Frontiers of Information Technology & Electronic Engineering
www.zju.edu.cn/jzus; engineering.cae.cn; www.springerlink.com
ISSN 2095-9184 (print); ISSN 2095-9230 (online)
E-mail: jzus@zju.edu.cn

# New computational treatment of optical wave propagation in lossy waveguides*

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Received Nov. 27, 2014; Revision accepted Apr. 18, 2015; Crosschecked July 8, 2015

**Abstract:** In this paper, the optical wave propagation in lossy waveguides is described by the Helmholtz equation with the complex refractive-index, and the Chebyshev pseudospectral method is used to discretize the transverse operator of the equation. Meanwhile, an operator marching method, a one-way re-formulation based on the Dirichlet-to-Neumann (DtN) map, is improved to solve the equation. Numerical examples show that our treatment is more efficient.

**Key words:** Adjoint operator, Orthogonal, Chebyshev, Pseudospectral method, Dirichlet-to-Neumann map
**doi:**10.1631/FITEE.1400406 **Document code:** A **CLC number:** O42

## 1 Introduction

It is known that high-precision computation of the optical wave propagation in an optical waveguide is very helpful to the optimal design of optoelectronic devices (Vassallo, 1991; März, 1995; Silva *et al.*, 2014). Its mathematical model is considered as a variable coefficient Helmholtz equation. The transverse length scale is typically much smaller than the propagation distance, but still much larger than the characteristic wavelength. Some standard numerical methods, such as finite difference (FD) and finite element (FE) methods, give rise to very large systems since a certain number of grid points (or basis functions) are needed for each wavelength (Lu and Zhu, 2004; Zhu and Lu, 2004). These linear systems are difficult to solve even by iterative methods, since they are nonsymmetric and indefinite. Moreover, the discretized problems require a very large computer memory.

For structures like these, one-way methods are widely used to obtain approximate solutions quickly. However, since one-way methods solve a parabolic equation which is an approximation of the original Helmholtz equation, there is an error that cannot be eliminated. To overcome this weakness, the operator marching method (OMM), a one-way re-formulation based on the Dirichlet-to-Neumann (DtN) map, was developed (Lu and McLaughlin, 1996; Lu, 1999). The DtN map reduces the boundary value problem to an initial value problem which is exactly equivalent to the original Helmholtz equation, and then a marching scheme is used to solve the resulting operator equation.

In OMM, there is a local base transformation to be done in each marching step by searching for a coordinate matrix related to the characteristic problem. For the lossless waveguide (the refractive index is real), the eigenfunctions of the characteristic problem form an orthogonal basis. The local base transformation can be implemented easily. However, for a waveguide with loss (the refractive index is complex), such as the waveguide with an iron core, the orthogonal property of eigenfunctions disappears since

* Project supported by the National Natural Science Foundation of China (No. 11371319) and the Zhejiang Provincial Natural Science Foundation of China (No. LY13A010002)
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the characteristic operator of the Helmholtz equation is not self-adjoint. This leads to the difficulty of the local base transformation in OMM.

Zhu and Song (2009) proposed the adjoint operator method and the complex symmetric matrix method to overcome the difficulty. However, they used a uniform grid to discretize the transverse operator of the Helmholtz equation. This leads to low efficiency and places a restriction on the accuracy. In this paper, a new treatment is given to deal with the local base transform for lossy waveguides. Obviously, the complex symmetric matrix method is a special case of the new method. Furthermore, we will use the Chebyshev pseudospectral method to discretize the transverse operator, and this is much better than the FD method.

Some numerical examples are demonstrated and compared with the results computed by the complex symmetric matrix method. The details can be found in the following sections.

## 2 Background

In this section, we firstly present the mathematical foundation of the lossy waveguide (Fig. 1). Then the characteristic problem of the Helmholtz equation is introduced.

Consider a 2D Helmholtz equation, i.e.,

$$u_{xx} + u_{zz} + \kappa^2(x, z)u = 0, \tag{1}$$

with the boundary conditions (BCs) in the $x$-direction:

$$u(0, z) = u_0(z), \tag{2}$$

$$u_x(L, z) = \mathrm{i}\sqrt{\partial_z^2 + \kappa^2(L, z)}\,u(L, z), \tag{3}$$

where $u_0(z)$ is a given incident wave, $L$ is the propagation distance, $\kappa = \kappa_0 n(x, z)$, and $\mathrm{i} = \sqrt{-1}$. Here, $\kappa_0$ and $n(x, z)$ represent the wavenumber in vacuum and the refractive index of the waveguide, respectively. The square root operator

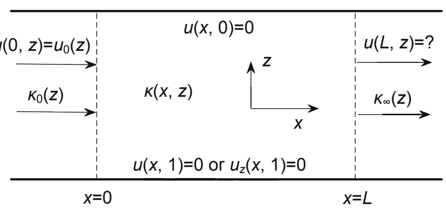


**Fig. 1 The problem sketch**

of $\sqrt{\partial_z^2 + \kappa^2(L, z)}$ has been well defined in Lu and McLaughlin (1996).

