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RESEARCH ARTICLE

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Option-like properties in the distribution of hedge fund returns

Abstract Hedge funds have recently become popular because of their low correlation with traditional investments and their ability to generate positive returns with a relatively low volatility. However, a close look at those high-performing hedge funds raises the questions on whether their performance is truly superior and whether the high management fees are justified. Incurring no alpha costs, passive hedge fund replication strategies raise the question on whether they can similarly perform by improving efficiency at reduced costs. Therefore, this study investigates two different model approaches for the equity long/short strategy, where weighted segmented linear regression models are employed and combined with two-state Markov switching models. The main finding proves a short put option structure, i.e., short equity market volatility, with the put structure present in all market states. We obtain an evidence that the hedge fund managers decrease their short-volatility profile during turbulent markets.

Keywords hedge funds, hedge fund index, segmented linear regression models, regime-switching models, mimicking portfolios, single factor-based hedge fund replication, equity long–short strategy

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1 Introduction

Victor Niederhoffer was a distinguished hedge fund trader who followed a strategy to sell seemingly innocuous out-of-the-money puts on the Standard and Poors 500 (S&P500) Index. Such trading activity creates regular income (through the selling of the puts) and rarely increases liabilities or losses because the market should drop by a substantial amount before losses are realized. Most of the backward-looking measures of risk and performance, including the Sharpe ratio, look excellent. In addition, through traditional portfolio theory, one is tempted to evaluate these strategies as superior to others that generate occasional losses. However, in 1997, Niederhoffer's fund collapsed after an extremely sharp decline in the S&P500, which generated margin calls he failed to satisfy. Under such cases, one should understand portfolio performance from a risk factor perspective and not merely from the statistical analysis of historical returns perspective.

In their classical work, Schneeweis and Spurgin (1998) and Fung and Hsieh (2004) analyzed the dependence of fund returns as a function of market variables such as equity indices, bond yields, and volatilities. Schneeweis and Spurgin (1998) proposed a number of factors to explain a broad range of managed assets' performance and the differences in investment return among commodity trading advisors, hedge funds, and traditional mutual funds. Fung and Hsieh (2004) proposed a seven-factor linear model (equity, bond, and trend-following factors) for hedge funds to explain the return variation of funds-of-funds and various hedge fund indices. The 2008 Crisis introduced nonlinear effects into those factor dependencies. We use a one-factor model to characterize a straightforward hedge fund (HF) strategy, that is, the equity long–short. To take nonlinearities into account, we examine the factor dependencies from a Markov switching perspective. Moreover, we address the applications of the classical risk factor characterization of hedge fund returns, that is, the topic of hedge fund replication, within the

Markov switching framework. Traditionally, the investment in long–short strategies is carried out through an investment in hedge funds. Fee structure (Kat and Amin, 2001) distinguishes hedge funds from other types of investment. Hedge fund managers receive two types of fees, namely, management fees and incentive or performance fees. In particular, a fund manager takes a percentage of the investment profits (10%–20%) in addition to the fixed management fee (1%–2% of the invested capital) for managing the investment. The cumulative fees can be substantially relative to the net investment returns to the investor. In addition to the fees, the investor is subject to a lock-up period. During this time, investors are restricted from redeeming or selling shares. The lock-up period helps portfolio managers avoid liquidity problems while capital is put to work in occasionally illiquid investments. The lock-up period may extend for years; and when it ends, investors may redeem their shares according to a set schedule, often quarterly, and must give a 30- to 90-day notice. Thus, a replicating strategy, which is occasionally offered in the form of an inexpensive exchange-traded fund (ETF), is economically appealing as the investor substantially reduces investment costs in addition to having a full control over the liquidity of the investment. A passive, fee-free replicating portfolio return is used as a benchmark to determine the value-added of the hedge fund investment with respect to a portfolio of securities that are traded following several simple trading rules. Passive hedge fund replication strategies attempt to achieve similar performance characteristics as the original hedge fund in an efficient manner by modeling risk and return patterns. From a regression perspective, the alpha of a hedge fund return cannot be replicated and is the source of value added offered by the most successful hedge funds. However, the beta can be obtained in a passive, inexpensive manner. Therefore, this paper aims to shed light on replication strategies beyond the traditional linear regression. Notably, although a short-option-like strategy may be suggested by factor analysis, a reason to assume that the portfolio in question employs such strategy is lacking. This paper particularly aims to provide factor characterization that allows to obtain relationships between strategies and only takes the returns into account (which can be observed) instead of focusing on the actual portfolio holdings (which is inevident).

The factor-based and the payoff distribution approaches are the two classical approaches for hedge fund replication (Amenc et al., 2008). Factor-based replication attempts to demonstrate the time series of hedge fund returns with a clone time series, which is composed of suitable risk factors that minimize the tracking error with respect to the individual hedge fund or hedge fund index to be replicated. By contrast, payoff distribution approach attempts to model the probability distribution of the hedge fund returns. The two approaches are commonly characterized by their purpose to generate a replicating portfolio for hedge fund returns. However, the applied methods are based on completely different concepts from probability theory. By developing a Markov switching framework, we aim to address one of the stylistic characteristics of hedge fund returns, namely, skewness (see Brunner and Hafner (2005) and Jaeger and Wagner (2005) for features of hedge funds' returns). Popular cases of trading strategies where the manager simply sells deep out of the money options to create low-volatility income streams exist. These cases increase Sharpe ratios. Furthermore, when extreme events occur in the markets, the fund blows up, which is similar to the case of Niederhoffer.

These topics are part of a broad issue to determine whether hedge fund products produce performance numbers that justify high fees. Such issues have been addressed by numerous authors who followed various perspectives (Treynor and Mazuy, 1966; Henriksson and Merton, 1981; Ferson and Schadt, 1996; Mitchell and Pulvino, 2001; Chen, 2007; Cao et al., 2013; Chen et al., 2017).

Anson and Ho (2003) proposed an interesting methodology that uses segmented regression (Hudson, 1966) to identify this type of trading behavior synthetically by analyzing the time series of fund returns. This current paper aims to utilize an enhanced version of the segmented regression used by Anson and Ho (2003) and address the aforementioned three topics of interest, namely, the hedge fund return factor dependency, the nonlinearity of these dependencies, and the return replication. We refine the factor-based approach provided by Anson and Ho (2003). They identified and measured the short volatility exposure of merger arbitrage and event-driven strategies versus one risk factor. Furthermore, to reveal the short put option exposure of the two strategies, they created a mimicking portfolio. We use a one-factor model to describe the nonlinear relationship between the excess returns of the considered hedge fund strategy and the excess returns of the S&P500 Index. We demonstrate and analyze the equity long–short strategy that defines the risk factor as an exposure to the equity market risk represented by the S&P500 Index. We provide two different model approaches to gain detailed insights into the risk exposure of the considered hedge fund strategy. These approaches are the weighted segmented regression model with the aggregate market data and the segmented regression model within the Markov switching framework that is characterized by two market states, namely, normal and distressed. Our analysis shows clear evidence that the equity long–short strategy displays a return pattern that is consistent with a short put option structure on the equity index, both in normal market conditions and during market distress. However, the strategy synthetically shortens less put options during turbulent markets as the hedge fund manager reduces the portfolio's risk exposure to the equity market by decreasing leverage (gross exposure).

The remainder of the article is organized as follows. The developed model approaches and their mathematical descriptions are provided in Section 2. The mimicking portfolios are presented in Section 3. Section 4 provides a description of the employed data and presents an algorithm for estimating the model parameters. Section 5 discusses the applications of the developed model approaches to the considered hedge fund strategy and interprets the results. Section 6 presents the conclusions.

2 Model approaches

In contrast to a time series evaluation of the hedge fund strategy, the intention behind our model approaches is clarified by plotting the excess hedge fund returns versus the excess market returns, which is similar to Anson and Ho (2003). The exposure of the considered hedge fund strategy in each model approach is investigated with respect to the same single risk factor. The nonlinear relationship between the excess returns of the considered hedge fund strategy and an appropriate explanatory time series, i.e., excess returns of the S&P500 Index in our case, is revealed through two segmented linear regression models.

