Front. Electr. Electron. Eng. China 2010, 5(2): 207–217
DOI 10.1007/s11460-010-0013-2

RESEARCH ARTICLE

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Equivalence of stability criteria for time-delay systems

Abstract During the past few decades, there have been a number of new delay-dependent stability criteria based on linear matrix inequalities published in the literature. These criteria, to some extent, can reduce the conservativeness. In fact, some criteria are equivalence. In this paper, we aim to theoretically establish equivalence of two stability criteria. One was obtained by Sun et al. [Sun et al. In: Proceedings of IEEE Power & Energy Society General Meeting. 2009, 1–7] and the other was given by Xu et al. [Xu et al. IEEE Transactions on Automatic Control, 2005, 50(3): 384–387]. Also, we theoretically establish equivalence of the robust stability criterion presented by Jia et al. [Jia et al. Automation of Electric Power Systems, 2010, 34(3): 6–11] and the one given by Xu et al. (2005).

Keywords delay-dependent stability criterion, robust stability, time-delay systems, linear matrix inequality (LMI)

1 Introduction

During the past few years, considerable attention has been devoted to the problem of delay-dependent stability analysis and controller design for time-delay systems due to the fact that time delay is often the main cause for instability and poor performance of control systems [1–18]. Many stability results for time-delay systems have been proposed, see Refs. [3–5,9] and the references therein. It is noted that stability criteria for delay systems can be classified into two categories according to their dependence on the information of time delays, namely, delay-dependent stability criteria and delay-independent stability criteria. Usually, the former are less conservative than the latter since they make use of the information on the length of time delays. Therefore, considerable attention has been paid on the derivation of delay-dependent stability results. Also, various approaches have been proposed in recent years. For example, Ref. [17] developed a unified linear matrix inequality (LMI) approach to establish sufficient conditions for the neural networks to be globally, robustly and exponentially stable. Reference [10] gave a criterion for time-delay system with single time delay by introducing some slack variables. In the direction, Ref. [19] formed the so-called free-weighting matrices method and provided stability criteria for the system with two time delays. Further, the method was improved to be suitable for the time-varying delay in Refs. [4,8,9]. For instance, in Ref. [9], both terms $d(t)$ and $h-d(t)$ were enlarged as h. Because $d(t)$ and $h-d(t)$ have an important relationship that their sum is h, there is room for further investigation.

It was observed by us that there were some redundant variables in the free-weighting matrices method during its derivation; omitting such redundant variables, we got an improved delay-dependent stability criterion for the system with multiple time delays in Ref. [13]. Based on the method in Ref. [13], we can get time-delay stability criterion for a system with single time delay, see Lemma 2 in this paper. Through numerical study, we can find that its conservativeness is the same as that in Ref. [10]; however, its effectiveness is significantly improved. Furthermore, we will theoretically establish the equivalence of two different criteria in Ref. [10] and in this paper or in Ref. [20].

Notation Throughout this paper, for real symmetric matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$, the notation $\boldsymbol{X} \geqslant \boldsymbol{Y}$ (respectively, $\boldsymbol{X} > \boldsymbol{Y}$) means that matrix $\boldsymbol{X}-\boldsymbol{Y}$ is positive semidefinite (respectively, positive definite). $\boldsymbol{I}$ is an identity matrix with appropriate dimension. Superscript "T" represents the transpose. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2 Main results

Consider the following time-delay system:

$$(\Sigma):\begin{cases}\dot{\boldsymbol{x}}(t)=\boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{A}_\tau\boldsymbol{x}(t-\tau), & (1)\\ \boldsymbol{x}(t)=\boldsymbol{\varphi}(t), & (2)\end{cases}$$

Received January 17, 2010; accepted March 9, 2010

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where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the vector of state variable, and $\boldsymbol{\varphi}(t)$ is the initial condition. The scalar $\tau > 0$ is the constant delay of the system, $\boldsymbol{A}$ and $\boldsymbol{A}_\tau$ are known real constant matrices.

