$-T_{\rm I}\boldsymbol{\mathcal{L}}\otimes\boldsymbol{B}\boldsymbol{K}_2\int_0^t\boldsymbol{x}(\tau){\rm d}\tau-T_{\rm D}\boldsymbol{\mathcal{L}}\otimes\boldsymbol{B}\boldsymbol{K}_2\frac{{\rm d}\boldsymbol{x}(t)}{{\rm d}t}. \tag{4}$$

Definition 1 Consensus in multi-agent system (4) can be achieved if for any initial condition, the following equation holds:

$$\lim_{t\to\infty}\|\boldsymbol{x}_i(t)-\boldsymbol{x}_j(t)\|=0. \tag{5}$$

Remark 1 Definition 1 requires only that the state differences among different agents go asymptotically to zero, no matter whether the states themselves converge to zero or not.

3 Consensus analysis

3.1 Problem transformation

This section converts the consensus problem into an asymptotic stability problem and gives another definition for consensus of MAS.

For any Laplacian matrix of graph $\boldsymbol{G}$, there exists a nonsingular constant matrix $\boldsymbol{u}\in\mathbb{C}^{N\times N}$ to obtain

$$\boldsymbol{u}^{-1}\boldsymbol{\mathcal{L}}\boldsymbol{u}=\boldsymbol{J}_{\boldsymbol{\mathcal{L}}}, \tag{6}$$

where $\boldsymbol{J}_{\boldsymbol{\mathcal{L}}}$ is the Jordan normal form. Assume that the graph contains a spanning tree. Then the eigenvalues of $\boldsymbol{\mathcal{L}}$ satisfy $0={\rm Re}(\lambda_1)<{\rm Re}(\lambda_2)\leq\cdots\leq{\rm Re}(\lambda_N)$ (Ren and Beard, 2005).

The differential part of Eq. (4) can be moved to the left side. Then one can obtain

$$\begin{aligned}&(\boldsymbol{I}_N\otimes\boldsymbol{I}_n+T_{\rm D}\boldsymbol{\mathcal{L}}\otimes\boldsymbol{B}\boldsymbol{K}_2)\dot{\boldsymbol{x}}(t)\\&=[\boldsymbol{I}_N\otimes(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}_1)-T_{\rm p}\boldsymbol{\mathcal{L}}\otimes\boldsymbol{B}\boldsymbol{K}_2]\boldsymbol{x}(t)\\&\quad-T_{\rm I}\boldsymbol{\mathcal{L}}\otimes\boldsymbol{B}\boldsymbol{K}_2\int_0^t\boldsymbol{x}(\tau){\rm d}\tau.\end{aligned} \tag{7}$$

Let $\tilde{\boldsymbol{\mathcal{L}}}=\boldsymbol{I}_N\otimes\boldsymbol{I}_n+T_{\rm D}\boldsymbol{\mathcal{L}}\otimes\boldsymbol{B}\boldsymbol{K}_2$. Assume that state vector $\boldsymbol{x}(t)$ has a second derivative and that $\tilde{\boldsymbol{\mathcal{L}}}$ is invertible. Then Eq. (7) can be depicted as

$$\ddot{\boldsymbol{x}}(t)=\tilde{\boldsymbol{\mathcal{L}}}^{-1}[\boldsymbol{I}_N\otimes(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}_1)-T_{\rm p}*\boldsymbol{\mathcal{L}}\otimes\boldsymbol{B}\boldsymbol{K}_2]\dot{\boldsymbol{x}}(t)$$
$$-T_{\rm I}*\tilde{\boldsymbol{\mathcal{L}}}^{-1}(\boldsymbol{\mathcal{L}}\otimes\boldsymbol{B}\boldsymbol{K}_2)\boldsymbol{x}(t). \tag{8}$$

The description using the state space of Eq. (8) is

$$\begin{bmatrix}\dot{\boldsymbol{x}}(t)\\\ddot{\boldsymbol{x}}(t)\end{bmatrix}=\begin{bmatrix}\boldsymbol{0}_N\otimes\boldsymbol{0}_n & \boldsymbol{I}_N\otimes\boldsymbol{I}_n\\-T_{\rm I}\tilde{\boldsymbol{\mathcal{L}}}^{-1}(\boldsymbol{\mathcal{L}}\otimes\boldsymbol{B}\boldsymbol{K}_2) & \begin{array}{c}\tilde{\boldsymbol{\mathcal{L}}}^{-1}[\boldsymbol{I}_N\otimes(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}_1)\\-T_{\rm p}\boldsymbol{\mathcal{L}}\otimes\boldsymbol{B}\boldsymbol{K}_2]\end{array}\end{bmatrix}\cdot\begin{bmatrix}\boldsymbol{x}(t)\\\dot{\boldsymbol{x}}(t)\end{bmatrix}. \tag{9}$$

Let $\boldsymbol{v}(t)=\dot{\boldsymbol{x}}(t)$. Then Eq. (9) is equivalent to

$$\begin{bmatrix}\dot{\boldsymbol{x}}(t)\\\dot{\boldsymbol{v}}(t)\end{bmatrix}=\begin{bmatrix}\boldsymbol{0}_N\otimes\boldsymbol{0}_n & \boldsymbol{I}_N\otimes\boldsymbol{I}_n\\-T_{\rm I}\tilde{\boldsymbol{\mathcal{L}}}^{-1}(\boldsymbol{\mathcal{L}}\otimes\boldsymbol{B}\boldsymbol{K}_2) & \begin{array}{c}\tilde{\boldsymbol{\mathcal{L}}}^{-1}[\boldsymbol{I}_N\otimes(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}_1)\\-T_{\rm p}\boldsymbol{\mathcal{L}}\otimes\boldsymbol{B}\boldsymbol{K}_2]\end{array}\end{bmatrix}\cdot\begin{bmatrix}\boldsymbol{x}(t)\\\boldsymbol{v}(t)\end{bmatrix}. \tag{10}$$

According to Eq. (6), we have

$$\begin{aligned}\boldsymbol{J}_{\tilde{\boldsymbol{\mathcal{L}}}}&=(\boldsymbol{u}\otimes\boldsymbol{I}_n)^{-1}(\boldsymbol{I}_N\otimes\boldsymbol{I}_n+T_{\rm D}\boldsymbol{\mathcal{L}}\otimes\boldsymbol{B}\boldsymbol{K}_2)(\boldsymbol{u}\otimes\boldsymbol{I}_n)\\&=\boldsymbol{u}^{-1}\boldsymbol{I}_N\boldsymbol{u}\otimes\boldsymbol{I}_n^{-1}\boldsymbol{I}_n\boldsymbol{I}_n\\&\quad+T_{\rm D}\boldsymbol{u}^{-1}\boldsymbol{\mathcal{L}}\boldsymbol{u}\otimes\boldsymbol{I}_n^{-1}\boldsymbol{B}\boldsymbol{K}_2\boldsymbol{I}_n\\&=(\boldsymbol{I}_N\otimes\boldsymbol{I}_n+T_{\rm D}\boldsymbol{J}_{\boldsymbol{\mathcal{L}}}\otimes\boldsymbol{B}\boldsymbol{K}_2)\\&={\rm diag}\left(\boldsymbol{J}_{\tilde{\boldsymbol{\mathcal{L}}}_1},\boldsymbol{J}_{\tilde{\boldsymbol{\mathcal{L}}}_2},\cdots,\boldsymbol{J}_{\tilde{\boldsymbol{\mathcal{L}}}_N}\right),\end{aligned} \tag{11}$$