2.1 Model 1

We refine the model of Anson and Ho (2003) by applying weighted regression models. Our approach of weighted segmented linear regression was motivated by the fact that the areas of data points with high concentration should obtain more confidence than low concentrated areas. The excess returns of the considered hedge fund strategy $\tilde{R}_t^{\mathrm{s}}$ are expressed as follows:

$$\text{Model 1}: \begin{aligned} \tilde{R}_t^{\mathrm{s}} &= \mathbf{1}_{\{\tilde{R}_t \leqslant \alpha\}} f_1(\tilde{R}_t, \boldsymbol{\beta}_1) + \mathbf{1}_{\{\alpha < \tilde{R}_t\}} f_2(\tilde{R}_t, \boldsymbol{\beta}_2) + \varepsilon_t \\ &= \mathbf{1}_{\{\tilde{R}_t \leqslant \alpha\}} (\beta_{1,0} + \beta_{1,1}\tilde{R}_t) + \mathbf{1}_{\{\alpha < \tilde{R}_t\}} (\beta_{2,0} + \beta_{2,1}\tilde{R}_t) + \varepsilon_t, \end{aligned} \tag{1}$$

where the residuals ε_t are assumed to be i.i.d. (independent and identically distributed), $N(0, \sigma_t^2)$-distributed, $t \in \mathbb{T} := \{1,..., T\}$. Thus, the intercept $\beta_{1,0}$ and the slope $\beta_{1,1}$ define the regression line to the left of the breakpoint α. The intercept $\beta_{2,0}$ and the slope $\beta_{2,1}$ define the regression line to the right of the breakpoint. $\mathbf{1}_{\{\tilde{R}_t \leqslant \alpha\}}$ denotes the indicator function that takes the value one for all excess markets returns $\tilde{R}_t \leqslant \alpha$, and zero otherwise[1]. Note that the strategy was also demonstrated with a weighted linear regression model without a breakpoint and a weighted segmented linear regression model with two breakpoints. These models can be obtained as a simplification and extension of the stated model, respectively. A detailed description of both other models will not be further discussed here because the optionality properties are best described with a model approach with one breakpoint regarding minimum residual sum of squares and the coefficient of determination R^2 (see Denk (2014) for additional details). For the estimation of the parameter vector $\boldsymbol{\theta}^{\mathrm{M1}} = (\beta_{1,0}, \beta_{1,1}, \beta_{2,0}, \beta_{2,1}, \alpha)$, the method of least square estimation can be used. In addition, the constraints of continuity:

- $f_1(\alpha, \boldsymbol{\beta}_1) = f_2(\alpha, \boldsymbol{\beta}_2)$,
- $\tilde{R}_l \leqslant \alpha < \tilde{R}_{l+1}$,

should be fulfilled. By employing the method of Lagrange multipliers, the nonlinear optimization problem to be solved can be stated as follows (see Appendix for details of integrating the constraints with the objective function):

$$\min_{\boldsymbol{\beta}_1,\ \boldsymbol{\beta}_2,\ \alpha,\ l} \Big(WRSS_1(\boldsymbol{\beta}_1, \alpha, l) + WRSS_2(\boldsymbol{\beta}_2, \alpha, l)\Big),$$

$$WRSS_1 = \sum\nolimits_{t=1}^{l} \big(w_t(\tilde{R}_t^{\mathrm{s}} - f_1(\tilde{R}_t, \boldsymbol{\beta}_1))\big)^2,$$

$$WRSS_2 = \sum\nolimits_{t=l+1}^{T} \big(w_t(\tilde{R}_t^{\mathrm{s}} - f_2(\tilde{R}_t, \boldsymbol{\beta}_2))\big)^2, \tag{2}$$

where we apply the weighting:

$$w_t = \frac{w_t^{(1)}}{\sum_{j=1}^{T} w_j^{(1)}},$$

$$w_t^{(1)} := \sum\nolimits_{j=1}^{T} \mathbf{1}_{\{\tilde{R}_t - \delta \leqslant \tilde{R}_j < \tilde{R}_t + \delta\}}, \tag{3}$$

for all $t \in \mathbb{T}$ and $\delta = 0.03$[2]. The Lagrange method provides the unique minimum. Given the weighted segmented linear regression model with one breakpoint, we accounted for the changing properties, i.e., the nonlinear properties in the

1) For sake of readability, we describe the unweighted case here. Let $\tilde{\boldsymbol{R}} = (\tilde{R}_t)$, $\tilde{\boldsymbol{R}}^i = (\tilde{R}_t^{\mathrm{s}}) \in \mathbb{R}^{\mathrm{T}}$, with $4 \leqslant T$ and $l \in \mathbb{N}$ such that $2 \leqslant l$ and $2 + l \leqslant T$ (note that T is the number of data points). Let RSS denote the overall residual sum of squares that is comprised of the individual residual sum of squares RSS_1 and RSS_2 by employing $(\tilde{R}_1, \tilde{R}_1^{\mathrm{s}}),..., (\tilde{R}_l, \tilde{R}_l^{\mathrm{s}})$ for segment 1 and $(\tilde{R}_{l+1}, \tilde{R}_{l+1}^{\mathrm{s}}),..., (\tilde{R}_T, \tilde{R}_T^{\mathrm{s}})$ for segment 2. Furthermore, the RSS should be minimized, where each segment is modeled with an individual polynomial of maximum degree 1, i.e., $f_i(\boldsymbol{\beta}_i, \tilde{R}) := \boldsymbol{\beta}_{i,0} + \boldsymbol{\beta}_{i,1}\tilde{R}$, where $i = 1, 2$. If the following two conditions are fulfilled, then α is called a breakpoint.
- $\alpha \in [\tilde{R}_l, \tilde{R}_{l+1})$ given that the RSS is minimal;
- Continuity constraint at breakpoint α: $f_1(\boldsymbol{\beta}_1, \alpha) = f_2(\boldsymbol{\beta}_2, \alpha)$.

The intervals $I_1 := [\tilde{R}_1, \alpha]$ and $I_2 := [\alpha, \tilde{R}_T]$ are called segments with respect to the breakpoint α.

2) The parameter δ was empirically estimated, and several δ were investigated. $\delta = 0.03$ delivered the best results.

hedge fund excess returns with respect to the market excess returns. Therefore, the weighting played a major role. Through weighting, the regression lines are significantly determined by the excess returns lying in a high-density environment of data points.

To decide whether a simple linear regression with no breakpoint (null hypothesis H_0) or a segmented linear regression with one breakpoint (alternative hypothesis H_1) should be used, Quandt (1958) proposed the following test statistics:

$$T_1 = \frac{\sum_{i=1}^{l} \frac{(\beta_{1,0}+\beta_{1,1}x_i-y_i)^2}{\sigma^2}}{\sum_{i=l+1}^{n} \frac{(\beta_{1,0}+\beta_{1,1}x_i-y_i)^2}{\sigma^2}},$$

$$\text{and } T_2 = \frac{\sum_{i=1}^{l} \frac{(\beta_{2,0}+\beta_{2,1}x_i-y_i)^2}{\sigma^2}}{\sum_{i=l+1}^{n} \frac{(\beta_{2,0}+\beta_{2,1}x_i-y_i)^2}{\sigma^2}}. \quad (4)$$

Evidently, T_k follows an F-distribution with $n-l-1$ and $l-1$ ($l-1$ and $n-l-1$) degrees of freedom. The null hypothesis is rejected if T_k has a p value greater than the chosen significance level. In each case, the p value of the test statistics is 0, and the null hypothesis is rejected.

2.2 Model 2

To characterize the investment style in different market states, we apply the concept of the Markov switching and combine it with our segmented linear regression models.

To determine the order of an r-state Markov regime-switching autoregressive process, we refer to Hauptmann et al. (2014) who introduced the AIC (Akaike Information Criterion) (Akaike, 1974) and BIC (Bayesian Information Criterion) (Schwarz, 1978) criteria to the whole time series of the S&P500 returns. Both criteria prefer an autoregressive (AR)(1) model over higher order AR(m) models with $m = 2,..., 8$. They use the result of Zhang and Stine (2001) that a univariate weakly stationary r-state Markov regime-switching autoregressive process of order m admits an ARMA (autoregressive moving average)(p, q) representation with $p \leqslant rm^2$ and $q \leqslant rm^2-1$. Subsequently, they select the parameters of an ARMA(p, q) model using again the AIC and BIC criteria. Both result in a Markov regime-switching autoregressive process of order 1 with at least two states. Moreover, they show that the model with two states is more robust out of sample than a model with three states. Hence, we use a Markov regime-switching autoregressive process of order 1 with two states here.

The market returns are modeled with a two-state first-order discrete time Markov switching process that depends on an unobservable state $(S_t)_{t\in\mathbb{T}}$ with state space $\{0, 1\}$. The discrete returns $(\tilde{R}_t)_{t\in\mathbb{T}}$ are assumed to be normally distributed with regime-dependent mean μ_{S_t} and regime-dependent volatility $\sigma_{S_t} > 0$. Based on Hauptmann et al. (2014)'s definition, we use the following approach:

$$\tilde{R}_t = \mu_{S_t} + \sigma_{S_t}\varepsilon_t, \quad (5)$$

where $\varepsilon_t \sim N(0,\ 1)$ i.i.d. The state process S is demonstrated as a time-homogeneous Markov chain with fixed transition matrix:

$$\Pi = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix}, \quad (6)$$

where $p := \mathbb{P}(S_t = 0 \mid S_{t-1} = 0)$, and $q := \mathbb{P}(S_t = 1 \mid S_{t-1} = 1)$. The initial distribution of the Markov chain is given by:

$$\boldsymbol{v} = (\nu_1,\ 1-\nu_1)^{\mathrm{T}} := \Big(\mathbb{P}(S_0 = 0),\ \mathbb{P}(S_0 = 1)\Big)^{\mathrm{T}}. \quad (7)$$

Given the parameter vector $\boldsymbol{\theta}^{\mathrm{ML}} = (\mu_1, \mu_2, \sigma_1, \sigma_2, p, q, \nu_1)$, the model is completely defined. We apply the method of maximum likelihood estimation in order to determine the parameter vector $\boldsymbol{\theta}^{\mathrm{ML}}$. However, $(S_t)_{t\in\mathbb{T}}$ is unobservable. Hence, the transition matrix $\boldsymbol{P}$ and the initial distribution $\boldsymbol{v}$ are unknown. Therefore, a general procedure for the determination of maximum-likelihood estimators of probabilistic models known as the expectation maximization (EM) algorithm is applied. The algorithm for the special case of hidden Markov models was developed by Baum (1972). Initially, the hidden Markov model with two regimes, regime-dependent mean μ_{S_t} and regime-dependent volatility σ_{S_t} from Eq. (5), is successively estimated for the market returns $\tilde{R}_t$, $t \in \mathbb{T}$. Subsequently, the set of data points in the "turbulent" and "calm" state is obtained as follows:

$$\mathbb{T}^{(0)} = \{t \in \mathbb{T} : \mathbb{P}(S_t = 0 \mid \tilde{R}_t,\ \hat{\boldsymbol{\theta}}_t) > 0.5\},$$

$$\mathbb{T}^{(1)} = \mathbb{T}\backslash\mathbb{T}^{(0)}. \quad (8)$$

The probability cutoff of 0.5 was proposed by Viterbi (1967) and simply refers to the maximum probability state in a two-state world. The intuition behind the proposed procedure is the attempt to analyze the HF strategy in high and low volatility times of the market underlying the trading strategies rather than searching for best fit in distribution.