Lemma 1 [10] The time-delay system (Σ) is asymptotically stable for any delay τ satisfying $0 < \tau \leqslant \bar{\tau}$ if there exist matrices $\boldsymbol{P} > 0$, $\boldsymbol{Q} > 0$, $\boldsymbol{Z} > 0$, $\boldsymbol{Y}$ and $\boldsymbol{W}$ such that the LMI shown in the following inequality holds:

$$\begin{bmatrix} \boldsymbol{PA} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{Y} + \boldsymbol{Y}^{\mathrm{T}} + \boldsymbol{Q} & \boldsymbol{PA}_\tau - \boldsymbol{Y} + \boldsymbol{W}^{\mathrm{T}} & -\bar{\tau}\boldsymbol{Y} & \tau\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Z} \\ \boldsymbol{A}_\tau^{\mathrm{T}}\boldsymbol{P} - \boldsymbol{Y}^{\mathrm{T}} + \boldsymbol{W} & -\boldsymbol{Q} - \boldsymbol{W} - \boldsymbol{W}^{\mathrm{T}} & -\bar{\tau}\boldsymbol{W} & \tau\boldsymbol{A}_\tau^{\mathrm{T}}\boldsymbol{Z} \\ \bar{\tau}\boldsymbol{Y} & -\bar{\tau}\boldsymbol{W} & \bar{\tau}\boldsymbol{Z} & 0 \\ \tau\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Z} & \tau\boldsymbol{A}_\tau^{\mathrm{T}}\boldsymbol{Z} & 0 & \bar{\tau}\boldsymbol{Z} \end{bmatrix} < 0. \tag{3}$$

Lemma 2 The time-delay system (Σ) is asymptotically stable for any delay τ satisfying $0 < \tau \leqslant \bar{\tau}$ if there exist matrices $\boldsymbol{P} > 0$, $\boldsymbol{Q} > 0$, $\boldsymbol{Z} > 0$, $\boldsymbol{Y}$ and $\boldsymbol{W}$ such that the LMIs given in the following two inequalities hold:

$$\begin{bmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}_{12} & \boldsymbol{N}_1 \\ \boldsymbol{X}_{12}^{\mathrm{T}} & \boldsymbol{X}_{22} & \boldsymbol{N}_2 \\ \boldsymbol{X}_{13}^{\mathrm{T}} & \boldsymbol{X}_{23}^{\mathrm{T}} & \boldsymbol{W} \end{bmatrix} > 0, \tag{4}$$

$$\begin{bmatrix} \boldsymbol{PA} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{Q}_1 + \boldsymbol{N}_1 + \boldsymbol{N}_1^{\mathrm{T}} + \bar{\tau}\boldsymbol{X}_{11} + \bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{WA} & \boldsymbol{PA}_\tau - \boldsymbol{N}_1 + \boldsymbol{N}_2^{\mathrm{T}} + \bar{\tau}\boldsymbol{X}_{12} + \bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{WA}_\tau \\ \boldsymbol{A}_\tau^{\mathrm{T}} - \boldsymbol{N}_1^{\mathrm{T}} + \boldsymbol{N}_2 + \bar{\tau}\boldsymbol{X}_{12}^{\mathrm{T}} + \bar{\tau}\boldsymbol{A}_\tau^{\mathrm{T}}\boldsymbol{WA} & -\boldsymbol{Q}_1 - \boldsymbol{N}_2 - \boldsymbol{N}_2^{\mathrm{T}} + \bar{\tau}\boldsymbol{X}_{22} + \bar{\tau}\boldsymbol{A}_\tau^{\mathrm{T}}\boldsymbol{WA}_\tau \end{bmatrix} < 0. \tag{5}$$

Consider a time-delay system in the form of Eq. (1) with

$$\boldsymbol{A} = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad \boldsymbol{A}_\tau = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$$

Using Lemma 1 and Lemma 2, we can get the same results, that is, when system (Σ) is stable, the maximum time delays are both $\bar{\tau} = 4.47$.

Further, let us consider the uncertain time-delay system $(\hat{\Sigma})$ as follows:

$$(\hat{\Sigma}): \begin{cases} \dot{\boldsymbol{x}}(t) = [\boldsymbol{A}_0 + \Delta\boldsymbol{A}_0(t)]\boldsymbol{x}(t) + [\boldsymbol{A}_1 + \Delta\boldsymbol{A}_1(t)]\boldsymbol{x}(t - \tau_1), \\ \boldsymbol{x}(t) = \boldsymbol{\varphi}(t), \quad \forall t \in [-\tau, 0], \end{cases} \tag{6}$$

with $[\Delta\boldsymbol{A}_0(t) \quad \Delta\boldsymbol{A}_1(t)] = \boldsymbol{DF}(t)[\boldsymbol{E}_0 \quad \boldsymbol{E}_1]$, $\boldsymbol{F}^{\mathrm{T}}(t)\boldsymbol{F}(t) \leqslant \boldsymbol{I}$ for any t.