where $\boldsymbol{J}_{\tilde{\mathcal{L}}}$ is the Jordan normal form of $\tilde{\mathcal{L}}$ and $\boldsymbol{J}_{\tilde{\mathcal{L}}_i}$ is the Jordan block of $\boldsymbol{J}_{\tilde{\mathcal{L}}}$. We can easily obtain $\boldsymbol{J}_{\tilde{\mathcal{L}}_i}=\boldsymbol{I}_n+T_{\rm D}\lambda_i\boldsymbol{BK}_2$. Then the inverse matrix of $\boldsymbol{J}_{\tilde{\mathcal{L}}}$ can be obtained as

$$\begin{aligned}\tilde{\mathcal{L}}^{-1}&=\left[(\boldsymbol{u}\otimes\boldsymbol{I}_n)\boldsymbol{J}_{\tilde{\mathcal{L}}}(\boldsymbol{u}\otimes\boldsymbol{I}_n)^{-1}\right]^{-1}\\&=(\boldsymbol{u}\otimes\boldsymbol{I}_n)\boldsymbol{J}_{\tilde{\mathcal{L}}}^{-1}(\boldsymbol{u}\otimes\boldsymbol{I}_n)^{-1}\\&=(\boldsymbol{u}\otimes\boldsymbol{I}_n)\begin{bmatrix}\boldsymbol{J}_1^{-1}&&&\\&\boldsymbol{J}_2^{-1}&&\\&&\ddots&\\&&&\boldsymbol{J}_N^{-1}\end{bmatrix}(\boldsymbol{u}\otimes\boldsymbol{I}_n)^{-1}.\end{aligned}\tag{12}$$

Let $\tilde{\boldsymbol{x}}(t)=(\boldsymbol{u}^{-1}\otimes\boldsymbol{I}_n)\boldsymbol{x}(t)$ and $\tilde{\boldsymbol{v}}(t)=(\boldsymbol{u}^{-1}\otimes\boldsymbol{I}_n)\boldsymbol{v}(t)$. Then we obtain

$$\begin{bmatrix}\tilde{\boldsymbol{x}}(t)\\\tilde{\boldsymbol{v}}(t)\end{bmatrix}=\begin{bmatrix}\boldsymbol{u}^{-1}\otimes\boldsymbol{I}_n&\\&\boldsymbol{u}^{-1}\otimes\boldsymbol{I}_n\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(t)\\\boldsymbol{v}(t)\end{bmatrix}.\tag{13}$$

For simplicity, let $\boldsymbol{T}=\begin{bmatrix}\boldsymbol{u}\otimes\boldsymbol{I}_n&\\&\boldsymbol{u}\otimes\boldsymbol{I}_n\end{bmatrix}$. The inverse matrix of $\boldsymbol{T}$ can be written as $\boldsymbol{T}^{-1}=\begin{bmatrix}\boldsymbol{u}^{-1}\otimes\boldsymbol{I}_n&\\&\boldsymbol{u}^{-1}\otimes\boldsymbol{I}_n\end{bmatrix}$. Let $\boldsymbol{H}=\begin{bmatrix}\boldsymbol{0}_N\otimes\boldsymbol{0}_n&\boldsymbol{I}_N\otimes\boldsymbol{I}_n\\-T_{\rm I}\tilde{\mathcal{L}}^{-1}(\mathcal{L}\otimes\boldsymbol{BK}_2)&\begin{matrix}\tilde{\mathcal{L}}^{-1}[\boldsymbol{I}_N\otimes(\boldsymbol{A}+\boldsymbol{BK}_1)\\-T_{\rm p}\mathcal{L}\otimes\boldsymbol{BK}_2]\end{matrix}\end{bmatrix}$. Then one can obtain

$$\boldsymbol{T}^{-1}\boldsymbol{HT}=\begin{bmatrix}\boldsymbol{0}_N\otimes\boldsymbol{0}_n&\boldsymbol{I}_N\otimes\boldsymbol{I}_n\\-T_{\rm I}\boldsymbol{J}_{\tilde{\mathcal{L}}}^{-1}(\boldsymbol{J}_{\mathcal{L}}\otimes\boldsymbol{BK}_2)&\begin{matrix}\boldsymbol{J}_{\tilde{\mathcal{L}}}^{-1}[\boldsymbol{I}_N\otimes(\boldsymbol{A}+\boldsymbol{BK}_1)\\-T_{\rm p}\boldsymbol{J}_{\mathcal{L}}\otimes\boldsymbol{BK}_2]\end{matrix}\end{bmatrix}.\tag{14}$$

Thus, the time derivative of Eq. (10) is as follows:

$$\begin{bmatrix}\dot{\tilde{\boldsymbol{x}}}(t)\\\dot{\tilde{\boldsymbol{v}}}(t)\end{bmatrix}=\begin{bmatrix}\boldsymbol{0}_N\otimes\boldsymbol{0}_n&\boldsymbol{I}_N\otimes\boldsymbol{I}_n\\-T_{\rm I}\boldsymbol{J}_{\tilde{\mathcal{L}}}^{-1}(\boldsymbol{J}_{\mathcal{L}}\otimes\boldsymbol{BK}_2)&\begin{matrix}\boldsymbol{J}_{\tilde{\mathcal{L}}}^{-1}[\boldsymbol{I}_N\otimes(\boldsymbol{A}+\boldsymbol{BK}_1)\\-T_{\rm p}\boldsymbol{J}_{\mathcal{L}}\otimes\boldsymbol{BK}_2]\end{matrix}\end{bmatrix}\cdot\begin{bmatrix}\tilde{\boldsymbol{x}}(t)\\\tilde{\boldsymbol{v}}(t)\end{bmatrix},\tag{15}$$

where $\boldsymbol{J}_{\tilde{\mathcal{L}}}^{-1}=\text{diag}\left(\boldsymbol{J}_{\tilde{\mathcal{L}}_1}^{-1},\boldsymbol{J}_{\tilde{\mathcal{L}}_2}^{-1},\cdots,\boldsymbol{J}_{\tilde{\mathcal{L}}_N}^{-1}\right)$.

It follows from Eq. (15) that the consensus problem is converted into an asymptotic stability problem. To obtain the results, we need to transform Eq. (15) to another form.

Lemma 1 Let $\boldsymbol{x}=\begin{bmatrix}\boldsymbol{I}_N\otimes\boldsymbol{A}_n&\boldsymbol{I}_N\otimes\boldsymbol{C}_n\\\boldsymbol{I}_N\otimes\boldsymbol{D}_n&\boldsymbol{I}_N\otimes\boldsymbol{B}_n\end{bmatrix}$, $\tilde{\boldsymbol{x}}=\boldsymbol{I}_N\otimes\begin{bmatrix}\boldsymbol{A}_n&\boldsymbol{C}_n\\\boldsymbol{D}_n&\boldsymbol{B}_n\end{bmatrix}$. Then we can obtain $\tilde{\boldsymbol{x}}=\boldsymbol{FXG}$, where $\boldsymbol{F}$ and $\boldsymbol{G}$ are the products of finite elementary matrices.