Figure 1 shows for every excess market return which of the two states is present for the respective market return, i.e., $S_t = 0$ or $S_t = 1$ for all $t \in \mathbb{T}$. In addition, the plot clearly illustrates that the model detects the financial crises in the year 2007 to 2009 and the market turbulences in the year 2011, which is reflected by the gray time frames in Fig. 1. Using the Markov switching approach, we separated the data of the S&P500 returns into two disjoint data sets. From the descriptive statistics of the two data sets, we

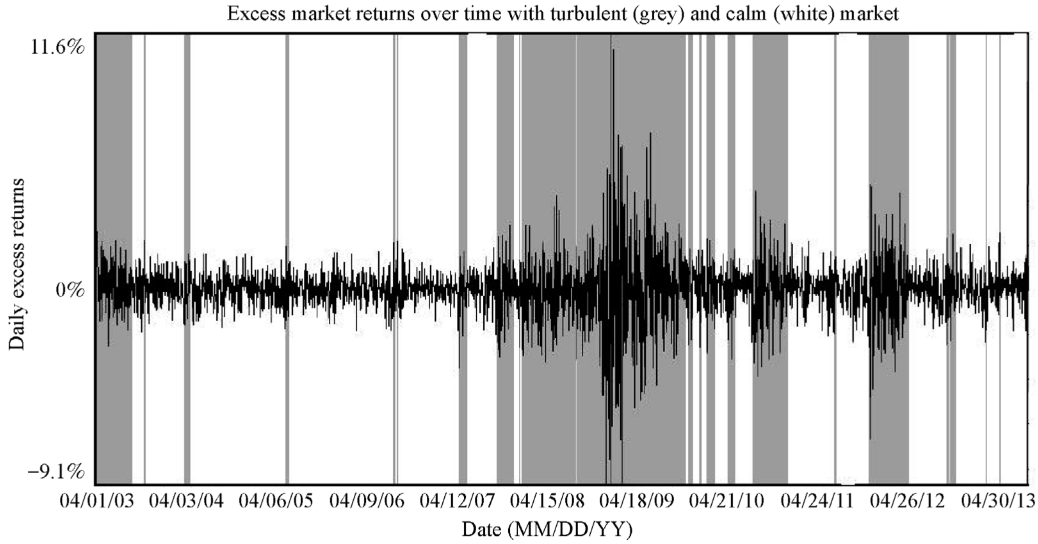

Fig. 1 Excess returns of S&P500 from 04/01/2003 to 04/30/2013.

name them as calm and turbulent markets. The turbulent market is characterized by a negative annualized mean return of -3.6% and an annualized standard deviation of 31.27%. Compared with the turbulent market, the calm market shows a completely different picture with a positive annualized mean return of 12.83%. Moreover, the calm market excess returns merely exhibit approximately one-third of the standard deviation of the turbulent market, i.e., an annualized standard deviation of 10.98%.

Second, we separate the series of hedge fund returns according to the set of indices $\mathbb{T}^{(0)}$ and $\mathbb{T}^{(1)}$. Consequently, the turbulent and calm hedge fund returns, i.e., $\left(\tilde{R}_t^{\mathrm{s}}\right)_{t\in\mathbb{T}^{(0)}}$ and $\left(\tilde{R}_t^{\mathrm{s}}\right)_{t\in\mathbb{T}^{(1)}}$, are modeled with the series of turbulent market returns $\left(\tilde{R}_t\right)_{t\in\mathbb{T}^{(0)}}$ and the series of calm market returns $\left(\tilde{R}_t\right)_{t\in\mathbb{T}^{(1)}}$, respectively. Given the parameter vector $\left(\alpha^{(k)},\ \beta_{1,0}^{(k)},\ \beta_{1,1}^{(k)},\ \beta_{2,0}^{(k)},\ \beta_{2,1}^{(k)}\right)$, for $k\in\{0,\ 1\}$, we estimate the excess returns of the considered hedge fund strategy as follows (similar to what we applied in Model 1):

$$\tilde{R}_t^{\mathrm{s}} = \mathbf{1}_{\left\{\tilde{R}_t\leqslant\alpha^{(k)}\right\}}\left(\beta_{1,0}^{(k)}+\beta_{1,1}^{(k)}\tilde{R}_t\right) + \mathbf{1}_{\left\{\alpha^{(k)}<\tilde{R}_t\right\}}\left(\beta_{2,0}^{(k)}+\beta_{2,1}^{(k)}\tilde{R}_t\right)+\varepsilon_t, \tag{9}$$

where $t\in\mathbb{T}^{(k)}$, $k\in\{0,1\}$, and $\varepsilon_t\sim N(0,\ \sigma_t^2)$. Note that the breakpoints of the calm and the turbulent market fail to necessarily coincide a priori. To provide a consistent model, we implement a market constraint given that the breakpoint of each market (calm and turbulent) coincides with the breakpoint estimated in Model 1. In particular, the parameters of each market are estimated in such a way the market-weighted sum of the two kinked regression lines amounts to the kinked regression line from Model 1. Thus, the parameter estimates of the two markets consistently depend on each other and consequently depend on the parameter estimates of the regression in Model 1. To link the calm and turbulent markets to the total market, we employ the following constraints to minimize the weighted residual sum of squares:

• The regression lines for the calm and turbulent market share the same breakpoint with the market as a whole. The breakpoint is estimated once in the model for the total market (Model 1). Subsequently, the breakpoint is assumed to be given in the model for the separated markets, wherein the continuity constraints are fulfilled in the separated markets (see the Appendix):

$$\left(\boldsymbol{\beta}_1^{(0)\mathrm{T}}-\boldsymbol{\beta}_2^{(0)\mathrm{T}}\right)\boldsymbol{q}=0,$$

$$\left(\boldsymbol{\beta}_1^{(1)\mathrm{T}}-\boldsymbol{\beta}_2^{(1)\mathrm{T}}\right)\boldsymbol{q}=0, \tag{10}$$

where the breakpoint vector is defined as $\boldsymbol{q}:=(1,\ \alpha)^{\mathrm{T}}$.

• For each segment in the calm and turbulent markets, a corresponding weight is determined according to its number of data points. The market weights are determined as the ratio of data points in the respective segment and the total number of data points in this segment over the two markets:

$$w_i^{(k)}:=\frac{n_i^{(k)}}{n_i^{(0)}+n_i^{(1)}},\ k\in\{0,\ 1\},\ i\in\{1,\ 2\}. \tag{11}$$

• The weighted segment lines sum up to the corresponding segment line of the entire market, which results in the weighted market constraints:

$$\boldsymbol{\beta}_1=w_1^{(0)}\boldsymbol{\beta}_1^{(0)}+w_1^{(1)}\boldsymbol{\beta}_1^{(1)},$$

$$\boldsymbol{\beta}_2=w_2^{(0)}\boldsymbol{\beta}_2^{(0)}+w_2^{(1)}\boldsymbol{\beta}_2^{(1)}. \tag{12}$$

Solving the respective nonlinear optimization problem (see the Appendix) results in the parameter vector $\boldsymbol{\theta}^{\mathrm{M2}} = \left(\beta_{1,0}^{(k)}, \beta_{1,1}^{(k)}, \beta_{2,0}^{(k)}, \beta_{2,1}^{(k)}, \alpha\right)$ and the excess return of the considered hedge fund strategy is estimated as:

$$\text{Model 2}: \ \tilde{R}_t^{\mathrm{s}} = \mathbf{1}_{\{\tilde{R}_t \leqslant \alpha\}}\left(\beta_{1,0}^{(k)} + \beta_{1,1}^{(k)}\tilde{R}_t\right) + \mathbf{1}_{\{\alpha < \tilde{R}_t\}}\left(\beta_{2,0}^{(k)} + \beta_{2,1}^{(k)}\tilde{R}_t\right) + \varepsilon_t, \quad (13)$$

where $k \in \{0, 1\}$ and $\varepsilon_t \sim N(0, \sigma_t^2)$ for $t \in \mathbb{T}^{(k)}$.