Two types of BCs in the $z$-direction are studied. One is

$$u(x, 0) = 0, \quad u(x, 1) = 0, \tag{4}$$

and the other is

$$u(x, 0) = 0, \quad u_z(x, 1) = 0. \tag{5}$$

Define the transverse operator of the Helmholtz equation at a fixed point $x$ as

$$D(x) = \partial_z^2 + \kappa^2(x, z). \tag{6}$$

Then the characteristic problem of Eq. (1) is

$$D(x)\phi(z) = \lambda\phi(z), \quad 0 \leq z \leq 1, \tag{7}$$

with BCs

$$\phi(0) = 0, \quad \phi(1) = 0, \tag{8}$$

corresponding to BCs (4), or

$$\phi(0) = 0, \quad \phi_z(1) = 0, \tag{9}$$

corresponding to BCs (5), where $\lambda$ is the eigenvalue in the characteristic problem.

The square root $D^{\frac{1}{2}}(x) = \sqrt{\partial_z^2 + \kappa^2(x, z)}$ is defined to satisfy

$$D^{\frac{1}{2}}(x)\phi(z) = \sqrt{\lambda}\phi(z), \quad 0 \leq z \leq 1, \tag{10}$$

where $\lambda$ and $\phi(z)$ are the same as in Eq. (7). The branch of $\sqrt{\lambda}$ is chosen by $(-\pi, \pi]$.

The characteristic problem is very important in optics, since it is the mathematical foundation of modal analysis. What is more, the propagation problems are based on the function expansion in the series of eigenfunctions.

If the refractive index is real, then the eigenfunctions of $D(x)$ will form a complete and orthogonal basis corresponding to the inner product

$$(f, g) = \int_0^1 g^* f \mathrm{d}z, \tag{11}$$

where $^*$ means conjugate. However, if the refractive index is complex, then the eigenfunctions lose the orthogonality since the operator is not self-adjoint.

## 3 Two methods for local transform

The local base transform in OMM needs to be done in each marching step by searching for a coordinate matrix $\boldsymbol{N}$ satisfying $\boldsymbol{V}_1 = \boldsymbol{V}_0\boldsymbol{N}$, where $\boldsymbol{V}_0$ and $\boldsymbol{V}_1$ are known eigenvector matrices of $\boldsymbol{D}(x)$ at different locations. When $\boldsymbol{D}(x)$ is Hermitian, $\boldsymbol{N}$ can be easily obtained by $\boldsymbol{N} = \boldsymbol{V}_0^{\mathrm{H}}\boldsymbol{V}_1$ (the superscript H means conjugate transpose). Hence, the local base transform is easily done for the lossless waveguides, where $\boldsymbol{D}(x)$ is self-adjoint. For the lossy waveguides, $\boldsymbol{D}(x)$ is non-self-adjoint and the eigenfunctions of $\boldsymbol{D}(x)$ are not orthogonal. It makes the local base transform in OMM fail. The most intuitive treatment is to find the inverse matrix of $\boldsymbol{v}_0$, i.e., $\boldsymbol{N} = \boldsymbol{V}_0^{-1}\boldsymbol{V}_1$. It is time-consuming and unstable. However, to address this, two techniques, namely the conjugate differentiation matrix method and the adjoint operator method, are described in this section.

### 3.1 Conjugate differentiation matrix method

Zhu and Song (2009) proposed the complex symmetric matrix method to deal with the local base transformation for the symmetric differentiation matrix. Denote the differentiation matrix of $\boldsymbol{D}(x)$ as $\boldsymbol{M}$. If $\boldsymbol{D}^{\mathrm{T}} = \boldsymbol{M}$, then $\boldsymbol{M}$ is diagonalizable if and only if there is an eigenvector matrix $\boldsymbol{V}$ such that

$$\boldsymbol{V}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{V} = \boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n), \tag{12}$$

and $\boldsymbol{V}^{\mathrm{T}}\boldsymbol{V} = \boldsymbol{I}$, where $\boldsymbol{I}$ is an $n \times n$ identity matrix. This property can be used in the local base transformation easily if the differentiation matrix is symmetric. However, for many efficient numerical methods, such as the Chebyshev pseudospectral method, the matrix which is used to approximate the transverse operator is not symmetric. To overcome this difficulty, we propose the conjugate differentiation matrix method here.

**Theorem 1** If matrix $\boldsymbol{M}$ is diagonalizable, suppose $\boldsymbol{V}$ is the eigenvector matrix of $\boldsymbol{M}$, and $\widehat{\boldsymbol{V}}$ is the eigenvector matrix of $\boldsymbol{M}^{\mathrm{H}}$. Then

$$\widehat{\boldsymbol{V}}^{\mathrm{H}}\boldsymbol{V} = \boldsymbol{I}, \tag{13}$$

where $\boldsymbol{M}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{\Lambda}$, $\boldsymbol{M}^{\mathrm{H}}\widehat{\boldsymbol{V}} = \widehat{\boldsymbol{V}}\boldsymbol{\Lambda}^{\mathrm{H}}$, and $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

**Proof** It is easy to verify that $\boldsymbol{M}^{\mathrm{H}}$ is the adjoint operator of $\boldsymbol{M}$ in the Hilbert space with the inner product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{y}^{\mathrm{H}}\boldsymbol{x}. \tag{14}$$

Thus,

$$\langle \boldsymbol{V}(i,:), \widehat{\boldsymbol{V}}(j,:) \rangle = 0, \quad i \neq j. \tag{15}$$

This turns out as Eq. (13).