Let $\boldsymbol{J}=\begin{bmatrix}\boldsymbol{0}_N\otimes\boldsymbol{0}_n&\boldsymbol{I}_N\otimes\boldsymbol{I}_n\\-T_{\rm I}\boldsymbol{J}_{\tilde{\mathcal{L}}}^{-1}(\boldsymbol{J}_{\mathcal{L}}\otimes\boldsymbol{BK}_2)&\begin{matrix}\boldsymbol{J}_{\tilde{\mathcal{L}}}^{-1}[\boldsymbol{I}_N\otimes(\boldsymbol{A}+\boldsymbol{BK}_1)\\-T_{\rm p}\boldsymbol{J}_{\mathcal{L}}\otimes\boldsymbol{BK}_2]\end{matrix}\end{bmatrix}$. Then we can obtain

$$\boldsymbol{J}=\boldsymbol{T}^{-1}\boldsymbol{HT},\ \begin{bmatrix}\dot{\tilde{\boldsymbol{x}}}(t)\\\dot{\tilde{\boldsymbol{v}}}(t)\end{bmatrix}=\boldsymbol{J}\begin{bmatrix}\tilde{\boldsymbol{x}}(t)\\\tilde{\boldsymbol{v}}(t)\end{bmatrix}.\tag{16}$$

According to Eq. (16), each variable satisfies

$$\begin{cases}\dot{\tilde{\boldsymbol{x}}}_i(t)=\tilde{\boldsymbol{v}}_i(t),\\\dot{\tilde{\boldsymbol{v}}}_i(t)=-T_{\rm I}\lambda_i\boldsymbol{J}_{\tilde{\mathcal{L}}_i}^{-1}\boldsymbol{BK}_2\tilde{\boldsymbol{x}}_i(t)\\\quad+\boldsymbol{J}_{\tilde{\mathcal{L}}_i}^{-1}(\boldsymbol{A}+\boldsymbol{BK}_1-T_{\rm p}\lambda_i\boldsymbol{BK}_2)\tilde{\boldsymbol{v}}_i(t),\end{cases}\tag{17}$$
$$i=1,2,\cdots,N.$$

Let $\tilde{\boldsymbol{\xi}}_i(t)=\left[\tilde{\boldsymbol{x}}_i^{\rm T}(t),\tilde{\boldsymbol{v}}_i^{\rm T}(t)\right]^{\rm T}$, $\boldsymbol{J}_{\mathcal{L}}=\text{diag}(0,\bar{\boldsymbol{J}}_{\mathcal{L}})$, where $\bar{\boldsymbol{J}}_{\mathcal{L}}=\text{diag}(\lambda_2,\lambda_3,\cdots,\lambda_N)$. Then it is easy to obtain

$$\dot{\tilde{\boldsymbol{\xi}}}_1(t)=\begin{bmatrix}\boldsymbol{0}_n&\boldsymbol{I}_n\\\boldsymbol{0}_n&\boldsymbol{A}+\boldsymbol{BK}_1\end{bmatrix}\tilde{\boldsymbol{\xi}}_1(t),\tag{18}$$

$$\dot{\tilde{\boldsymbol{\xi}}}_i(t)=\begin{bmatrix}\boldsymbol{0}_n&\boldsymbol{I}_n\\-T_{\rm I}*(\lambda_i\boldsymbol{J}_{\tilde{\mathcal{L}}_i}^{-1}\boldsymbol{BK}_2)&\boldsymbol{J}_{\tilde{\mathcal{L}}_i}^{-1}(\boldsymbol{A}+\boldsymbol{BK}_1-T_{\rm p}\lambda_i\boldsymbol{BK}_2)\end{bmatrix}\tilde{\boldsymbol{\xi}}_i(t),\tag{19}$$
$$i=2,\cdots,N.$$

Let $\tilde{\boldsymbol{\xi}}(t)=\left[\tilde{\boldsymbol{\xi}}_1^{\rm T}(t),\tilde{\boldsymbol{\xi}}_2^{\rm T}(t),\cdots,\tilde{\boldsymbol{\xi}}_N^{\rm T}(t)\right]^{\rm T}$. According to Lemma 1, we can obtain

$$\tilde{\boldsymbol{\xi}}(t)=\boldsymbol{F}\begin{bmatrix}\tilde{\boldsymbol{x}}(t)\\\tilde{\boldsymbol{v}}(t)\end{bmatrix}.\tag{20}$$

Then it is easy to obtain

$$\begin{aligned}\dot{\tilde{\boldsymbol{\xi}}}(t)&=\boldsymbol{F}\begin{bmatrix}\dot{\tilde{\boldsymbol{x}}}(t)\\\dot{\tilde{\boldsymbol{v}}}(t)\end{bmatrix}=\boldsymbol{FJ}\begin{bmatrix}\tilde{\boldsymbol{x}}(t)\\\tilde{\boldsymbol{v}}(t)\end{bmatrix}\\&=\boldsymbol{FJF}^{-1}\tilde{\boldsymbol{\xi}}(t)=\boldsymbol{FJG}\tilde{\boldsymbol{\xi}}(t).\end{aligned}\tag{21}$$

Let $\tilde{\boldsymbol{J}}=\boldsymbol{FJG}$. Then we have

$$\tilde{\boldsymbol{J}}=\text{diag}(\tilde{\boldsymbol{J}}_1,\tilde{\boldsymbol{J}}_2,\cdots),\tag{22}$$

where $\tilde{\boldsymbol{J}}_1 = \begin{bmatrix} \boldsymbol{0}_n & \boldsymbol{I}_n \\ \boldsymbol{0}_n & \boldsymbol{A}+\boldsymbol{BK}_1 \end{bmatrix}$, $\tilde{\boldsymbol{J}}_i = \begin{bmatrix} \boldsymbol{0}_n & \boldsymbol{I}_n \\ -T_{\rm I}(\lambda_i \boldsymbol{J}_{\tilde{\boldsymbol{\mathcal{L}}}_i}{}^{-1}\boldsymbol{BK}_2) & \boldsymbol{J}_{\tilde{\boldsymbol{\mathcal{L}}}_i}^{-1}[\boldsymbol{A}+\boldsymbol{BK}_1 - T_{\rm p}\lambda_i \boldsymbol{BK}_2] \end{bmatrix}$, $i=2,3,\cdots,N$. Then we can obtain

$$\dot{\tilde{\boldsymbol{\xi}}}(t) = \tilde{\boldsymbol{J}}\tilde{\boldsymbol{\xi}}(t). \tag{23}$$

Through these transformations the consensus problem of Eq. (4) is converted into an asymptotic stability problem of Eq. (23). Furthermore, we can obtain

$$\begin{aligned}\dot{\tilde{\boldsymbol{\xi}}}(t) &= \boldsymbol{FJF}^{-1}\tilde{\boldsymbol{\xi}}(t) \\ &= \boldsymbol{FT}^{-1}\boldsymbol{HTF}^{-1}\tilde{\boldsymbol{\xi}}(t) \\ &= (\boldsymbol{FT}^{-1}\boldsymbol{F}^{-1})\boldsymbol{FHF}^{-1}(\boldsymbol{FTF}^{-1})\tilde{\boldsymbol{\xi}}(t) \\ &= (\boldsymbol{G}^{-1}\boldsymbol{T}^{-1}\boldsymbol{F}^{-1})\boldsymbol{FHF}^{-1}(\boldsymbol{FTG})\tilde{\boldsymbol{\xi}}(t) \\ &= (\boldsymbol{FTG})^{-1}(\boldsymbol{FHG})(\boldsymbol{FTG})\tilde{\boldsymbol{\xi}}(t).\end{aligned} \tag{24}$$

Let $\boldsymbol{V} = \boldsymbol{FTG}$. Then $\boldsymbol{V} = \boldsymbol{U}\otimes \boldsymbol{I}_{2n}$. So, we can obtain

$$\dot{\tilde{\boldsymbol{\xi}}}(t) = \boldsymbol{V}^{-1}(\boldsymbol{FHG})\boldsymbol{V}\tilde{\boldsymbol{\xi}}(t). \tag{25}$$

Let $\boldsymbol{V}=\boldsymbol{FTG}$, $\bar{\boldsymbol{H}}=\boldsymbol{FHG}$. It is easy to obtain

$$\begin{cases}\dot{\boldsymbol{\xi}}(t) = \bar{\boldsymbol{H}}\boldsymbol{\xi}(t), \\ \boldsymbol{\xi}_i(t) = \left[\boldsymbol{x}_i^{\rm T}(t), \boldsymbol{v}_i^{\rm T}(t)\right]^{\rm T}, \ i=1,2,\cdots,N.\end{cases} \tag{26}$$

Through Eq. (26) we can obtain another definition of consensus for system (4):

Definition 2 For any given initial $\boldsymbol{\xi}(0)$ which is bounded, if there exists a vector-valued function $\boldsymbol{c}(t)\in\mathbb{R}^{2n}$ related with $\boldsymbol{\xi}(0)$ to satisfy $\lim\limits_{t\to\infty}[\boldsymbol{\xi}(t) - \boldsymbol{1}\otimes\boldsymbol{c}(t)] = 0$, then system (4) reaches consensus and $\boldsymbol{c}(t)$ is called the 'consensus function'.