Lemma 3 [10] The uncertain time-delay system $(\hat{\Sigma})$ is robustly asymptotically stable for any delay τ satisfying $0 < \tau < \bar{\tau}$ if there exist a scalar $\varepsilon > 0$ and matrices $\boldsymbol{P} > 0$, $\boldsymbol{Q} > 0$, $\boldsymbol{Z} > 0$, $\boldsymbol{Y}$ and $\boldsymbol{W}$ such that the LMI given in the following inequality holds:

$$\begin{bmatrix} \boldsymbol{PA} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{Y} + \boldsymbol{Y}^{\mathrm{T}} + \boldsymbol{Q} + \varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E} & \boldsymbol{PA}_\tau - \boldsymbol{Y} + \boldsymbol{W}^{\mathrm{T}} + \varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}_\tau & -\bar{\tau}\boldsymbol{Y} & \bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Z} & \boldsymbol{PD} \\ \boldsymbol{A}_\tau^{\mathrm{T}}\boldsymbol{P} - \boldsymbol{Y}^{\mathrm{T}} + \boldsymbol{W} + \varepsilon\boldsymbol{E}_\tau^{\mathrm{T}}\boldsymbol{E} & -\boldsymbol{Q} - \boldsymbol{W} - \boldsymbol{W}^{\mathrm{T}} + \varepsilon\boldsymbol{E}_\tau^{\mathrm{T}}\boldsymbol{E}_\tau & -\bar{\tau}\boldsymbol{W} & -\bar{\tau}\boldsymbol{A}_\tau^{\mathrm{T}}\boldsymbol{Z} & 0 \\ -\bar{\tau}\boldsymbol{Y}^{\mathrm{T}} & -\bar{\tau}\boldsymbol{W}^{\mathrm{T}} & -\bar{\tau}\boldsymbol{Z} & 0 & 0 \\ \bar{\tau}\boldsymbol{ZA} & \bar{\tau}\boldsymbol{ZA}_\tau & 0 & -\bar{\tau}\boldsymbol{Z} & \bar{\tau}\boldsymbol{ZD} \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} & 0 & 0 & -\bar{\tau}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{Z} & -\varepsilon\boldsymbol{I} \end{bmatrix} < 0. \tag{7}$$

Lemma 4 [20] The uncertain time-delay system $(\hat{\Sigma})$ is robustly asymptotically stable for any delay τ satisfying $0 < \tau < \bar{\tau}$ if there exist a scalar $\varepsilon > 0$ and matrices $\boldsymbol{P} > 0, \boldsymbol{Q} > 0, \boldsymbol{Z} > 0$, $\boldsymbol{Y}$ and $\boldsymbol{W}$ such that the LMIs given in the following two inequalities hold:

$$\begin{bmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}_{12} & \boldsymbol{N}_1 \\ \boldsymbol{X}_{12}^{\mathrm{T}} & \boldsymbol{X}_{22} & \boldsymbol{N}_2 \\ \boldsymbol{X}_{13}^{\mathrm{T}} & \boldsymbol{X}_{23}^{\mathrm{T}} & \boldsymbol{W} \end{bmatrix} > 0, \tag{8}$$