**Remark 1** The complex symmetric matrix method is a special case of our method.

### 3.2 Adjoint operator method

The adjoint operator method was introduced by Zhu and Song (2009). Suppose $\{\lambda_i, \phi_i(z)\}$ and $\{\lambda_j^*, \psi_j(z)\}$ are the eigen-pairs of $D(x)$ and its adjoint operator $G(x)$, respectively. Then

$$(\phi_i(z), \psi_j(z)) = \int_0^1 \psi_j^*(z)\phi_i(z)\mathrm{d}z = 0, \quad i \neq j. \tag{16}$$

The bi-orthogonal property corresponds to the eigenfunctions of $D(x)$ and $G(x)$ rather than the differentiation matrix. However, for the uniform grid, this property is useless since there is no high-precision quadrature. Thus, to take advantage of this property, an efficient quadrature is needed. We use the Chebyshev points of the second kind (Trefethen, 2013) to discretize the operators, and correspondingly, Clenshaw-Curtis quadrature is used to calculate the following integral:

$$\int_0^1 \psi_j^*(z)\phi_i(z)\mathrm{d}z \approx \sum_{k=0}^{n} \psi_j^*(z_k)w_k\phi_i(z_k), \tag{17}$$

where $\{w_k\}_{k=0}^n$ are the weights of Clenshaw-Curties quadrature, and $\{z_k\}_{k=0}^n$ are the Chebyshev points of the second kind. The weights $\{w_k\}$ can be computed efficiently with time complexity $O(n\log n)$ by the fast Fourier transform (FFT) (Waldvogel, 2006). We do not need to compute $\psi_i(z)$ since

$$G(x)\overline{v}(z) = \overline{D(x)v(z)}, \tag{18}$$

which means $\psi_i(z) = \overline{\phi}(z)$ (overline represents conjugate operation). By the property (16), we find that

$$\boldsymbol{V}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{V} = \boldsymbol{I}, \tag{19}$$

where $\boldsymbol{W} = \mathrm{diag}(w_0, w_1, \ldots, w_n)$.

**Remark 2** The complex symmetric matrix method is a special case of the adjoint operator method since the quadrature weights for the uniform grid are identical.

# 4 Numerical methods

By the forward preparations, new formulas can be obtained with a small modification of OMM (Lu, 1999). We also take segment $(x_0, x_1)$ as an example. The marching formulas indicate the relation of operators from $x_1$ to $x_0$.

## 4.1 Finite difference method

For Eqs. (7) and (8), variable $z$ is discretized by

$$z_j = j\delta,\quad \delta = 1/n,\quad j = 1, 2, \ldots, n-1 \tag{20}$$

for the finite difference method. Then the differential operator $\partial_z^2$ is approximated by

$$\boldsymbol{A} = \frac{1}{\delta^2}\begin{pmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & -2 & 1 \\ & & 1 & -2 \end{pmatrix}. \tag{21}$$

For Eqs. (7) and (9), variable $z$ is discretized by

$$z_j = j\delta,\quad \delta = 1/(n-0.5),\quad j = 1, 2, \ldots, n-1 \tag{22}$$

for the finite difference method. Then the differential operator $\partial_z^2$ is approximated by

$$\boldsymbol{A} = \frac{1}{\delta^2}\begin{pmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & -2 & 1 \\ & & 1 & -1 \end{pmatrix}. \tag{23}$$

## 4.2 Chebyshev pseudospectral method

The idea behind the Chebyshev pseudospectral method is that the derivation of the Chebyshev interpolation is an approximation for deriving the interpolated function. The derivative operator on a polynomial space is a linear transformation between finite-dimensional spaces, and therefore can be represented by a matrix (differentiation matrix). Denote the differentiation matrix of the first and the second order in domain $[a, b]$ as $\boldsymbol{D}_1 = (d_{ij})_{(n+1)\times(n+1)}$ and $\boldsymbol{D}_2 = \boldsymbol{D}_1^2$. Then

$$\frac{\partial^2}{\partial z^2}\begin{bmatrix} \phi(z_0) \\ \phi(z_1) \\ \vdots \\ \phi(z_n) \end{bmatrix} \approx \boldsymbol{D}_2 \begin{bmatrix} \phi(z_0) \\ \phi(z_1) \\ \vdots \\ \phi(z_n) \end{bmatrix}, \tag{24}$$

where

$$z_i = a + \frac{b-a}{2}(1 - \cos(i\pi/n)),\quad i = 0, 1, \cdots, n. \tag{25}$$

Let the rows and columns of $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ be indexed from 0 to $n$. Then the entries of $\boldsymbol{D}_1$ are (Trefethen, 2000)

$$d_{ij} = \frac{-2}{b-a}\begin{cases} \dfrac{2n^2+1}{6}, & i = j = 0, \\ \dfrac{c_i}{c_j}\dfrac{(-1)^{i+j}}{(y_i - y_j)}, & i \neq j, \\ \dfrac{-y_i}{2(1-y_i)}, & 1 \leq i = j \leq n-1, \\ -\dfrac{2n^2+1}{6}, & i = j = n, \end{cases} \tag{26}$$

where $y_i = \cos(i\pi/n)$ and $c_i = 2$ for $i = 0$ or $n$, and otherwise $c_i = 1$.