3.2 Main results

3.2.1 Consensus condition

Now we introduce two subspaces of $\mathbb{C}^{2Nn}$.

Definition 3 Let $\boldsymbol{u}_i$ be the column vectors of $\boldsymbol{U}$ and assume $\boldsymbol{c}(t)\in\mathbb{R}^{2n}$ and $k=1,2,\cdots,2n$ are linearly independent eigenvectors of $\tilde{\boldsymbol{J}}_i$, $i=1,2,\cdots,N$. Thus, we can know $\boldsymbol{p}_j = \boldsymbol{u}_i\otimes\boldsymbol{c}_k \in \mathbb{C}^{2Nn}$ $(j=2(i-1)n+k;\ i=1,2,\cdots,N;\ k=1,2,\cdots,2n)$ are linearly independent eigenvectors and generalized eigenvectors of $\bar{\boldsymbol{H}}$, where $\boldsymbol{p}_j$ $(j=1,2,\cdots,2n)$ and $\boldsymbol{p}_j$ $(j=2n+1,2n+2,\cdots,2Nn)$ are the corresponding eigenvectors of the eigenvalues of $\tilde{\boldsymbol{J}}_1$ and $\tilde{\boldsymbol{J}}_i$, respectively. Let $\boldsymbol{p} = [\boldsymbol{p}_1,\boldsymbol{p}_2,\cdots,\boldsymbol{p}_{2Nn}]\in\mathbb{C}^{2Nn\times 2Nn}$. Then $\boldsymbol{P}$ is a nonsingular matrix which satisfies $\boldsymbol{P}^{-1}\bar{\boldsymbol{H}}\boldsymbol{P} = \boldsymbol{J}_{\bar{\boldsymbol{H}}}$, where $\boldsymbol{J}_{\bar{\boldsymbol{H}}}$ is the Jordan form of $\bar{\boldsymbol{H}}$. The subspace spanned by $\boldsymbol{p}_j$ $(j=1,2,\cdots,2n)$ is called a uniform subspace and $\bar{\mathbb{C}}(\bar{\boldsymbol{H}})$ spanned by $\bar{\boldsymbol{H}}$, where $\boldsymbol{p}_j$ $(j=2n+1,2n+2,\cdots,2Nn)$, is called a uniformly complemented subspace. Since $\boldsymbol{P}$ is nonsingular, Lemma 2 holds.

Lemma 2 $\mathbb{C}(\bar{\boldsymbol{H}})\oplus\bar{\mathbb{C}}(\bar{\boldsymbol{H}}) = \mathbb{C}^{2Nn}$.

According to Lemma 2, there exist scalars $\alpha_j(t)$ $(j=1,2,\cdots,2Nn)$ satisfying

$$\boldsymbol{\xi}(t) = \boldsymbol{\xi}_C(t) + \boldsymbol{\xi}_{\bar{C}}(t), \tag{27}$$

where $\boldsymbol{\xi}_C(t) = \sum\limits_{j=1}^{2n}\alpha_j(t)\boldsymbol{p}_j$, $\boldsymbol{\xi}_{\bar{C}}(t) = \sum\limits_{j=2n+1}^{2Nn}\alpha_j(t)\boldsymbol{p}_j$. Obviously, $\boldsymbol{\xi}_C(t)$ and $\boldsymbol{\xi}_{\bar{C}}(t)$ are linearly independent.

Theorem 1 If the graph contains a spanning tree, for given $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$, system (4) achieves consensus if and only if $\tilde{\boldsymbol{J}}_i$ $(i=2,3,\cdots,N)$ is Hurwitz.

Proof According to the structure of $\boldsymbol{p}_j$ $(j=1,2,\cdots,2Nd)$, along with $\boldsymbol{u}_1=\boldsymbol{1}$, one can obtain

$$\boldsymbol{\xi}_C(t) = \sum_{j=1}^{2n}\alpha_j(t)\boldsymbol{p}_j = (\boldsymbol{U}\otimes\boldsymbol{I}_{2n})\Big(\boldsymbol{e}_1\otimes\sum_{j=1}^{2n}\alpha_j(t)\boldsymbol{c}_j\Big), \tag{28}$$

$$\begin{aligned}\boldsymbol{\xi}_{\bar{C}}(t) &= \sum_{j=2n+1}^{2Nn}\alpha_j(t)\boldsymbol{p}_j \\ &= (\boldsymbol{U}\otimes\boldsymbol{I}_{2n})\sum_{i=2}^{N}\sum_{k=1}^{2n}\alpha_{2(i-1)n+k}(t)(\boldsymbol{e}_i\otimes\boldsymbol{c}_k),\end{aligned} \tag{29}$$

where $\boldsymbol{e}_i$ is a vector with 1 as its ith element and 0 elsewhere. Since $\tilde{\boldsymbol{\xi}}(t) = \boldsymbol{V}^{-1}\boldsymbol{\xi}(t)$ and $\boldsymbol{\xi}(t) = \boldsymbol{\xi}_C(t) + \boldsymbol{\xi}_{\bar{C}}(t)$, we can obtain

$$\boldsymbol{\xi}_C(t) = (\boldsymbol{U}\otimes\boldsymbol{I}_{2n})[\tilde{\boldsymbol{\xi}}_1^{\rm T}(t), 0]^{\rm T} = \boldsymbol{1}\otimes\tilde{\boldsymbol{\xi}}_1(t), \tag{30}$$

$$\boldsymbol{\xi}_{\bar{C}}(t) = (\boldsymbol{U}\otimes\boldsymbol{I}_{2n})[\boldsymbol{0}, \bar{\boldsymbol{\xi}}^{\rm T}(t)]^{\rm T}, \tag{31}$$

where $\tilde{\boldsymbol{\xi}}_1(t) = \sum\limits_{j=1}^{2n}\alpha_j(t)\boldsymbol{c}_j$, $\bar{\boldsymbol{\xi}}^{\rm T}(t) = [\tilde{\boldsymbol{\xi}}_2^{\rm T}(t), \tilde{\boldsymbol{\xi}}_3^{\rm T}(t), \cdots, \tilde{\boldsymbol{\xi}}_N^{\rm T}(t)]^{\rm T} = \left[\sum\limits_{k=1}^{2n}\alpha_{2n+k}(t)c_k^{\rm T}, \sum\limits_{k=1}^{2n}\alpha_{2\times 2n+k}(t)c_k^{\rm T}, \cdots, \sum\limits_{k=1}^{2n}\alpha_{2(N-1)n+k}(t)\,c_k^{\rm T}\right]^{\rm T}$. According to Lemma 2, system (4) achieves consensus if and only if $\lim\limits_{t\to\infty}\boldsymbol{\xi}_{\bar{C}}(t) = 0$, namely $\lim\limits_{t\to\infty}\bar{\boldsymbol{\xi}}(t) = \boldsymbol{0}$. Thus, $\tilde{\boldsymbol{J}}_i$ $(i=2,3,\cdots,N)$ is Hurwitz.