$$\begin{bmatrix} \boldsymbol{PA} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{Q}_1 + \boldsymbol{N}_1 + \boldsymbol{N}_1^{\mathrm{T}} + \bar{\tau}\boldsymbol{X}_{11} + \varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E} & \boldsymbol{PA}_1 - \boldsymbol{N}_1 + \boldsymbol{N}_2^{\mathrm{T}} + \bar{\tau}\boldsymbol{X}_{12} + \varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}_1 & \bar{\tau}\boldsymbol{W}_1\boldsymbol{A}^{\mathrm{T}} & \boldsymbol{PD} \\ \boldsymbol{A}_1^{\mathrm{T}} - \boldsymbol{N}_1^{\mathrm{T}} + \boldsymbol{N}_2 + \bar{\tau}\boldsymbol{X}_{12}^{\mathrm{T}} + \varepsilon\boldsymbol{E}_1^{\mathrm{T}}\boldsymbol{E} & -\boldsymbol{Q}_1 - \boldsymbol{N}_2 - \boldsymbol{N}_2^{\mathrm{T}} + \bar{\tau}\boldsymbol{X}_{22} + \varepsilon\boldsymbol{E}_1^{\mathrm{T}}\boldsymbol{E}_1 & \bar{\tau}\boldsymbol{W}_1\boldsymbol{A}_1^{\mathrm{T}} & 0 \\ \bar{\tau}\boldsymbol{W}_1\boldsymbol{A} & \bar{\tau}\boldsymbol{W}_1\boldsymbol{A}_1 & -\bar{\tau}\boldsymbol{W}_1 & \bar{\tau}\boldsymbol{W}_1\boldsymbol{D} \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} & 0 & \bar{\tau}\boldsymbol{W}_1\boldsymbol{D}^{\mathrm{T}} & -\varepsilon\boldsymbol{I} \end{bmatrix} < 0. \tag{9}$$

Consider the uncertain time-delay system with

$$\boldsymbol{A} = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix}, \quad \boldsymbol{A}_1 = \begin{bmatrix} -1 & 0 \\ -0.8 & -1 \end{bmatrix},$$

$$\boldsymbol{D} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad \boldsymbol{E} = \boldsymbol{E}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Table 1 shows the results given by us and those by Refs. [10,12,21]. It can be found that our results are also the same as those in Ref. [10]. In the following, we will theoretically establish equivalence of Lemma 1 and Lemma 2, Lemma 3 and Lemma 4, respectively.

Table 1 Comparison of robust delay-dependent stability conditions

λ	$\bar{\tau}$ in Ref. [12]	$\bar{\tau}$ in Ref. [21]	$\bar{\tau}$ in Ref. [10]	$\bar{\tau}$ in this paper
0.30	0.9288	0.9288	0.9514	0.9514
0.35	0.8324	0.8324	0.8711	0.8711
0.40	0.7342	0.7342	0.7950	0.7950
0.45	0.6242	0.6242	0.7210	0.7210
0.50	0.4903	0.4903	0.6426	0.6426
0.55	0.3201	0.3201	0.5325	0.5325
0.60	0.1027	0.1027	0.2087	0.2087

Although the aforementioned LMI-based stability criteria are obtained via different methods and with different appearances, their results are equivalent. To show this, we first show that Lemma 1 is equivalent to Lemma 2, which is given in Theorem 1 as follows.

Theorem 1 There exist matrices $\boldsymbol{P} > 0, \boldsymbol{Q}_1 > 0, \boldsymbol{N}_1, \boldsymbol{N}_2, \boldsymbol{X}_{ij}\ (i,j = 1,2)$ and $\boldsymbol{W}$ such that both Eq. (4) and Eq. (5) hold if and only if there exist matrices $\boldsymbol{P} > 0$, $\boldsymbol{Q} > 0$, $\boldsymbol{Z} > 0$, $\boldsymbol{Y}$ and $\boldsymbol{W}$ such that Eq. (3) holds.

Proof Consider the following three inequalities:

$$\begin{bmatrix} \boldsymbol{PA}_0 + \boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{Q}_1 + \boldsymbol{N}_1 + \boldsymbol{N}_1^{\mathrm{T}} & \boldsymbol{PA}_1 - \boldsymbol{N}_1 + \boldsymbol{N}_2^{\mathrm{T}} & \bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{W} & \bar{\tau}\boldsymbol{N}_1 \\ \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{N}_2 - \boldsymbol{N}_1^{\mathrm{T}} & -\boldsymbol{Q}_1 - \boldsymbol{N}_2 - \boldsymbol{N}_2^{\mathrm{T}} & \bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{W} & \bar{\tau}\boldsymbol{N}_2 \\ \bar{\tau}\boldsymbol{WA} & \bar{\tau}\boldsymbol{WA}_1 & -\bar{\tau}\boldsymbol{W} & 0 \\ \bar{\tau}\boldsymbol{N}_1^{\mathrm{T}} & \bar{\tau}\boldsymbol{N}_2^{\mathrm{T}} & 0 & \bar{\tau}\boldsymbol{W} \end{bmatrix} < 0, \tag{10}$$