For Eqs. (7) and (8), variable $z$ is discretized by

$$z_i = (1 - \cos(i\pi/n))/2,\quad i = 0, 1, \cdots, n \tag{27}$$

for the Chebyshev pseudospectral method. Then the differential operator $\partial_z^2$ is approximated by

$$\boldsymbol{A} = \boldsymbol{D}_2(1:n-1, 1:n-1). \tag{28}$$

For Eqs. (7) and (8), the grid is just the same as in the first case. The treatment of BCs is a little more complex than that of the Dirichlet BCs. By computation, the differential operator $\partial_z^2$ is approximated by

$$\begin{aligned} \boldsymbol{A} =& \boldsymbol{D}_2(1:n-1, 1:n-1) \\ &- \frac{1}{\boldsymbol{D}_1(n,n)}\boldsymbol{D}_2(1:n-1, n)\boldsymbol{D}_1(n, 1:n-1). \end{aligned} \tag{29}$$

## 4.3 Modified operator marching method

The transverse operator $D(x_{1/2}) = \partial_z^2 + \kappa^2(x_{1/2}, z)$ is approximated by

$$\boldsymbol{M} = \boldsymbol{A} + \mathrm{diag}(\kappa_1^2, \kappa_2^2, \ldots, \kappa_{n-1}^2), \tag{30}$$

where $\boldsymbol{A}$ is an $(n-1)\times(n-1)$ matrix mentioned above, and $\kappa_i = \kappa(x_{1/2}, z_i)$.

Denote the differentiation matrix of $D(x_{1/2})$ and $D(x_{3/2})$ as $\boldsymbol{M}_0$ and $\boldsymbol{M}_1$, respectively. Their truncated local eigenvector decompositions are

$$\boldsymbol{M}_0\boldsymbol{V}_{0m} = \boldsymbol{V}_{0m}\boldsymbol{\Lambda}_{0m},\quad \boldsymbol{M}_1\boldsymbol{V}_{1m} = \boldsymbol{V}_{1m}\boldsymbol{\Lambda}_{1m}, \tag{31}$$

where $\boldsymbol{V}_{im}$ $(i = 0, 1)$ are the $(n-1) \times m$ eigenvector matrices and $\boldsymbol{\Lambda}_{im}$ are the $m \times m$ diagonal eigenvalue matrices.

For the conjugate differentiation matrix method, we need to find out the two decompositions of $\boldsymbol{M}_0^{\mathrm{H}}$ and $\boldsymbol{M}_1^{\mathrm{H}}$. Suppose

$$\boldsymbol{M}_0^{\mathrm{H}}\widehat{\boldsymbol{V}}_{0m} = \widehat{\boldsymbol{V}}_{0m}\boldsymbol{\Lambda}_{0m}^{\mathrm{H}}, \quad \boldsymbol{M}_1^{\mathrm{H}}\widehat{\boldsymbol{V}}_{1m} = \widehat{\boldsymbol{V}}_{1m}\boldsymbol{\Lambda}_{1m}^{\mathrm{H}}. \tag{32}$$

We have

$$\widehat{\boldsymbol{V}}_{im}^{\mathrm{H}}\boldsymbol{V}_{im} = \boldsymbol{I}, \quad i = 0, 1. \tag{33}$$

The local base transform can be done by the conjugate differentiation matrix method as

$$\boldsymbol{N} = \widehat{\boldsymbol{V}}_{0m}^{\mathrm{H}}\boldsymbol{V}_{1m}. \tag{34}$$

Alternatively, the local base transform can be done by the adjoint operator method.

For BCs (4),

$$\boldsymbol{N} = \boldsymbol{V}_{0m}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{V}_{1m}, \tag{35}$$

where $\boldsymbol{W} = \mathrm{diag}(w_1, w_2, \ldots, w_{n-1})$.

For BCs (5), Eq. (35) is still applied; however, $\boldsymbol{W}$ should be a minor modification. With a view to

$$\phi(z_n) \approx -\frac{\boldsymbol{R}}{\boldsymbol{D}_1(n,n)}(\phi(z_1), \phi(z_2), \ldots, \phi(z_{n-1}))^{\mathrm{T}}, \tag{36}$$

we have

$$\boldsymbol{W} = \mathrm{diag}(w_1, w_2, \ldots, w_{n-1}) + \frac{\boldsymbol{R}^{\mathrm{T}}\boldsymbol{R}}{\boldsymbol{D}_1^2(n,n)}, \tag{37}$$

where $\boldsymbol{D}_1$ is a Chebyshev differentiation matrix of the first order, and $\boldsymbol{R} = \boldsymbol{D}_1(n, 1:n-1)$.