The following theorem gives a kind of explicit expression of the consensus function:

Theorem 2 If system (4) can achieve consensus, so can system (26). Then consensus function $\boldsymbol{c}(t)$ satisfies

$$\lim_{t\to\infty}\Big(\boldsymbol{c}(t)-\mathrm{e}^{\tilde{\boldsymbol{J}}_1 t}[\boldsymbol{I}_{2n},\boldsymbol{0},\cdots,\boldsymbol{0}]\boldsymbol{P}_{\mathbb{C}(H),\bar{\mathbb{C}}(H)}\boldsymbol{\xi}(0)\Big)=\boldsymbol{0}, \tag{32}$$

where $\boldsymbol{P}_{\mathbb{C}(H),\bar{\mathbb{C}}(H)}=[\boldsymbol{p}_1,\boldsymbol{p}_2,\cdots,\boldsymbol{p}_{2n},\boldsymbol{0},\cdots,\boldsymbol{0}]\boldsymbol{P}^{-1}$ denotes the projection operator to space $\mathbb{C}(\bar{\boldsymbol{H}})$ along $\bar{\mathbb{C}}(\bar{\boldsymbol{H}})$.

Proof According to Lemma 2, the initial state vector $\boldsymbol{\xi}(0)$ can be resolved uniquely into $\boldsymbol{\xi}(0)=\boldsymbol{\xi}_C(0)+\boldsymbol{\xi}_{\bar{C}}(0)$, where $\boldsymbol{\xi}_C(0)\in\mathbb{C}(\bar{\boldsymbol{H}})$ and $\boldsymbol{\xi}_{\bar{C}}(0)\in\bar{\mathbb{C}}(\bar{\boldsymbol{H}})$. When system (26) reaches consensus, according to Lemma 2, we know that the system response activated by $\boldsymbol{\xi}_{\bar{C}}(0)$ should satisfy $\lim_{t\to\infty}\boldsymbol{\xi}_{\bar{C}}(t)=0$. Therefore, the consensus function is totally determined by $\boldsymbol{\xi}_C(0)$. Moreover, since $\mathbb{C}(\bar{\boldsymbol{H}})\oplus\bar{\mathbb{C}}(\bar{\boldsymbol{H}})=\mathbb{C}^{2Nn}$, there exists an oblique projector operator $\boldsymbol{P}_{\mathbb{C}(H),\bar{\mathbb{C}}(H)}=[\boldsymbol{p}_1,\boldsymbol{p}_2,\cdots,\boldsymbol{p}_{2n},\boldsymbol{0},\cdots,\boldsymbol{0}]\boldsymbol{P}^{-1}$ to make $\boldsymbol{\xi}_C(0)=\boldsymbol{P}_{\mathbb{C}(H),\bar{\mathbb{C}}(H)}\boldsymbol{\xi}(0)$. From equation $\boldsymbol{\xi}_C(t)=\boldsymbol{1}\otimes\tilde{\boldsymbol{\xi}}_1(t)$, we can obtain $\tilde{\boldsymbol{\xi}}_1(0)=[\boldsymbol{I}_{2n},\boldsymbol{0},\cdots,\boldsymbol{0}]\boldsymbol{P}_{\mathbb{C}(H),\bar{\mathbb{C}}(H)}\boldsymbol{\xi}(0)$. Thus, the solution of differential Eq. (18) is

$$\tilde{\boldsymbol{\xi}}_1(t)=\mathrm{e}^{\tilde{\boldsymbol{J}}_1}\tilde{\boldsymbol{\xi}}_1(0)=\mathrm{e}^{\tilde{\boldsymbol{J}}_1}[\boldsymbol{I}_{2n},\boldsymbol{0},\cdots,\boldsymbol{0}]\boldsymbol{P}_{\mathbb{C}(H),\bar{\mathbb{C}}(H)}\boldsymbol{\xi}(0). \tag{33}$$

As mentioned above, $\lim_{t\to\infty}(\boldsymbol{c}(t)-\tilde{\boldsymbol{\xi}}_1(t))=\boldsymbol{0}$. The proof is completed.

Theorem 1 gives the sufficient condition under which MAS can reach consensus. Next we will give the conditions that gain matrices $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ should satisfy to guarantee consensus.

3.2.2 Range of gain matrices

Since the communication topology in this study is directed, the associated Laplacian matrix $\boldsymbol{\mathcal{L}}$ is not symmetric and may have complex eigenvalues. Let $\lambda_i=\sigma_i+\mathrm{j}\,\omega_i$. According to the inverse matrix of a nonsingular complex matrix, one can obtain

$$\begin{aligned}\boldsymbol{J}_{\tilde{\boldsymbol{\mathcal{L}}}_i}^{-1}&=(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2+\mathrm{j}\,T_{\mathrm{D}}\omega_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\\&=[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)+\\&\quad T_{\mathrm{D}}^2\omega_i^2\boldsymbol{B}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\big]^{-1}\\&\quad-\mathrm{j}\,T_{\mathrm{D}}\omega_i(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\\&\quad\cdot[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)+\\&\quad T_{\mathrm{D}}^2\omega_i^2\boldsymbol{B}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\big]^{-1}.\end{aligned} \tag{34}$$

Thus, the complex form of $\tilde{\boldsymbol{I}}_i$ is as follows:

$$\tilde{\boldsymbol{J}}_i=\boldsymbol{\Theta}_{\boldsymbol{1}}+\mathrm{j}\,\boldsymbol{\Theta}_2, \tag{35}$$

where $\boldsymbol{\Theta}_{\boldsymbol{1}}$ and $\boldsymbol{\Theta}_{\boldsymbol{2}}$ are shown on the next page.

Through Eq. (35), matrix $\tilde{\boldsymbol{J}}_i$ is divided into a real part and an imaginary part. A complex matrix can be expressed by another square matrix; for example, $\boldsymbol{Z}=\boldsymbol{A}+\mathrm{j}\,\boldsymbol{B}$ can be reformed by $\hat{\boldsymbol{Z}}=\begin{pmatrix}\boldsymbol{A}&-\boldsymbol{B}\\\boldsymbol{B}&\boldsymbol{A}\end{pmatrix}$. Therefore, the eigenvalues of $\boldsymbol{Z}$ can be replaced by $\hat{\boldsymbol{Z}}$. Only when the real parts of the eigenvalues of $\tilde{\boldsymbol{J}}_i$ are negative, can system (4) reach consensus. System (19) can be transformed into the following differential form:

$$\dot{\hat{\tilde{\boldsymbol{\xi}}}}_i(t)=\begin{bmatrix}\boldsymbol{\Theta}_1&-\boldsymbol{\Theta}_2\\\boldsymbol{\Theta}_2&\boldsymbol{\Theta}_1\end{bmatrix}\hat{\tilde{\boldsymbol{\xi}}}_i(t),\ i=2,3,\cdots,N, \tag{36}$$

where $\hat{\tilde{\boldsymbol{\xi}}}_i(t)=[\mathrm{Re}(\tilde{\boldsymbol{\xi}}_i(t))^{\mathrm{T}},\ \mathrm{Im}(\tilde{\boldsymbol{\xi}}_i(t))^{\mathrm{T}}]^{\mathrm{T}}$. Then we can conclude that if system (36) is asymptotically stable, Theorem 1 holds.

The following theorem gives a sufficient condition for consensus and a method to determine the value of $\boldsymbol{K}_2$:

Theorem 3 If the graph contains a spanning tree, system (4) can reach consensus if there exists a positive-definite symmetric matrix $\boldsymbol{R}\in\mathbb{R}^{n\times n}$, so that $\boldsymbol{K}_2$ satisfies the following linear matrix inequality:

$$\boldsymbol{\Xi}_i=\begin{bmatrix}\boldsymbol{0}&\boldsymbol{\Xi}_{i12}&\boldsymbol{0}&\boldsymbol{\Xi}_{i14}\\ *&\boldsymbol{\Xi}_{i22}&\boldsymbol{\Xi}_{i23}&\boldsymbol{\Xi}_{i24}\\ *&*&\boldsymbol{0}&\boldsymbol{\Xi}_{i34}\\ *&*&*&\boldsymbol{\Xi}_{i44}\end{bmatrix}\prec 0,\ i=2,3,\cdots,N, \tag{37}$$

where each element is shown on page 1078. Then system (4) can achieve consensus under protocol (2).