3 Mimicking portfolio

For the models with one breakpoint, we construct a mimicking portfolio that consists of the following three components, namely, the equity market index corresponding to the risk factor, a short put option on the underlying index, and the risk free security. The mimicking portfolio replicates the segmented regression line with these financial instruments. To determine the weights of these three securities, the return of our mimicking portfolio should equate the return of the estimated segmented regression line of the considered hedge fund strategy. The weights of the three securities (w^{I}, w^{P}, w^{B}) are determined with respect to the parameter estimates of the segmented regression line with one breakpoint as follows[1)]: $w^{\mathrm{P}} = \beta_{1,1} - \beta_{2,1}$, $w^{\mathrm{B}} = 1 - \beta_{1,1}$. Consequently, the weight of the S&P500 Index is $w^{\mathrm{I}} = 1 - w^{\mathrm{P}} - w^{\mathrm{B}}$.

At this point, the following observations should be conducted. That is, the proposed replication portfolio that combines an exposure to an equity market and a short put option on the underlying index is an approach to mimic the equity long–short strategy. This portfolio is one of the possibilities to interpret the nonlinearity observed in practice. Among them is the use of derivatives. Chen (2011) performed a study on a large sample of individual hedge funds and focused on the heterogeneity across individual hedge funds to find that the majority of hedge funds trade derivatives. Market timing and interim trading are other factors contributing to the existence of nonlinearities. The fund managers can differentiate the market exposure over time depending on the market conditions (Chen and Liang, 2007) and to rebalance the portfolio as frequently as needed contrary to a traditional index fund (Chen et al., 2010).

4 Employed data

In the present study of the option-like properties of hedge fund returns, we considered the equity long–short strategy. As a substitute for this strategy, we employed the HFRX Equity Hedge Index from HFR (Hedge Fund Research, Inc.). Equity long–short is an investment strategy that is generally associated with hedge funds. This strategy involves buying long equities that are expected to increase in value and selling short equities that are expected to decrease in value. Furthermore, this strategy is generally based on a "bottom-up" fundamental analysis of the individual companies, wherein investments are made. The strategy can be focused on countries or regions, industries or sectors or diversified across these classifications. In addition, the strategy may also focus on certain categories such as value or growth and large cap or small cap. A strategy manager typically attempts to reduce risk/volatility by either diversifying or hedging positions across individual regions, industries, sectors, and market capitalization. Moreover, numerous trading styles are evident from frequent traders on a large universe of stocks to several longer-term investors with a concentrated portfolio of stocks (Jacobs et al., 1999). A wide variation exists in the degree to which managers prioritize high returns, which may involve concentrated and leveraged portfolios. To illustrate, if the strategy manager is given \$100 to invest, he/she may choose to buy \$100 worth of stocks and short \$80. The portfolio is characterized as being long 100%, short 80%, net (long minus short) 20%, and gross 180%. Thus, the portfolio will consist of long positions and short positions in stocks and cash from the short proceeds. The net exposure is his/her directional exposure. Gross exposure is an additional dimension of exposure that can be varied while maintaining the same net exposure. Using the above example, the manager may choose for the same initial asset of \$100 to go long \$200 and short \$180 for the same net exposure of 20% but a gross exposure of 380%. This example introduces leverage into the strategy. The strategy performance is amplified up/down with an increase of the gross exposure while maintaining the same net exposure, if the strategy manager makes the right/wrong portfolio selection, respectively.

To apply the different types of segmented linear regression models, we should preprocess the time series of hedge fund indices. To this end, we follow the guideline presented in the paper. The analysis and all further investigations of the indices are conducted over the time period from 1 April 2003, to 30 April 2013, which is denoted by $\mathbb{T} := \{1,..., T\}$. For all $t \in \mathbb{T}$, R_t^{s} denotes the discrete return of the equity long–short strategy, and R_t is the discrete return of the S&P500 Index at time t. We refer to the returns of the S&P500 Index as market returns.

In all our model approaches, excess returns are used for the explanatory variable and the response variable. Thus, the excess return of a time series at time t is defined as:

1) Note that a positive w^{P} describes a short put position, i.e., a long index exposure, whereas a positive w^{I} (w^{B}) stands for a long stock (bond) exposure.

$\tilde{R}_t := R_t - r_t$, where r_t denotes the riskless rate at time t. As a substitute for the riskless rate, we use the three-month London Interbank Ofered Rate (LIBOR). Furthermore, let $\tilde{R}_t^s$ and $\tilde{R}_t$ denote the excess return of the considered hedge fund strategy and the S&P500 Index, respectively. We obtain the time series of hedge fund index returns from HFR. The time series for the S&P500 Index and the three-month LIBOR rate are derived from Bloomberg.

5 Analysis

Figures 2, 3 and 4 exhibit the results of applying the weighted segmented linear regression models for the total market approach (M1) and the separated markets approach (M2) and for calm and turbulent markets, respectively. Table 1 exhibits the regression parameter estimates for the two models.

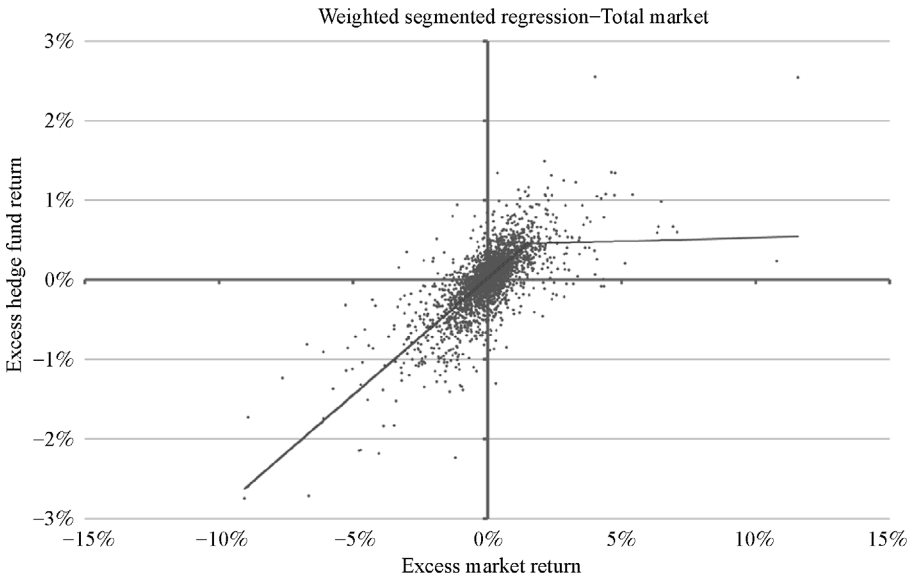

Fig. 2 HFRX Equity Hedge Index: Total market.

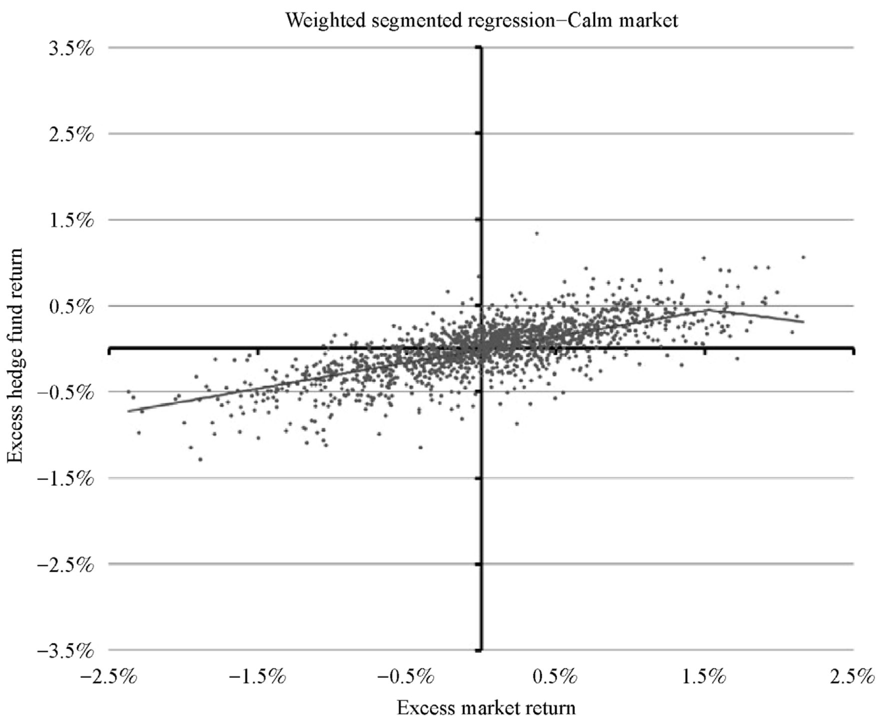

Fig. 3 HFRX Equity Hedge Index: Calm market.