Then the marching formulas from $x_1$ to $x_0$ in matrix form (Lu, 1999; Zhu and Song, 2009) are

$$\boldsymbol{S} = \boldsymbol{N}\boldsymbol{S}_1\boldsymbol{N}^{-1}, \tag{38}$$

$$\boldsymbol{P}_1 = (\mathrm{i}\sqrt{\boldsymbol{\Lambda}_{0m}} + \boldsymbol{S})^{-1}(\mathrm{i}\sqrt{\boldsymbol{\Lambda}_{0m}} - \boldsymbol{S}), \tag{39}$$

$$\boldsymbol{P}_0 = \mathrm{e}^{\mathrm{i}h\sqrt{\boldsymbol{\Lambda}_{0m}}}\boldsymbol{P}_1\mathrm{e}^{\mathrm{i}h\sqrt{\boldsymbol{\Lambda}_{0m}}}, \tag{40}$$

$$\boldsymbol{U} = (\boldsymbol{I} - \boldsymbol{P}_0)(\boldsymbol{I} + \boldsymbol{P}_0)^{-1}, \tag{41}$$

$$\boldsymbol{S}_0 = \mathrm{i}\sqrt{\boldsymbol{\Lambda}_{0m}}\boldsymbol{U}, \tag{42}$$

$$\boldsymbol{Z}_0 = \boldsymbol{N}\boldsymbol{Z}_1\boldsymbol{N}^{-1}(\boldsymbol{I} + \boldsymbol{P}_1)\mathrm{e}^{\mathrm{i}h\sqrt{\boldsymbol{\Lambda}_{0m}}}(\boldsymbol{I} + \boldsymbol{P}_0)^{-1}, \tag{43}$$

where $\boldsymbol{S}_1$ and $\boldsymbol{Z}_1$ are known at $x_1$, and $h = x_1 - x_0$. By the marching formulas, $\boldsymbol{S}_0$ and $\boldsymbol{Z}_0$ at $x_0$ are obtained. The same process from $x = L$ to $x = 0$ can be done step by step. $\boldsymbol{S}_1$ and $\boldsymbol{Z}_1$ at $x = L$ are treated as the initial values, where $\boldsymbol{Z}_1 = \boldsymbol{I}$ and $\boldsymbol{S}_1 = \boldsymbol{\Lambda}_{Lm}$. Then, the wave field at $x = L$ is solved by

$$\boldsymbol{u} = \boldsymbol{V}_{0m}\boldsymbol{Z}_0\widehat{\boldsymbol{V}}_{0m}^{\mathrm{H}}\boldsymbol{u}_0 \tag{44}$$

or

$$\boldsymbol{u} = \boldsymbol{V}_{0m}\boldsymbol{Z}_0\boldsymbol{V}_{0m}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{u}_0 \tag{45}$$

for the conjugate differentiation matrix method or the adjoint operator marching method, respectively.

## 5 Numerical results and discussion

In this section, several examples are solved to show the performance of the Chebyshev pseudospectral method. The equations are solved by Eqs. (38)–(43) together with Eq. (34). It is noticed that the results of Eqs. (38)–(43) together with Eq. (35) are very similar. Thus, they are omitted for conciseness. Problem (1) with BCs (4) and with BCs (5) are denoted as cases 1 and 2, respectively. For each case, both homogeneous and inhomogeneous media are considered. The homogeneous media are used to verify the validity of our method, because the exact solutions can be obtained. The Chebyshev differentiation matrices $\boldsymbol{D}_1$, $\boldsymbol{D}_2$ and the weights $w_i$ for the quadrature are obtained by the toolbox Chebfun (http://www.chebfun.org).

For the homogeneous media, let

$$\kappa^2(x,z) = (1 + \alpha\mathrm{i})\kappa_0^2,$$

and for the inhomogeneous media, let

$$\kappa^2(x,z) = (1+\alpha\mathrm{i})\kappa_0^2[1+0.05\mathrm{e}^{-20(x/L-0.5)^2}\sin^2(\pi z)],$$

where $\kappa_0 = 10$, and $\alpha$ is an adjustable factor related to the absorbing strength. In the following examples, unless otherwise stated, we take $\alpha = 0.01$ only, but for other values, the results are similar. The propagation distance $L$ is 10 μm for both cases, where the units of $x$ and $z$ are the same.

### 5.1 Homogeneous media

First, we consider homogeneous media.

Case 1: Problem (1) with BCs (4)

The starting field at $x = 0$ is $u_0(z) = \sin(2\pi z)$, which is the third propagation mode. Then case 1 has an exact solution:

$$u(x,z) = \mathrm{e}^{\mathrm{i}\lambda_3 x}\sin(2\pi z), \tag{46}$$

where $\lambda_3 = \sqrt{\kappa^2 - (2\pi)^2}$ is the third eigenvalue of Eqs. (7) and (8). The field in Eq. (46) is used as reference.