Proof In terms of system (36), consider the following Lyapunov function:

$$\boldsymbol{V}_i(t)=\hat{\tilde{\boldsymbol{\xi}}}_i^{\mathrm{T}}(t)(\boldsymbol{I}_4\otimes\boldsymbol{R})\hat{\tilde{\boldsymbol{\xi}}}_i(t). \tag{38}$$

It is easy to find that if $\boldsymbol{R}$ is a positive-definite matrix, so is $\boldsymbol{I}_4\otimes\boldsymbol{R}$. Taking the time derivative of $\boldsymbol{V}_i(t)$

$$\boldsymbol{\Theta}_1=\begin{bmatrix}\boldsymbol{0}_n & \boldsymbol{I}_n\\ \begin{array}{c}-T_{\mathrm{I}}\sigma_i\Big[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{BK}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{BK}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{BK}_2)^{-1}\boldsymbol{BK}_2\Big]^{-1}\boldsymbol{BK}_2-T_{\mathrm{I}}T_{\mathrm{D}}\omega_i^2(\boldsymbol{I}_n\\ +T_{\mathrm{D}}\sigma_i\boldsymbol{BK}_2)^{-1}\boldsymbol{BK}_2\big[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{BK}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{BK}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{BK}_2)^{-1}\boldsymbol{BK}_2\big]^{-1}\boldsymbol{BK}_2\end{array} & \begin{array}{c}\Big[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{BK}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{BK}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{BK}_2)^{-1}\boldsymbol{BK}_2\Big]^{-1}(\boldsymbol{A}+\boldsymbol{BK}_1-T_{\mathrm{p}}\sigma_i\boldsymbol{BK}_2)-T_{\mathrm{D}}T_{\mathrm{p}}\omega_i^2(\boldsymbol{I}_n\\ +T_{\mathrm{D}}\sigma_i\boldsymbol{BK}_2)^{-1}\boldsymbol{BK}_2\big[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{BK}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{BK}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{BK}_2)^{-1}\boldsymbol{BK}_2\big]^{-1}\boldsymbol{BK}_2\end{array}\end{bmatrix}.$$

$$\boldsymbol{\Theta}_2=\begin{bmatrix}\boldsymbol{0}_n & \boldsymbol{0}_n\\ \begin{array}{c}T_{\mathrm{I}}T_{\mathrm{D}}\sigma_i\omega_i(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{BK}_2)^{-1}\boldsymbol{BK}_2\big[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{BK}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{BK}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{BK}_2)^{-1}\boldsymbol{BK}_2\big]^{-1}\boldsymbol{BK}_2\\ -T_{\mathrm{I}}\omega_i\Big[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{BK}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{BK}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{BK}_2)^{-1}\boldsymbol{BK}_2\Big]^{-1}\boldsymbol{BK}_2\end{array} & \begin{array}{c}-T_{\mathrm{p}}\omega_i\Big[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{BK}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{BK}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{K}_2)^{-1}\boldsymbol{BK}_2\Big]^{-1}\boldsymbol{BK}_2-T_{\mathrm{D}}\omega_i(\boldsymbol{I}_n\\ +T_{\mathrm{D}}\sigma_i\boldsymbol{BK}_2)^{-1}\boldsymbol{BK}_2\big[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{BK}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{BK}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{BK}_2)^{-1}\boldsymbol{BK}_2\big]^{-1}\\ \cdot(\boldsymbol{A}+\boldsymbol{BK}_1-T_{\mathrm{p}}\sigma_i\boldsymbol{BK}_2)\end{array}\end{bmatrix}.$$

along the trajectory of system (36), we have

$$\begin{aligned}\dot{\boldsymbol{V}}_i(t)&=\dot{\hat{\boldsymbol{\xi}}}_i^{\mathrm{T}}(t)(\boldsymbol{I}_4\otimes\boldsymbol{R})\hat{\boldsymbol{\xi}}_i(t)+\hat{\boldsymbol{\xi}}_i^{\mathrm{T}}(t)(\boldsymbol{I}_4\otimes\boldsymbol{R})\dot{\hat{\boldsymbol{\xi}}}_i(t)\\ &=\hat{\boldsymbol{\xi}}_i^{\mathrm{T}}(t)\begin{bmatrix}\boldsymbol{\Theta}_1^{\mathrm{T}} & \boldsymbol{\Theta}_2^{\mathrm{T}}\\ -\boldsymbol{\Theta}_2^{\mathrm{T}} & \boldsymbol{\Theta}_1^{\mathrm{T}}\end{bmatrix}(\boldsymbol{I}_4\otimes\boldsymbol{R})\hat{\boldsymbol{\xi}}_i(t)\\ &\quad+\hat{\boldsymbol{\xi}}_i^{\mathrm{T}}(t)(\boldsymbol{I}_4\otimes\boldsymbol{R})\begin{bmatrix}\boldsymbol{\Theta}_1 & -\boldsymbol{\Theta}_2\\ \boldsymbol{\Theta}_2 & \boldsymbol{\Theta}_1\end{bmatrix}\hat{\boldsymbol{\xi}}_i(t) &(39)\\ &=\hat{\boldsymbol{\xi}}_i^{\mathrm{T}}(t)\left(\begin{bmatrix}\boldsymbol{\Theta}_1^{\mathrm{T}} & \boldsymbol{\Theta}_2^{\mathrm{T}}\\ -\boldsymbol{\Theta}_2^{\mathrm{T}} & \boldsymbol{\Theta}_1^{\mathrm{T}}\end{bmatrix}(\boldsymbol{I}_4\otimes\boldsymbol{R}) \right. &(40)\\ &\quad\left.+(\boldsymbol{I}_4\otimes\boldsymbol{R})\begin{bmatrix}\boldsymbol{\Theta}_1 & -\boldsymbol{\Theta}_2\\ \boldsymbol{\Theta}_2 & \boldsymbol{\Theta}_1\end{bmatrix}\right)\hat{\boldsymbol{\xi}}_i(t).\end{aligned}$$

When inequality (37) holds, we have $\dot{\boldsymbol{V}}_i(t)<0$ $(i=2,3,\cdots,N)$. Substituting variables $\boldsymbol{\Theta}_1$ and $\boldsymbol{\Theta}_2$ into Eq. (39), we obtain the result. According to the Lyapunov-Krasovskii stability theorem, it is known that system (36) is asymptotically stable. Therefore, we infer that system (4) can reach consensus asymptotically if Theorem 2 holds.

According to Theorem 1, the consensus of the general linear MAS is simultaneously determined by the protocol, agent dynamics, and communication topology.

4 Simulation analysis

Simulations have been conducted to verify the effectiveness of the consensus protocol proposed. Consider an MAS with six nodes, whose communication topology is shown in Fig. 1. It can be seen that node 1 is the globally reachable node which can be the root node of the spanning tree.