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{Y}+\boldsymbol{Y}^{\mathrm{T}}+\boldsymbol{Q} & \boldsymbol{PA}_1-\boldsymbol{Y}+\boldsymbol{W}^{\mathrm{T}} & -\bar{\tau}\boldsymbol{Y} & \bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Z} \\ \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{Y}^{\mathrm{T}}+\boldsymbol{W} & -\boldsymbol{Q}-\boldsymbol{W}-\boldsymbol{W}^{\mathrm{T}} & -\bar{\tau}\boldsymbol{W} & \bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{Z} \\ -\bar{\tau}\boldsymbol{Y}^{\mathrm{T}} & -\bar{\tau}\boldsymbol{W}^{\mathrm{T}} & -\bar{\tau}\boldsymbol{Z} & 0 \\ -\bar{\tau}\boldsymbol{ZA} & -\bar{\tau}\boldsymbol{ZA}_1 & 0 & -\bar{\tau}\boldsymbol{Z} \end{bmatrix} < 0, \tag{11}$$

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{Y}+\boldsymbol{Y}^{\mathrm{T}}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{ZA}+\boldsymbol{Q} & \boldsymbol{PA}_1-\boldsymbol{Y}+\boldsymbol{W}^{\mathrm{T}}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{ZA}_1 & -\bar{\tau}\boldsymbol{Y} \\ \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{Y}^{\mathrm{T}}+\boldsymbol{W}+\bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{ZA} & \bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{ZA}_1-\boldsymbol{Q}-\boldsymbol{W}-\boldsymbol{W}^{\mathrm{T}} & -\bar{\tau}\boldsymbol{W} \\ -\bar{\tau}\boldsymbol{Y}^{\mathrm{T}} & -\bar{\tau}\boldsymbol{W}^{\mathrm{T}} & -\bar{\tau}\boldsymbol{Z} \end{bmatrix} < 0. \tag{12}$$

We can show that there exist matrices $\boldsymbol{P}>0$, $\boldsymbol{Q}_1>0$, $\boldsymbol{N}_1$, $\boldsymbol{N}_2$, $\boldsymbol{X}_{ij}$ $(i,j=1,2)$ and $\boldsymbol{W}$ such that both Eq. (4) and Eq. (5) hold if and only if there exist matrices $\boldsymbol{P}>0$, $\boldsymbol{Q}>0$, $\boldsymbol{Z}>0$, $\boldsymbol{Y}$ and $\boldsymbol{W}$ such that Eq. (11) holds. We can get Eq. (10) from Eqs. (4) and (5). At the same time, we give Eq. (12) by applying Schur complement to Eq. (11). Thus, in order to prove Theorem 1, we only need to show that Eq. (10) is equivalent to Eq. (12).

First, we show that Eq. (10) implies Eq. (12). Pre- and post-multiplying Eq. (10) by

$$\begin{bmatrix} \boldsymbol{I} & 0 & \boldsymbol{A}^{\mathrm{T}} & 0 \\ 0 & \boldsymbol{I} & 0 & 0 \\ 0 & 0 & \boldsymbol{I} & 0 \\ 0 & 0 & 0 & \boldsymbol{I} \end{bmatrix}$$

and its transpose, respectively, we can obtain

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{Q}_1+\boldsymbol{N}_1+\boldsymbol{N}_1^{\mathrm{T}}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{WA}_1 & \boldsymbol{PA}_1-\boldsymbol{N}_1+\boldsymbol{N}_2^{\mathrm{T}}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{WA}_1 & 0 & \bar{\tau}\boldsymbol{N}_1 \\ \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{N}_1^{\mathrm{T}}+\boldsymbol{N}_2+\bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{WA} & -\boldsymbol{Q}_1-\boldsymbol{N}_2-\boldsymbol{N}_2^{\mathrm{T}} & \bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{W} & \bar{\tau}\boldsymbol{N}_2 \\ 0 & \bar{\tau}\boldsymbol{WA}_1 & -\bar{\tau}\boldsymbol{W} & 0 \\ \bar{\tau}\boldsymbol{N}_1^{\mathrm{T}} & \bar{\tau}\boldsymbol{N}_2^{\mathrm{T}} & 0 & -\bar{\tau}\boldsymbol{W} \end{bmatrix} < 0. \tag{13}$$