Numerical fields $u(L, z)$ (denoted as uL) obtained with step size $h = 1$ and the reference fields in Eq. (46) are plotted in Fig. 2. Relative errors in $L^2$ norm for different $n$'s are displayed in Fig. 3. It is easy to see that the Chebyshev pseudospectral method with 30 points is much better than the FD method with 300 points. In fact, since the accuracy of the FD method we use here is $O(n^{-2})$, thousands of points are needed to achieve 10-digit accuracy. However, for the Chebyshev pseudospectral method, hundreds of points are enough.

In practice, to expand $\sin(2\pi z)$ by the Chebyshev series in domain $[0, 1]$, only 22 terms are needed

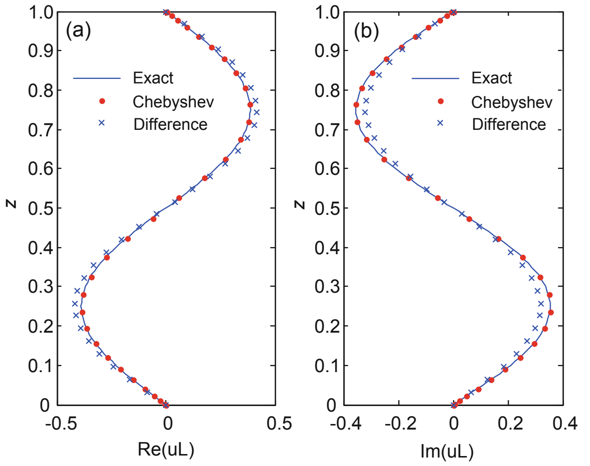


**Fig. 2 Comparison of two methods for case 1: (a) real part of uL; (b) imaginary part of uL**

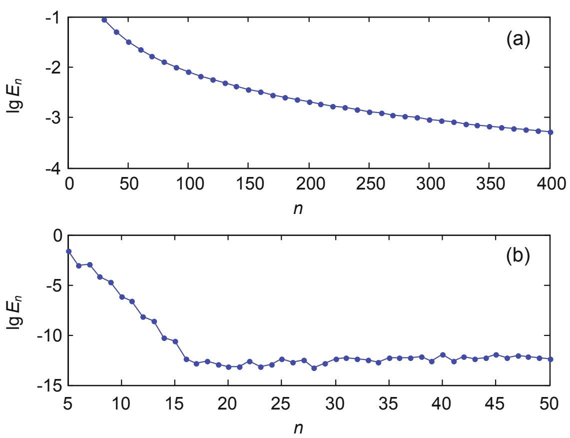


**Fig. 3 Relative errors for different $n$'s for case 1: (a) finite difference method; (b) Chebyshev pseudospectral method**

to achieve 15-digit accuracy. Thus, 30 points are enough for $z$-direction discretization. Any more points are meaningless because it will not achieve a higher accuracy.

The numerical schemes connect with the characteristic problem closely, and the accuracy of eigenvalues and eigenfunctions of Eq. (7) seriously affects the propagation result. Although the sparsity of the resulting matrix (the resulting matrix is tridiagonal in the FD method, while it is dense in the pseudospectral method) may affect the computational efficiency, it will not change the fact that the Chebyshev pseudospectral method is superior to the FD method in terms of both accuracy and efficiency (Canuto *et al.*, 1988; Boyd, 2001). In fact, the error of the $k$th eigenvalue obtained by the FD method is typically $O(k^4 n^{-2})$ rather than $O(n^{-2})$ (Andrew, 2000). Only about 10% to 20% of the eigenvalues obtained by the FD method are accurate, while this proportion is about $2/\pi$ for the Chebyshev pseudospectral method (Costa *et al.*, 2007; Zhang, 2010).

Case 2: Problem (1) with BCs (5)

The starting field at $x = 0$ is $u_0(z) = \sin(2.5\pi z)$ corresponding to the third propagation mode of Eqs. (7) and (9). Then problem (1) with (5) has an exact solution:

$$u(x, z) = e^{i\lambda_3 \sin(2.5\pi z)}, \tag{47}$$

where $\lambda_3 = \sqrt{\kappa^2 - (2.5\pi)^2}$ is the third eigenvalue. The field in Eq. (47) is used as reference in this case.

Numerical results are shown in Figs. 4 and 5. For case 1, the solution solved by the Chebyshev grid is much better than the uniform grid. The BCs for the uniform grid are approximated by the central difference method, which also reduces the accuracy. Three hundred points of the uniform grid can achieve only 3-digit accuracy. To achieve 10-digit accuracy, thousands of points are needed for the FD method. However, 12-digit accuracy is obtained by only 30 points of the Chebyshev grid.