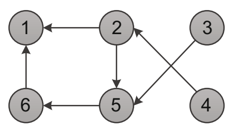

Fig. 1 Topological structure of a multi-agent system

Assume that system matrix $\boldsymbol{A}$ and input matrix $\boldsymbol{B}$ are

$$\boldsymbol{A}=\begin{bmatrix}0.2 & 0 & 0 & 0\\ 0.5 & 0.1 & 0.1 & -2.5\\ -1.5 & 0 & -0.4 & 0\\ -1.0 & 2.0 & 0 & 1.0\end{bmatrix},\ \boldsymbol{B}=\begin{bmatrix}0 & 1\\ 1 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}.$$

The initial states of each agent are $\boldsymbol{x}_1(0)=[15,-15,-10,-5]^{\mathrm{T}}$, $\boldsymbol{x}_2(0)=[10,0,-4,5]^{\mathrm{T}}$, $\boldsymbol{x}_3(0)=[5,-5,1,-15]^{\mathrm{T}}$, $\boldsymbol{x}_4(0)=[25,15,-4,-9]^{\mathrm{T}}$, $\boldsymbol{x}_5(0)=[-5,10,-8,15]^{\mathrm{T}}$, and $\boldsymbol{x}_6(0)=[-9,2,-10,-3]^{\mathrm{T}}$.

Example 1 This scenario is to verify the effectiveness of the proposed protocol and make comparisons with the control method based on the PD controller

$$\begin{aligned}\boldsymbol{\Xi}_{i12} =& -T_{\mathrm{I}}\sigma_i\boldsymbol{K}_2^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\left\{\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{B}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\right]^{-1}\right\}^{\mathrm{T}}\boldsymbol{R}\\ &- T_{\mathrm{I}}T_{\mathrm{D}}\omega_i^2\boldsymbol{K}_2^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\left\{\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{B}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\right]^{-1}\right\}^{\mathrm{T}}\\ &\cdot\boldsymbol{K}_2^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\right]^{\mathrm{T}}\boldsymbol{R}+\boldsymbol{R}.\end{aligned}$$

$$\begin{aligned}\boldsymbol{\Xi}_{i14} =& T_{\mathrm{I}}T_{\mathrm{D}}\sigma_i\omega_i\boldsymbol{K}_2^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\left\{\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{B}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\right]^{-1}\right\}^{\mathrm{T}}\\ &\cdot\boldsymbol{K}_2^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\right]^{\mathrm{T}}\boldsymbol{R}-T_{\mathrm{I}}\omega_i\boldsymbol{K}_2^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\Big\{[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)\\ &+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{B}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2]^{-1}\Big\}^{\mathrm{T}}\boldsymbol{R}.\end{aligned}$$

$$\begin{aligned}\boldsymbol{\Xi}_{i22} =& (\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}_1-T_{\mathrm{p}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{\mathrm{T}}\left\{\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{B}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\right]^{-1}\right\}^{\mathrm{T}}\boldsymbol{R}\\ &- T_{\mathrm{D}}T_{\mathrm{p}}\omega_i^2\boldsymbol{K}_2^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\left\{\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{B}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\right]^{-1}\right\}^{\mathrm{T}}\boldsymbol{K}_2^{\mathrm{T}}\\ &\cdot\boldsymbol{B}^{\mathrm{T}}\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\right]^{\mathrm{T}}\boldsymbol{R}+\boldsymbol{R}\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{B}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\right]^{-1}\\ &\cdot(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}_1-T_{\mathrm{p}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)-T_{\mathrm{D}}T_{\mathrm{p}}\omega_i^2\boldsymbol{K}(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\\ &\cdot\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{B}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\right]^{-1}\boldsymbol{B}\boldsymbol{K}_2.\end{aligned}$$

$$\begin{aligned}\boldsymbol{\Xi}_{i23} =& -T_{\mathrm{I}}T_{\mathrm{D}}\sigma_i\omega_i\boldsymbol{R}(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{B}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\right]^{-1}\\ &\cdot\boldsymbol{B}\boldsymbol{K}_2+T_{\mathrm{I}}\omega_i\boldsymbol{R}\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{B}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\right]^{-1}\boldsymbol{B}\boldsymbol{K}_2.\end{aligned}$$

$$\begin{aligned}\boldsymbol{\Xi}_{i24} =& -T_{\mathrm{p}}\omega_i\boldsymbol{K}_2^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\left\{\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{B}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\right]^{-1}\right\}^{\mathrm{T}}\boldsymbol{R}\\ &- T_{\mathrm{D}}\omega_i(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}_1-T_{\mathrm{p}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{\mathrm{T}}\left\{\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{B}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\right]^{-1}\right\}^{\mathrm{T}}\\ &\cdot\boldsymbol{K}_2^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\right]^{\mathrm{T}}\boldsymbol{R}+T_{\mathrm{p}}\omega_i\boldsymbol{R}\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{R}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\right]^{-1}\\ &\cdot\boldsymbol{B}\boldsymbol{K}_2+T_{\mathrm{D}}\omega_i\boldsymbol{K}(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{B}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\right]^{-1}\\ &\cdot(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}_1-T_{\mathrm{p}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2).\end{aligned}$$

$$\begin{aligned}\boldsymbol{\Xi}_{i34} =& -T_{\mathrm{I}}\sigma_i\boldsymbol{K}_2^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\left\{\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{B}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\right]^{-1}\right\}^{\mathrm{T}}\boldsymbol{R}\\ &- T_{\mathrm{I}}T_{\mathrm{D}}\omega_i^2\boldsymbol{K}_2^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\left\{\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{B}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\right]^{-1}\right\}^{\mathrm{T}}\boldsymbol{K}_2^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\\ &\cdot\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\right]^{\mathrm{T}}\boldsymbol{R}+\boldsymbol{R}.\end{aligned}$$

$$\begin{aligned}\boldsymbol{\Xi}_{i44} =& (\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}_1-T_{\mathrm{p}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{\mathrm{T}}\left\{\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{B}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\right]^{-1}\right\}^{\mathrm{T}}\boldsymbol{R}\\ &- T_{\mathrm{D}}T_{\mathrm{p}}\omega_i^2\boldsymbol{K}_2^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\left\{\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{B}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\right]^{-1}\right\}^{\mathrm{T}}\boldsymbol{K}_2^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\\ &\cdot\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\right]^{\mathrm{T}}\boldsymbol{R}+\boldsymbol{R}\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{B}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\right]^{-1}\\ &\cdot(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}_1-T_{\mathrm{p}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)-T_{\mathrm{D}}T_{\mathrm{p}}\omega_i^2\boldsymbol{K}(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\\ &\cdot\left[(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)+T_{\mathrm{D}}^2\omega_i^2\boldsymbol{B}\boldsymbol{K}_2(\boldsymbol{I}_n+T_{\mathrm{D}}\sigma_i\boldsymbol{B}\boldsymbol{K}_2)^{-1}\boldsymbol{B}\boldsymbol{K}_2\right]^{-1}\boldsymbol{B}\boldsymbol{K}_2.\end{aligned}$$

proposed by Ji and Liao (2013). Yang *et al.* (2014) proposed an optimal algorithm based on a genetic algorithm to obtain the optimal PID parameters. The optimization indices include rise time, output energy of controllers, and tracking error. Since there are too many parameters to be determined in the control system, it is not very meaningful to calculate the optimal PID selection. According to the approach proposed by Lin *et al.* (2015), we choose proper PID parameters by considering both adjusting time and vibration amplitude. Then the parameters of the PID are chosen as $T_{\rm P} = 0.8516$, $T_{\rm I} = 0.4998$, and $T_{\rm D} = 0.1250$. In another experiment the same proportional parameter and derivative parameter will be used. Through solving inequality (37), we can choose a feasible gain matrix $\boldsymbol{K}_2$. Then we obtain

$$\boldsymbol{K}_1 = \begin{bmatrix} 4.7158 & -0.2303 & -2.9133 & -4.6176 \\ -4.2423 & -0.6593 & -1.4864 & -3.5140 \end{bmatrix},$$

$$\boldsymbol{K}_2 = \begin{bmatrix} -3.7227 & 3.0269 & -0.1578 & 3.0844 \\ 0.5896 & -0.8861 & 0.2478 & -0.0932 \end{bmatrix}.$$

To illustrate the effect of the comparison accurately, we employ the norm of the state error of MAS as the index of judgment. Assume that $e_k(t) = 0.2\sum_{i=2}^{6} \|\boldsymbol{x}_{1_k} - \boldsymbol{x}_{i_k}\|$ $(k = 1, 2, 3, 4)$ and $\boldsymbol{E}(t) = [e_1(t), e_2(t), e_3(t), e_4(t)]$. The simulation results are shown in Figs. 2–5.