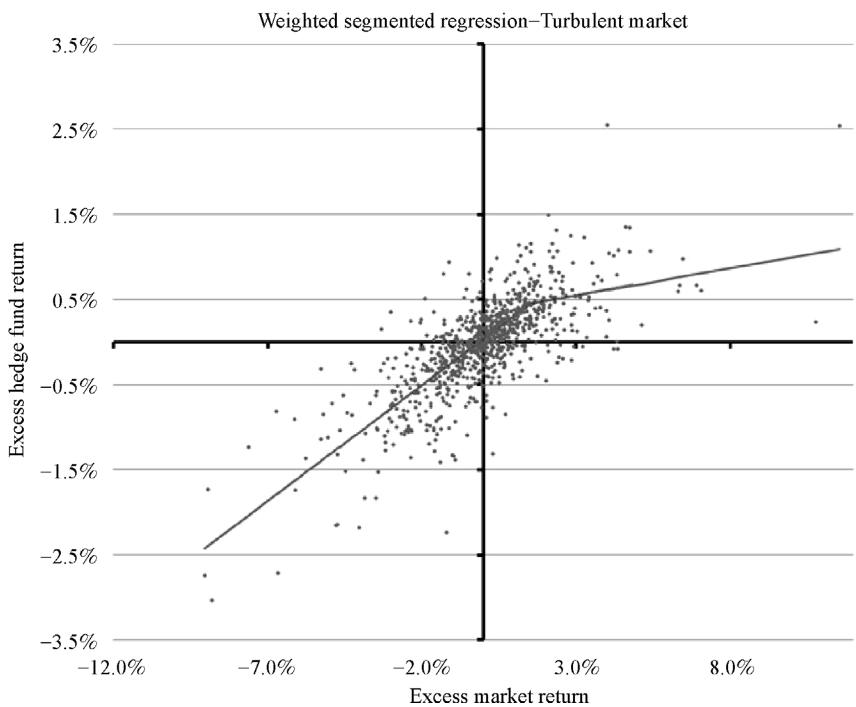

Fig. 4 HFRX Equity Hedge Index: Turbulent market.

Table 1 HFRX Equity Hedge Index

	M1	M2(0) (turbulent)	M2(1) (calm)
$\beta_{1,0}$[%]	0.00	0.04	−0.01
$\beta_{1,1}$[%]	0.29	0.27	0.30
$\beta_{2,0}$[%]	0.44	0.35	0.78
$\beta_{2,1}$[%]	0.01	0.06	−0.21
α_1[%]	1.53	1.53	1.53
RSS	0.0000	0.0119	0.0076
R^2[%]	42.14	46.58	47.60

Each model approach exhibits similarities and differences when the two segments are compared. In all the models, the left segment is characterized by a positive slope, while the slope sign for the right segment varies with the model. The left segment is essentially the area when the market returns are negative. In particular, low market returns result in low strategy returns. Significant negative market returns are associated with high market volatility. Therefore, the left segment indicates a strategy that is short volatility.

For our first model approach M1, Fig. 2 illustrates that the strategy shortens volatility through the sale of synthetic put options to collect the option premium. The apparent risk exposure of the strategy is different on the two sides of the breakpoint, that is, increased market exposure (β) on the left segment and decreased to no market exposure (β) on the right segment. If market returns increase/decrease by 1%, an increase/decrease of $\beta_{1,1}$ = 0.29% occurs for the equity long–short strategy excess returns on the left segment. However, the strategy's risk exposure changes when market excess returns exceed α_1 = 1.53%. When the excess returns of the equity long–short strategy remain on a constant level independent of the size of the market returns. The independence of the market and the equity long–short strategy becomes apparent through the slope of the second segment ($\beta_{2,1}$ = 0.01%). Given that market returns are greater than the breakpoint, the strategy provides a constant premium (estimated by the intercept) to the investor regardless of the magnitude of the market returns.

In our model approach M2, when we separate the two regimes, we observe that the strategy remains short in volatility on the left segment with similar slopes to the Model 1. However, on the right segment, the strategy exhibits a negative slope with a higher premium (intercept) for the calm market than for the turbulent market. This negative slope is related to the long/short nature of the strategy. In particular, the short part of the portfolio hinders the overall performance relative to a merely long portfolio.

The model with the single linear regression has an R^2 equals to 45.24%. The model with one breakpoint has an R^2 equals to 42.14%. By distinguishing calm and turbulent markets, the model with one breakpoint has an R^2 equals to 47.60% in the calm and an R^2 equals to 46.58% in the turbulent market. Both values are higher than those in the model with single linear regression.

Table 2 further corroborates the aforementioned results

Table 2 Weights of the mimicking portfolio for the HFRX Equity Hedge Index

	M1	M2 (turbulent)	M2 (calm)
w^P[%]	28.30	20.78	51.63
w^I[%]	0.88	6.40	−21.50
w^B[%]	70.82	72.83	69.87

and summarizes the weights of the portfolio that mimics the equity long–short strategy for each of the developed model approaches M1–2. Evidently, the strategy exhibits a short put option structure with varying degrees of exposure in each model approach. In addition, this strategy allocates a substantial weight to the risk-free security in all market conditions. During the calm market, the strategy significantly increases the short position in the synthetic put option ($w^P = 51.63\%$) compared to the turbulent market ($w^P = 20.78\%$), while hedging the risk away (to some extent) by shorting the underlying index ($w^I = -21.50\%$). Interestingly, the aggregate exposure to short put and the underlying index remains roughly the same for the turbulent and calm markets. These findings can be related to the equity long–short strategy as follows. Generally, the strategy consists of long and short portfolios of equity securities. The short portfolio proceeds contribute to the weight of the risk-free security. The strategy manager typically sets the gross (leverage) and net exposures of the portfolio in response to his/her perception of the market state. In turbulent markets, he/she tends to lower the portfolio risk by reducing the gross (leverage) and net exposures. This de-leveraging process is equivalent to writing few put options and reduced exposure to the underlying index.

6 Conclusions

We derived two model approaches by applying the theory of weighted segmented regression combined with Markov switching stochastic processes. Furthermore, we use the equity long–short strategy, as an application, to detect the option-like properties of this strategy. The weighted linear regression with Markov switching and dependent markets outperforms other model in terms of residual sum of squares and coefficient of determination and fits the data in a reasonable way, thereby clarifying the underlying hedge fund strategy. The equity long–short strategy earns a constant premium when the excess returns of the market exceed the breakpoint. We also derive a mimicking portfolio that consists of a short put option, the underlying stock, and the risk-free security for the replication of the respective hedge fund strategy. This mimicking portfolio illustrates the short volatility nature of this strategy. The equity long–short strategy in calm markets provides the investor with a higher premium than in turbulent markets by synthetically writing additional put options while lowering the portfolio risk by shorting the underlying index. By contrast, the strategy synthetically reduces put options during turbulent markets as the hedge fund manager decreases the portfolio's risk exposure to the equity market by lowering the leverage (gross exposure).

References

Akaike H (1974). A new look at the statistical model identification. IEEE Transactions on Automatic Control, 19(6): 716–723

Amenc N, Géhin W, Martellini L, Meyfredi J C (2008). Passive hedge fund replication: A critical assessment of existing techniques. Journal of Alternative Investments, 11(2): 69–83

Anson M J P, Ho H (2003). Short volatility strategies: Identification, measurement and risk. Journal of Investment Management, 1(2): 30–43

Baum L E (1972). An inequality and associated maximization technique in statistical estimation of probabilistic functions of Markov processes. Inequalities, 3(1): 1–8

Brunner B, Hafner R (2005). Cope with the characteristics. Finanz Business

Cao C, Chen Y, Liang B, Lo A (2013). Can hedge funds time market liquidity? Journal of Financial Economics, 109(2): 493–516

Chen Y (2007). Timing ability in the focus market of hedge funds. Journal of Investment Management, 5: 66–98

Chen Y, Cliff M, Zhao H (2017). Hedge funds: The good, the bad, and the lucky. Journal of Financial and Quantitative Analysis, 52(3): 1081–1109

Chen Y, Ferson W, Peters H (2010). Measuring the timing ability and performance of bond mutual funds. Journal of Financial Economics, 98(1): 72–89

Chen Y, Liang B (2007). Do market timing hedge funds time the market? Journal of Financial and Quantitative Analysis, 42(4): 827–856

Chen Y (2011). Derivatives use and risk taking: Evidence from the hedge fund industry. Journal of Financial and Quantitative Analysis, 46(4): 1073–1106

Denk K (2014). Optionality Properties in the Return Distribution of Hedge Fund Returns. Dissertation for the Master's Degree. München: Technische Universität München

Ferson W, Schadt R W (1996). Measuring fund strategy and performance in changing economic conditions. The Journal of Finance, 51(2): 425–461

Fung W, Hsieh D (2004). Hedge fund benchmarks: A risk based approach. Financial Analysts Journal, 60(5): 65–80

Hauptmann J, Hoppenkamps A, Min A, Ramsauer F, Zagst R (2014). Forecasting market turbulences using regime-switching models. Financial Markets and Portfolio Management, 28(2): 139–164

Henriksson R, Merton R (1981). On market timing and investment performance II: Statistical procedures for evaluating forecasting skills. Journal of Business, 54(4): 513–533

Hudson D J (1966). Fitting segmented curves whose join points have to be estimated. Journal of the American Statistical Association, 61(316): 1097–1129

Jacobs B I, Levy K N, Starer D (1999). Long–short portfolio management: An integrated approach. The Journal of Portfolio