### 5.2 Inhomogeneous media

Now we consider inhomogeneous media. For both cases, the starting field at $x = 0$ is

$$u_0(z) = \sum_{j=1}^{7} \sin(m_j z_0) \sin(m_j z) \Big/ \sqrt{10^2 - m_j^2}, \tag{48}$$

where $m_j = (j - \frac{1}{2})\pi$, $z_0 = 0.65$.

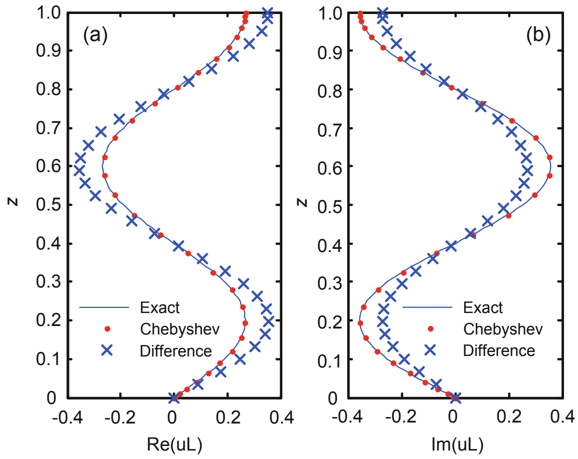


**Fig. 4 Comparison of two methods for case 2: (a) real part of uL; (b) imaginary part of uL**

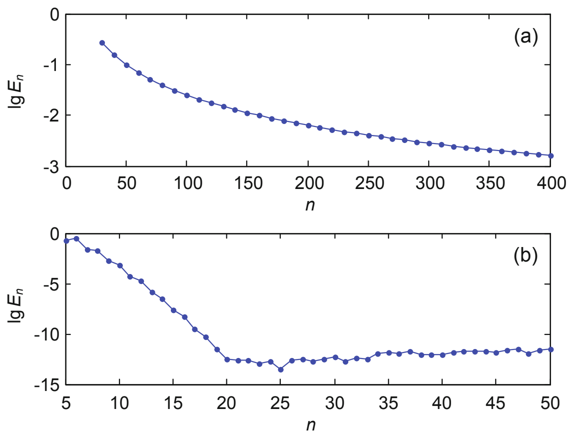


**Fig. 5 Relative errors for different $n$'s for case 2: (a) finite difference method; (b) Chebyshev pseudospectral method**

It is the same as that used in Zhu and Song (2009). This problem does not have an exact solution. The numerical fields with step size $h = 0.01$ and a $300 \times 300$ Chebyshev differentiation matrix are calculated as reference in both cases.

For both cases, to compare the Chebyshev pseudospectral method and the FD method, a fixed $x$-direction step $h = 0.01$ is chosen so that the dominant error is the $z$-direction truncation error.

Fig. 6 displays the relative errors of the Chebyshev pseudospectral method for different $n$'s, and Table 1 shows the relative errors of the FD method. By Fig. 6 and Table 1, it is evident that the Chebyshev pseudospectral method with 30 points is much better than the FD method with 400 points.

By a posteriori error estimation, there are 6-digit and 4-digit accuracies in the $x$-direction for

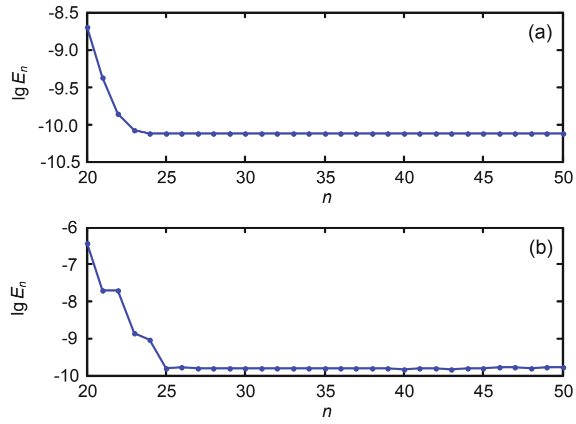


**Fig. 6 Relative errors of the Chebyshev pseudospectral method under different $n$'s: (a) case 1; (b) case 2**

**Table 1 Relative errors of the finite difference method under different $n$'s**

| $n$ | Relative error ($\times 10^{-3}$) Case 1 | Relative error ($\times 10^{-3}$) Case 2 | $n$ | Relative error ($\times 10^{-3}$) Case 1 | Relative error ($\times 10^{-3}$) Case 2 |
|---|---|---|---|---|---|
| 100 | 31.5 | 19.7 | 300 | 3.54 | 2.20 |
| 200 | 7.95 | 4.96 | 400 | 2.00 | 1.24 |

cases 1 and 2, respectively. Thus, for the FD method, the truncation error in the $z$-direction is the limitation of accuracy. However, for the spectral method, there is no problem since the high accuracy can be easily achieved.

To further check the superiority of the method, we give another example with a stronger loss. In this example,

$$\kappa^2(x, z) = 3.5(1 + \alpha\mathrm{i})\kappa_0^2[1 - 0.4(z - 0.5)\mathrm{e}^{-(x/L-0.5)^2}],$$

where $\kappa_0 = 2\pi/1.55$ and $\alpha = 0.1$. The other parameters such as $L$ and $u_0(z)$ are the same as in the previous example.