The state curves are shown in Figs. 2 and 3 while the state errors are shown in Figs. 4 and 5. The consensus function is described by the trajectory marked by circles. We can see that none of the agent states converge to zero because the MAS is not stable, but all the agents reach consensus under the control. From Figs. 2 and 3 we can see that both consensus algorithms can guarantee the consensus of MAS; however, distributed PID control can eliminate the state error of the MAS. As can be seen from Figs. 4 and 5, PID control can achieve consensus in a shorter time. More importantly, it can eliminate the

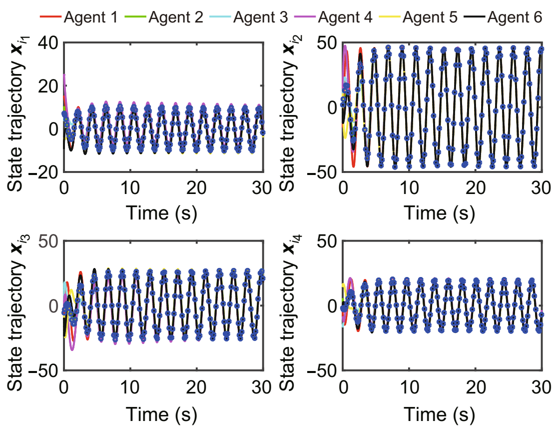

Fig. 2 State trajectories of the multi-agent system under PD control ($i = 1, 2, \ldots, 6$) (References to color refer to the online version of this figure)

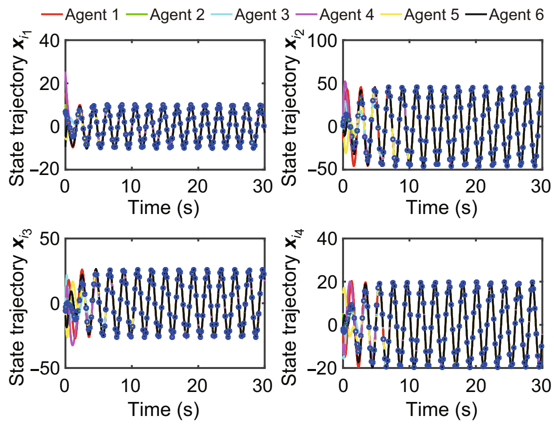

Fig. 3 State trajectories of the multi-agent system under PID control ($i = 1, 2, \ldots, 6$) (References to color refer to the online version of this figure)

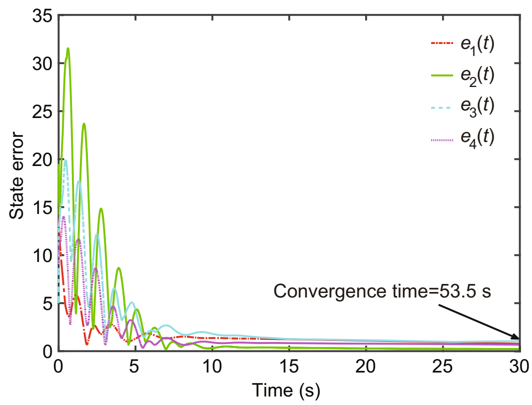

Fig. 4 State errors of the multi-agent system under PD control

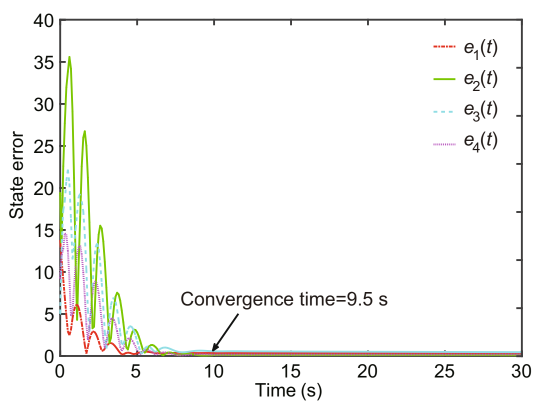

Fig. 5 State errors of the multi-agent system under PID control

steady error when all the agents reach a continuous wave.

Example 2 In the second experiment, the presence of disturbance is considered. Assume that the noise is $\boldsymbol{\omega}(t) = 10\,[\eta(t),\ \ \eta(t),\ \ \eta(t),\ \ \eta(t)]^{\mathrm{T}}$, where $\eta(t)$ is the white noise. The parameters are chosen as $T_{\mathrm{P}} = 10$, $T_{\mathrm{I}} = 0.4$, and $T_{\mathrm{D}} = 0.5$ by the former method. According to Theorem 3, $\boldsymbol{K}_2$ can keep unchanged. The results are shown in Fig. 6.

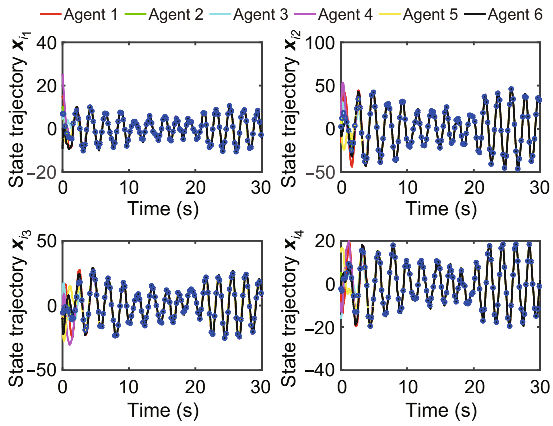

Fig. 6 State trajectories of the multi-agent system with disturbance under PID control ($i = 1, 2, \ldots, 6$) (References to color refer to the online version of this figure)

Obviously, the consensus algorithm proposed in this study can still achieve consensus with the presence of disturbance according to Figs. 6 and 7. Comparing Fig. 3 with Fig. 7, we can see that the noise has a bad influence on the vibration of the states of the MAS. The convergence speed of the MAS is also changed. After all, the consensus protocol based on the PID controller can achieve consensus with or without the presence of disturbance.

5 Conclusions

In this paper, we have proposed a novel consensus protocol for the general linear MAS under directed topology. Through transformations, the consensus problem was converted into an asymptotical stability problem of multiple low-dimensional subsystems, and the consensus function was given. Based on the graph theory, matrix theory, and Lyapunov stability theorem, the consensus was proved effective. Simulation results have shown that the consensus of the MAS is related to the topology property, gain matrices, and parameters

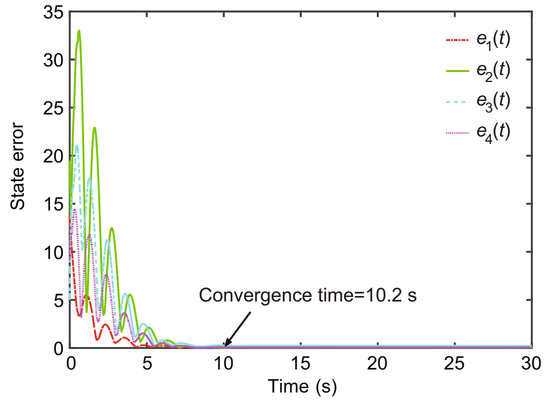

Fig. 7 State errors of the multi-agent system with disturbance under PID control

of PID control. Since the research object in this study is a general model, the proposed protocol has great representativeness. In the future, we will take communication delays and switched topology into consideration and conduct further research.

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