Management, 25(2): 23–32

Jaeger L, Wagner C (2005). Factor modeling and benchmarking of hedge funds: Can passive investments in hedge fund strategies deliver? Journal of Alternative Investments, 8(3): 9–36

Kat H M, Amin G S (2001). Hedge fund performance 1990–2000: Do the money machines really add value? In: EFMA 2001 Lugano Meetings

Königsberger K (2003). Analysis 2, vol. 5. Berlin: Springer

Mitchell M, Pulvino T (2001). Characteristics of risk and return in risk arbitrage. The Journal of Finance, 56(6): 2135–2175

Quandt R (1958). The estimation of the parameters of a linear regression system obeying two separate regimes. Journal of the American Statistical Association, 53(284): 873–880

Schneeweis T, Spurgin R (1998). Multifactor analysis of hedge fund, managed futures, and mutual fund return and risk characteristics. Journal of Alternative Investments, 1(2): 1–24

Schwarz G (1978). Estimating the dimension of a model. Annals of Statistics, 6(2): 461–464

Treynor J, Mazuy K (1966). Can mutual funds outguess the market? Harvard Business Review, 44: 131–136

Viterbi A (1967). Error bounds for convolutional codes and an asymptotically optimum decoding algorithm. IEEE Transactions on Information Theory, 13(2): 260–269

Zhang J, Stine R A (2001). Autocovariance structure of Markov regime switching models and model selection. Journal of Time Series Analysis, 22(1): 107–124

Appendix (Model estimation)

• Model 1

In the following context, let $\boldsymbol{x} = (x_i) \in \mathbb{R}^n$ denote the excess market returns and $\boldsymbol{y} = (y_i) \in \mathbb{R}^n$ the excess hedge fund returns with $4 \leqslant n$ and $l \in \mathbb{N}$ with $2 \leqslant l, l+2 \leqslant n$. We define $\boldsymbol{x}_1 := (x_1,..., x_l)^{\mathrm{T}}$, $\boldsymbol{x}_2 := (x_{l+1},..., x_n)^{\mathrm{T}}$, where $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ are defined analogously. Thus, $\boldsymbol{x} = (\boldsymbol{x}_1^{\mathrm{T}}, \boldsymbol{x}_2^{\mathrm{T}})^{\mathrm{T}}$ and $\boldsymbol{y} = (\boldsymbol{y}_1^{\mathrm{T}}, \boldsymbol{y}_2^{\mathrm{T}})^{\mathrm{T}}$. Let the matrix $\boldsymbol{F}$ be:

$$\boldsymbol{F} := \begin{pmatrix} \boldsymbol{F}_1 & 0 \\ 0 & \boldsymbol{F}_2 \end{pmatrix} \in \mathcal{M}(n, 4, \mathbb{R}),$$

$$\boldsymbol{F}_1 := \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_l \end{pmatrix}, \boldsymbol{F}_2 := \begin{pmatrix} 1 & x_{l+1} \\ 1 & x_{l+2} \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}, \tag{A1}$$

where $\boldsymbol{F}_1 \in \mathcal{M}(l, 2, \mathbb{R})$ and $\boldsymbol{F}_2 \in \mathcal{M}(n-l, 2, \mathbb{R})$. For each of the segments, we introduce an individual polynomial of degree 1: $f_i(\boldsymbol{\beta}_i, \boldsymbol{x}) := \sum_{j=0}^{1} \boldsymbol{\beta}_{i,j} x^j$ ($\boldsymbol{x}_i \in I_i$, $\boldsymbol{\beta}_{i,j} \in \mathbb{R}$, for i = 1, 2, j = 0, 1), and $I_1 := [x_1, \alpha]$, $I_2 := [\alpha, x_n]$. The associated parameter vectors for the polynomial coefficients are: $\boldsymbol{\beta}_1 := (\beta_{1,0}, \beta_{1,1})^{\mathrm{T}}$, $\boldsymbol{\beta}_2 := (\beta_{2,0}, \beta_{2,1})^{\mathrm{T}}$, $\boldsymbol{\beta} := (\boldsymbol{\beta}_1^{\mathrm{T}}, \boldsymbol{\beta}_2^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^4$.

Depending on the situation with respect to the breakpoint α, we distinguish the following two constraints of continuity. For $\alpha \in (x_l, x_{l+1})$, we have $f_1(\boldsymbol{\beta}_1, \alpha) - f_2(\boldsymbol{\beta}_2, \alpha) = \boldsymbol{\beta}_1^{\mathrm{T}} \boldsymbol{q}_\alpha - \boldsymbol{\beta}_2^{\mathrm{T}} \boldsymbol{q}_\alpha = \boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{Q}_\alpha = 0$, with $\boldsymbol{q}_\alpha := (1, \alpha)^{\mathrm{T}} \in \mathbb{R}^2$, $\boldsymbol{Q}_\alpha := (\boldsymbol{q}_\alpha^{\mathrm{T}}, -\boldsymbol{q}_\alpha^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^4$. For $\alpha = x_l$, we get the same equations $f_1(\boldsymbol{\beta}_1, x_l) = f_2(\boldsymbol{\beta}_2, x_l)$, where instead of $\boldsymbol{q}_\alpha$ and $\boldsymbol{Q}_\alpha$, we define, respectively, $\boldsymbol{q}_l := (1, x_l)^{\mathrm{T}} \in \mathbb{R}^2$ and $\boldsymbol{Q}_l := (\boldsymbol{q}_l^{\mathrm{T}}, -\boldsymbol{q}_l^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^4$. The overall residual sum of squares for the two segments can be expressed as: $RSS(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) := (\boldsymbol{y} - \boldsymbol{F}\boldsymbol{\beta})^{\mathrm{T}} (\boldsymbol{y} - \boldsymbol{F}\boldsymbol{\beta}) = RSS_1(\boldsymbol{\beta}_1) + RSS_2(\boldsymbol{\beta}_2)$, with $RSS_i(\boldsymbol{\beta}_i) := (\boldsymbol{y}_i - \boldsymbol{F}_i \boldsymbol{\beta}_i)^{\mathrm{T}} (\boldsymbol{y}_i - \boldsymbol{F}_i \boldsymbol{\beta}_i)$, for i = 1, 2.

We can currently provide the theorems that deal with the two cases for the breakpoint α. First, we consider the case Theorem 1 where the breakpoint α lies between two consecutive data points, i.e., $\alpha \in (x_l, x_{l+1})$ for some l. Next, we consider the case Theorem 2 where the breakpoint α coincides with certain data point x_l.

[Theorem 1] If the breakpoint $\alpha \in (x_l, x_{l+1})$ and $RSS(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ is minimized for some $\boldsymbol{\beta} = \boldsymbol{\beta}^* := (\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*)$ under the continuity constraint:

$$f_1(\boldsymbol{\beta}_1^*, \alpha) = f_2(\boldsymbol{\beta}_2^*, \alpha), \tag{A2}$$

and the derivatives of the polynomials are not continuous:

$$\frac{\partial f_1(\boldsymbol{\beta}_1^*, \alpha)}{\partial \alpha} \neq \frac{\partial f_2(\boldsymbol{\beta}_2^*, \alpha)}{\partial \alpha}. \tag{A3}$$

Then the solutions $\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*$ for each segment can be derived as given the lack of constraints:

$$-\boldsymbol{F}_i^{\mathrm{T}} \boldsymbol{y}_i + \boldsymbol{F}_i^{\mathrm{T}} \boldsymbol{F}_i \boldsymbol{\beta}_i^* = 0 \ (i = 1, 2),$$
$$\text{or } \boldsymbol{\beta}_i^* = (\boldsymbol{F}_i^{\mathrm{T}} \boldsymbol{F}_i)^{-1} \boldsymbol{F}_i^{\mathrm{T}} \boldsymbol{y}_i \ (i = 1, 2). \tag{A4}$$

The overall residual sum of squares is expressed as:

$$RSS(\boldsymbol{\beta}^*) = RSS_1(\boldsymbol{\beta}_1^*) + RSS_2(\boldsymbol{\beta}_2^*),$$
$$RSS_i(\boldsymbol{\beta}_i^*) = \boldsymbol{y}_i^{\mathrm{T}} (\boldsymbol{y}_i - \boldsymbol{F}_i \boldsymbol{\beta}_i^*). \tag{A5}$$

Moreover, for $\boldsymbol{\beta} \neq \boldsymbol{\beta}^*$, we have $RSS(\boldsymbol{\beta}) > RSS(\boldsymbol{\beta}^*)$.

We apply the method of Lagrange factor (Königsberger, 2003) to minimize the overall residual sum of squares and fulfill the continuity constraint. The function:

$$\Lambda(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \lambda) = (\boldsymbol{y}_1 - \boldsymbol{F}_1 \boldsymbol{\beta}_1)^{\mathrm{T}} (\boldsymbol{y}_1 - \boldsymbol{F}_1 \boldsymbol{\beta}_1) + (\boldsymbol{y}_2 - \boldsymbol{F}_2 \boldsymbol{\beta}_2)^{\mathrm{T}} (\boldsymbol{y}_2 - \boldsymbol{F}_2 \boldsymbol{\beta}_2) + 2\lambda(\boldsymbol{\beta}_1^{\mathrm{T}} \boldsymbol{q}_\alpha - \boldsymbol{\beta}_2^{\mathrm{T}} \boldsymbol{q}_\alpha), \tag{A6}$$

should be minimized with respect to the two vector parameters $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ and the two scalars α and λ. Given the discontinuity assumption represented by Eq. (A3), the derivative of Λ with respect to the breakpoint α leads to the Lagrange factor $\lambda = 0$. Consequently, the two vector equations involving the derivatives of Λ with respect to $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ become the equations for an unconstrained linear regression in the segments I_1 and I_2 for which we can obtain the unique solutions $\boldsymbol{\beta}_1^*$ and $\boldsymbol{\beta}_2^*$, respectively.