Fig. 7 displays the relative errors of the Chebyshev pseudospectral method for different $n$'s, and Table 2 shows the relative errors of the FD method. Again, we find that the Chebyshev pseudospectral method is much better than the FD method.

## 6 Conclusions

We propose a new method for the computation of propagation in lossy waveguides. To improve accuracy and efficiency, the Chebyshev pseudospectral method is used. The step size depends on the wavenumber and the order of the marching method

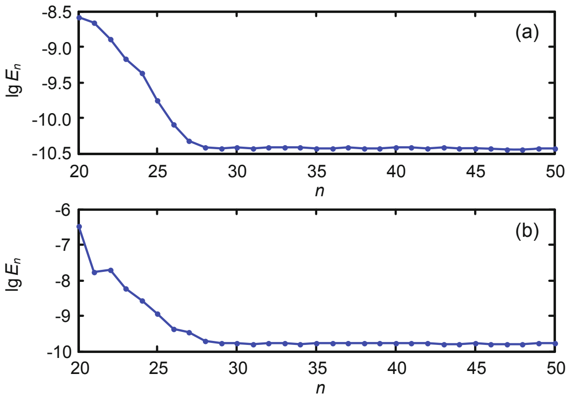


**Fig. 7 Relative errors of the Chebyshev pseudospectral method under different $n$'s: (a) case 1; (b) case 2**

**Table 2 Relative errors of the finite difference method under different $n$'s**

| $n$ | Relative error ($\times 10^{-4}$) | | $n$ | Relative error ($\times 10^{-4}$) | |
|---|---|---|---|---|---|
| | Case 1 | Case 2 | | Case 1 | Case 2 |
| 100 | 19.6 | 4.88 | 300 | 2.20 | 0.545 |
| 200 | 4.93 | 1.23 | 400 | 1.24 | 0.307 |

in the $x$-direction, which is not taken into account in this paper.

The method is also suitable for some complex problems such as a waveguide with curved interfaces. For complex problems, it is hard to compute the adjoint operator and tedious to find the weights for quadrature. Thus, the conjugate differentiation matrix method is suitable with its elegant formulation.

## References


Andrew, A.L., 2000. Twenty years of asymptotic correction for eigenvalue computation. *ANZIAM J.*, **42**:96-116.

Boyd, J.P., 2001. Chebyshev and Fourier Spectral Methods (2nd Ed.). Dover Publications, Inc., USA.

Canuto, C., Hussaini, M.Y., Quarteroni, A., *et al.*, 1988. Spectral Methods in Fluid Dynamics. Springer-Verlag Berlin Heidelberg, USA.

Costa, B., Don, W.S., Simas, A., 2007. Spatial resolution properties of mapped spectral Chebyshev methods. Proc. SCPDE: Recent Progress in Scientific Computing, p.179-188.

Lu, Y.Y., 1999. One-way large range step methods for Helmholtz waveguides. *J. Comput. Phys.*, **152**(1):231-250. [doi:10.1006/jcph.1999.6243]

Lu, Y.Y., McLaughlin, J.R., 1996. The Riccati method for the Helmholtz equation. *J. Acoust. Soc. Am.*, **100**(3): 1432-1446. [doi:10.1121/1.415990]

Lu, Y.Y., Zhu, J.X., 2004. A local orthogonal transform for acoustic waveguides with an internal interface. *J. Comput. Acoust.*, **12**1:37-53. [doi:10.1142/S0218396X04002183]

März, R., 1995. Integrated Optics: Design and Modeling. Artech House, USA.

Silva, A., Monticone, F., Castaldi, G., *et al.*, 2014. Performing mathematical operations with metamaterials. *Science*, **343**(6167):160-163. [doi:10.1126/science.1242818]

Trefethen, L.N., 2000. Spectral Methods in MATLAB. Society for Industrial and Applied Mathematics, USA.

Trefethen, L.N., 2013. Approximation Theory and Approximation Practice. Society for Industrial and Applied Mathematics, USA.

Vassallo, C., 1991. Optical Waveguide Concepts. Elsevier, Amsterdam.

Waldvogel, J., 2006. Fast construction of the Fejér and Clenshaw-Curtis quadrature rules. *BIT Numer. Math.*, **46**(1):195-202. [doi:10.1007/s10543-006-0045-4]

Zhang, X., 2010. Mapped barycentric Chebyshev differentiation matrix method for the solution of regular Sturm-Liouville problems. *Appl. Math. Comput.*, **217**(5): 2266-2276. [doi:10.1016/j.amc.2010.07.027]

Zhu, J., Lu, Y.Y., 2004. Validity of one-way models in the weak range dependence limit. *J. Comput. Acoust.*, **12**(1):55-66. [doi:10.1142/S0218396X0400216X]

Zhu, J., Song, R., 2009. Fast and stable computation of optical propagation in micro-waveguides with loss. *Microelectron. Reliab.*, **49**(12):1529-1536. [doi:10.1016/j.microrel.2009.06.004]