[Theorem 2] Suppose the breakpoint $\alpha = x_l$ for some $l \in \{2,..., n-2\}$ and the overall residual sum of squares $RSS(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ is minimal for some $\boldsymbol{\beta} = \hat{\boldsymbol{\beta}}$ under the continuity constraint:

$$f_1(\boldsymbol{\beta}_1, x_l) = f_2(\boldsymbol{\beta}_2, x_l), \tag{A7}$$

and the derivatives of the polynomials are not continuous:

$$\frac{\partial f_1(\boldsymbol{\beta}_1^*, x_l)}{\partial x_l} \neq \frac{\partial f_2(\boldsymbol{\beta}_2^*, x_l)}{\partial x_l}. \tag{A8}$$

Then the solutions $\hat{\boldsymbol{\beta}}_1, \hat{\boldsymbol{\beta}}_2$ have the following characteristics:

$$\hat{\boldsymbol{\beta}}_1 = \boldsymbol{C}_1^{-1}(\boldsymbol{F}_1^{\mathrm{T}}\boldsymbol{y}_1 - \lambda \boldsymbol{q}_l) = \boldsymbol{\beta}_1^* - \lambda \boldsymbol{C}_1^{-1}\boldsymbol{q}_l,$$
$$\hat{\boldsymbol{\beta}}_2 = \boldsymbol{C}_2^{-1}(\boldsymbol{F}_2^{\mathrm{T}}\boldsymbol{y}_2 + \lambda \boldsymbol{q}_l) = \boldsymbol{\beta}_2^* + \lambda \boldsymbol{C}_2^{-1}\boldsymbol{q}_l, \tag{A9}$$

with $\boldsymbol{C}_i = \boldsymbol{F}_i^{\mathrm{T}}\boldsymbol{F}_i$ and $\boldsymbol{\beta}_i^* = \boldsymbol{C}_i^{-1}\boldsymbol{F}_i^{\mathrm{T}}\boldsymbol{y}_i$ from Theorem 1, $i = 1$, 2, and Lagrange multiplier:

$$\lambda = \frac{\boldsymbol{q}_l^{\mathrm{T}}(\boldsymbol{\beta}_1^* - \boldsymbol{\beta}_2^*)}{\boldsymbol{q}_l^{\mathrm{T}}(\boldsymbol{C}_1^{-1} + \boldsymbol{C}_2^{-1})\boldsymbol{q}_l}. \tag{A10}$$

For the overall residual sum of squares, the following equation holds:

$$RSS(\hat{\boldsymbol{\beta}}) = RSS(\boldsymbol{\beta}^*) + \frac{(\boldsymbol{Q}_l^{\mathrm{T}}\boldsymbol{\beta}^*)^2}{\boldsymbol{Q}_l^{\mathrm{T}}\boldsymbol{C}^{-1}\boldsymbol{Q}_l}. \tag{A11}$$

For each $\boldsymbol{\beta} \neq \hat{\boldsymbol{\beta}}$ satisfying the continuity constraint (A7), we have $RSS(\boldsymbol{\beta}) > RSS(\hat{\boldsymbol{\beta}})$.

The method of Lagrange factor is applied (Königsberger, 2003) to minimize the overall residual sum of squares *RSS* and fulfill the continuity constraint. The function:

$$\Lambda(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \lambda) = (\boldsymbol{y}_1 - \boldsymbol{F}_1\boldsymbol{\beta}_1)^{\mathrm{T}}(\boldsymbol{y}_1 - \boldsymbol{F}_1\boldsymbol{\beta}_1)$$
$$+(\boldsymbol{y}_2 - \boldsymbol{F}_2\boldsymbol{\beta}_2)^{\mathrm{T}}(\boldsymbol{y}_2 - \boldsymbol{F}_2\boldsymbol{\beta}_2) + 2\lambda(\boldsymbol{\beta}_1^{\mathrm{T}}\boldsymbol{q}_l - \boldsymbol{\beta}_2^{\mathrm{T}}\boldsymbol{q}_l), \tag{A12}$$

should be minimized with respect to the two vector parameters $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ and the scalar λ. This equation leads to the estimates and the minimal *RSS* stated in the theorem.

Our weighted segmented linear regression problem reveals that the characteristics of segmented linear regression considered in Theorems 1 and 2 remain valid. If the sample vectors $\boldsymbol{y}_i$, $\boldsymbol{y}$ and the matrices $\boldsymbol{F}_i$, $\boldsymbol{F}$ are replaced by $\boldsymbol{W}_i\boldsymbol{y}_i$, $\boldsymbol{W}\boldsymbol{y}$, $\boldsymbol{W}_i\boldsymbol{F}_i$, and $\boldsymbol{W}\boldsymbol{F}$, respectively, where $\boldsymbol{W}$ is an appropriately defined block diagonal matrix incorporating the weights used.

• Model 2

For model 2, a Markov-switching model is estimated on the market returns with the model described in Eq. (5). According to this market separation, the vector $\boldsymbol{x}$ is divided into $\boldsymbol{x}^{(0)} := (x_i^{(0)})$ defining the calm market and $\boldsymbol{x}^{(1)} := (x_i^{(1)})$ defining the turbulent market. Then, this separation is applied to the dependent variable vector $\boldsymbol{y}$ leading to the respective subsets, i.e., $\boldsymbol{y}^{(0)} := (y_i^{(0)})$ and $\boldsymbol{y}^{(1)} := (y_i^{(1)})$.

Next, we define the weighting matrices for each market to perform a weighted segmented linear regression model with the shared breakpoint α on each market.

Let $n_i^{(s)}$ denote the number of data points in the ith segment of market s and $n^{(s)}$ the total number of data points in market s. Thus, $n^{(0)} = n_1^{(0)} + n_2^{(0)}$, $n^{(1)} = n_1^{(1)} + n_2^{(1)}$, $n = n^{(0)} + n^{(1)}$. The index $n_1^{(s)}$ separates each vector $\boldsymbol{x}^{(s)}$ into two segments: $x_{n_1^{(s)}}^{(s)} \leqslant \alpha < x_{n_1^{(s)}+1}^{(s)}$.

The parameter estimates of the calm and the turbulent market model are defined as $\boldsymbol{\beta}_i^{(s)} := \left(\beta_{i,0}^{(s)}, \beta_{i,1}^{(s)}\right)^{\mathrm{T}}$, $i \in \{1, 2\}$ and $s \in \{0, 1\}$, where the upper index indicates the market and the lower index stands for the segment. This indexing is further used for all other variables and parameters to be introduced. The market weights are determined as $w_i^{(s)} := \dfrac{n_i^{(s)}}{n_i^{(0)} + n_i^{(1)}}$, $i \in \{1, 2\}$, $s \in \{0, 1\}$. It holds $w_i^{(0)} + w_i^{(1)} = 1$, $i \in \{1, 2\}$.

[Theorem 3] The optimization problem for Model 2 is given by the minimization of the following residuals sum of squares:

$$\min_{\boldsymbol{\beta}_1^{(0)}, \boldsymbol{\beta}_2^{(0)}, \boldsymbol{\beta}_1^{(1)}, \boldsymbol{\beta}_2^{(1)}} RSS\left(\boldsymbol{\beta}_1^{(0)}, \boldsymbol{\beta}_2^{(0)}, \boldsymbol{\beta}_1^{(1)}, \boldsymbol{\beta}_2^{(1)}\right), \tag{A13}$$

where the residual sum of squares is defined by:

$$RSS\left(\boldsymbol{\beta}_1^{(0)}, \boldsymbol{\beta}_2^{(0)}, \boldsymbol{\beta}_1^{(1)}, \boldsymbol{\beta}_2^{(1)}\right)$$
$$:= \sum_{i=1}^{n_1^{(0)}} \left(\rho_i^{(0)}\left(y_i^{(0)} - f_1^{(0)}\left(\boldsymbol{\beta}_1^{(0)}, x_i^{(0)}\right)\right)\right)^2$$
$$+ \sum_{i=n_1^{(0)}+1}^{n^{(0)}} \left(\rho_i^{(0)}\left(y_i^{(0)} - f_2^{(0)}\left(\boldsymbol{\beta}_2^{(0)}, x_i^{(0)}\right)\right)\right)^2$$
$$+ \sum_{i=1}^{n_1^{(1)}} \left(\rho_i^{(1)}\left(y_i^{(1)} - f_1^{(1)}\left(\boldsymbol{\beta}_1^{(1)}, x_i^{(1)}\right)\right)\right)^2$$
$$+ \sum_{i=n_1^{(1)}+1}^{n^{(1)}} \left(\rho_i^{(1)}\left(y_i^{(1)} - f_2^{(1)}\left(\boldsymbol{\beta}_2^{(1)}, x_i^{(1)}\right)\right)\right)^2, \tag{A14}$$

under the validity of the following constraints:

• The weighted market constraints:

$$\boldsymbol{\beta}_1 = w_1^{(0)}\boldsymbol{\beta}_1^{(0)} + w_1^{(1)}\boldsymbol{\beta}_1^{(1)},$$
$$\boldsymbol{\beta}_2 = w_2^{(0)}\boldsymbol{\beta}_2^{(0)} + w_2^{(1)}\boldsymbol{\beta}_2^{(1)}. \quad \text{(A15)}$$

• The continuity constraints:

$$\left(\boldsymbol{\beta}_1^{(0)\mathrm{T}} - \boldsymbol{\beta}_2^{(0)\mathrm{T}}\right)\boldsymbol{q} = 0,$$
$$\left(\boldsymbol{\beta}_1^{(1)\mathrm{T}} - \boldsymbol{\beta}_2^{(1)\mathrm{T}}\right)\boldsymbol{q} = 0. \quad \text{(A16)}$$

For each market s, $s \in \{0, 1\}$, the weight $\rho_i^{(s)}$, $i \in \{1, 2, ..., n^{(s)}\}$ is computed according to Eq. (3). The solution can be found by solving the following inhomogeneous linear regression problem for the vector of Lagrange factors $(\lambda_0, \lambda_1)^{\mathrm{T}}$:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}, \quad \text{(A17)}$$

where the definition of a_{kj}, $k, j \in \{1, 2\}$ and z_i, $i \in \{0, 1\}$ is given in the following proof.

Rewriting the residual sum of squares in matrix notation leads to the following equations:

$$RSS\left(\boldsymbol{\beta}_1^{(0)}, \boldsymbol{\beta}_2^{(0)}, \boldsymbol{\beta}_1^{(1)}, \boldsymbol{\beta}_2^{(1)}\right) =$$
$$\sum_{s=0}^{1}\sum_{i=1}^{2}\left(\left(\boldsymbol{y}_i^{(s)} - \boldsymbol{F}_i^{(s)}\boldsymbol{\beta}_i^{(s)}\right)\right)^{\mathrm{T}}\left(\left(\boldsymbol{y}_i^{(s)} - \boldsymbol{F}_i^{(s)}\boldsymbol{\beta}_i^{(s)}\right)\right), \quad \text{(A18)}$$

where $\boldsymbol{y}$ and $\boldsymbol{F}$ now involve the weights $\rho_i^{(s)}$, $i \in \{1, 2,..., n^{(s)}\}$ introduced above. We apply the method of Lagrange to find the least square estimates under the given continuity constraints:

$$\Lambda\left(\boldsymbol{\beta}_1^{(0)}, \boldsymbol{\beta}_2^{(0)}, \boldsymbol{\beta}_1^{(1)}, \boldsymbol{\beta}_2^{(1)}, \lambda_0, \lambda_1, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2\right)$$
$$= \sum_{s=0}^{1}\sum_{i=1}^{2}\left(\left(\boldsymbol{y}_i^{(s)} - \boldsymbol{F}_i^{(s)}\boldsymbol{\beta}_i^{(s)}\right)\right)^{\mathrm{T}}\left(\left(\boldsymbol{y}_i^{(s)} - \boldsymbol{F}_i^{(s)}\boldsymbol{\beta}_i^{(s)}\right)\right)$$
$$+ 2\lambda_0\left(\boldsymbol{\beta}_1^{(0)\mathrm{T}} - \boldsymbol{\beta}_2^{(0)\mathrm{T}}\right)\boldsymbol{q} + 2\lambda_1\left(\boldsymbol{\beta}_1^{(1)\mathrm{T}} - \boldsymbol{\beta}_2^{(1)\mathrm{T}}\right)\boldsymbol{q}$$
$$+ 2\left(w_1^{(0)}\boldsymbol{\beta}_1^{(0)\mathrm{T}} + w_1^{(1)}\boldsymbol{\beta}_1^{(1)\mathrm{T}} - \boldsymbol{\beta}_1^{\mathrm{T}}\right)\boldsymbol{\kappa}_1$$
$$+ 2\left(w_2^{(0)}\boldsymbol{\beta}_2^{(0)\mathrm{T}} + w_2^{(1)}\boldsymbol{\beta}_2^{(1)\mathrm{T}} - \boldsymbol{\beta}_2^{\mathrm{T}}\right)\boldsymbol{\kappa}_2. \quad \text{(A19)}$$

By introducing the following short notations:

$$\boldsymbol{C}_i^{(s)} := \boldsymbol{F}_i^{(s)\mathrm{T}}\boldsymbol{F}_i^{(s)},$$
$$\boldsymbol{d}_i^{(s)} := \boldsymbol{F}_i^{(s)\mathrm{T}}\boldsymbol{y}_i^{(s)},$$
$$b^{(s)} := \boldsymbol{q}^{\mathrm{T}}\left((\boldsymbol{C}_1^{(s)})^{-1} + (\boldsymbol{C}_2^{(s)})^{-1}\right)\boldsymbol{q},$$
$$a^{(s)} := \boldsymbol{q}^{\mathrm{T}}\left((\boldsymbol{C}_1^{(s)})^{-1}\boldsymbol{d}_1^{(s)} - (\boldsymbol{C}_2^{(s)})^{-1}\boldsymbol{d}_2^{(s)}\right),$$
$$\boldsymbol{D}_i := \left((w_i^{(0)})^2(\boldsymbol{C}_i^{(0)})^{-1} + (w_i^{(1)})^2(\boldsymbol{C}_i^{(1)})^{-1}\right),$$
$$\boldsymbol{\kappa}_i := (\boldsymbol{D}_i)^{-1}\left[\sum_{s=0}^{1} w_i^{(s)}(\boldsymbol{C}_i^{(s)})^{-1}\boldsymbol{d}_i^{(s)} - \boldsymbol{\beta}_i\right.$$
$$\left. - \sum_{s=0}^{1}\lambda_s(-1)^{i-1}w_i^{(s)}(\boldsymbol{C}_i^{(s)})^{-1}\boldsymbol{q}\right], \quad \text{(A20)}$$

we obtain the inhomogeneous linear equation for λ_0, λ_1:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix},$$

with

$$a_{kj} = \sum_{i=1}^{2}[w_i^{(k-1)}\boldsymbol{q}^{\mathrm{T}}(\boldsymbol{C}_i^{(k-1)})^{-1}\boldsymbol{D}_i^{-1}w_i^{(j-1)}(\boldsymbol{C}_i^{(j-1)})^{-1}\boldsymbol{q}]$$
$$- \delta_{kj}\boldsymbol{b}^{(k-1)}, \quad \text{(A21)}$$

i.e., the indices k, j reflect the calm and the turbulent market. The components of the vector (z_0, z_1) are given by:

$$z_0 = (-1)\left[a^{(0)} - w_1^{(0)}\boldsymbol{q}^{\mathrm{T}}(\boldsymbol{C}_1^{(0)})^{-1}\boldsymbol{D}_1^{-1}\right.$$
$$\left(\sum_{s=0}^{1} w_1^{(s)}(\boldsymbol{C}_1^{(s)})^{-1}\boldsymbol{d}_1^{(s)} - \boldsymbol{\beta}_1\right)$$
$$+ w_2^{(0)}\boldsymbol{q}^{\mathrm{T}}(\boldsymbol{C}_2^{(0)})^{-1}\boldsymbol{D}_2^{-1}$$
$$\left.\left(\sum_{s=0}^{1} w_2^{(s)}(\boldsymbol{C}_2^{(s)})^{-1}\boldsymbol{d}_2^{(s)} - \boldsymbol{\beta}_2\right)\right],$$
$$z_1 = (-1)\left[a^{(1)} - w_1^{(1)}\boldsymbol{q}^{\mathrm{T}}(\boldsymbol{C}_1^{(1)})^{-1}\boldsymbol{D}_1^{-1}\right.$$
$$\left(\sum_{s=1}^{1} w_1^{(s)}(\boldsymbol{C}_1^{(s)})^{-1}\boldsymbol{d}_1^{(s)} - \boldsymbol{\beta}_1\right)$$
$$+ w_2^{(1)}\boldsymbol{q}^{\mathrm{T}}(\boldsymbol{C}_2^{(1)})^{-1}\boldsymbol{D}_2^{-1}$$
$$\left.\left(\sum_{s=1}^{1} w_2^{(s)}(\boldsymbol{C}_2^{(s)})^{-1}\boldsymbol{d}_2^{(s)} - \boldsymbol{\beta}_2\right)\right]. \quad \text{(A22)}$$

By solving the linear inhomogeneous equation system for λ_1, λ_2, we can solve for $\boldsymbol{\beta}_i^{(s)}$:

$$\boldsymbol{\beta}_i^{(s)} = (\boldsymbol{C}_i^{(s)})^{-1}\left(\boldsymbol{d}_i^{(s)} - (-1)^{i-1}\lambda_s\boldsymbol{q} - w_i^{(s)}\boldsymbol{\kappa}_i\right). \quad \text{(A23)}$$