By Eq. (13), it is easy to see that there exists a scalar $\alpha>0$ such that the following inequality holds:

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{Q}_1+\boldsymbol{N}_1+\boldsymbol{N}_1^{\mathrm{T}}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{WA}_1+\alpha\boldsymbol{I} & 0 & \boldsymbol{PA}_1-\boldsymbol{N}_1+\boldsymbol{N}_2^{\mathrm{T}}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{WA}_1 & 0 & \bar{\tau}\boldsymbol{N}_1 \\ 0 & -\alpha\boldsymbol{I} & 0 & 0 & 0 \\ \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{N}_1^{\mathrm{T}}+\boldsymbol{N}_2+\bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{WA} & 0 & -\boldsymbol{Q}_1-\boldsymbol{N}_2-\boldsymbol{N}_2^{\mathrm{T}} & \bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{W} & \bar{\tau}\boldsymbol{N}_2 \\ 0 & 0 & \bar{\tau}\boldsymbol{WA}_1 & -\bar{\tau}\boldsymbol{W} & 0 \\ \bar{\tau}\boldsymbol{N}_1^{\mathrm{T}} & 0 & \bar{\tau}\boldsymbol{N}_2^{\mathrm{T}} & 0 & \bar{\tau}\boldsymbol{W} \end{bmatrix} < 0. \tag{14}$$

Pre- and post-multiplying Eq. (14) by

$$\begin{bmatrix} \boldsymbol{I} & 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{I} & \boldsymbol{I} & \boldsymbol{A}_1^{\mathrm{T}} & 0 \\ 0 & 0 & \boldsymbol{I} & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{I} & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{I} \end{bmatrix}$$

and its transpose, respectively. Then, the result is pre- and post-multiplied by

$$\begin{bmatrix} \boldsymbol{I} & 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\boldsymbol{I} \end{bmatrix}$$

and its transpose, respectively, the following inequality is true:

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{Q}_1+\boldsymbol{N}_1+\boldsymbol{N}_1^{\mathrm{T}}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{WA}+\alpha\boldsymbol{I} & \boldsymbol{PA}_1-\boldsymbol{N}_1+\boldsymbol{N}_2^{\mathrm{T}}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{WA}_1 & -\bar{\tau}\boldsymbol{N}_1 \\ \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{N}_1^{\mathrm{T}}+\boldsymbol{N}_2+\bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{WA} & -\alpha\boldsymbol{I}-\boldsymbol{Q}-\boldsymbol{N}_2-\boldsymbol{N}_2^{\mathrm{T}}+\bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{WA}_1 & -\bar{\tau}\boldsymbol{N}_2 \\ -\bar{\tau}\boldsymbol{N}_1^{\mathrm{T}} & -\bar{\tau}\boldsymbol{N}_2^{\mathrm{T}} & -\bar{\tau}\boldsymbol{W} \end{bmatrix} < 0. \tag{15}$$

Now, choose

$$\boldsymbol{N}_1=\boldsymbol{Y}, \quad \boldsymbol{N}_2=\boldsymbol{W}, \quad \boldsymbol{Q}=\boldsymbol{Q}_1+\alpha\boldsymbol{I}, \quad \boldsymbol{W}=\boldsymbol{Z}. \tag{16}$$

Then, by Eq. (16), it is easy to see that $\boldsymbol{Q}_1$, $\boldsymbol{Z}$, $\boldsymbol{Y}$ and $\boldsymbol{W}$ given in the following inequality satisfy (13):

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{Y}+\boldsymbol{Y}^{\mathrm{T}}+\boldsymbol{Q}_1+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{Z}}\boldsymbol{A} & \boldsymbol{P}\tilde{\boldsymbol{A}}_1-\boldsymbol{Y}+\boldsymbol{W}^{\mathrm{T}}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{Z}}\tilde{\boldsymbol{A}}_1 & \bar{\tau}\boldsymbol{Y} \\ \tilde{\boldsymbol{A}}_1^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{Y}^{\mathrm{T}}+\boldsymbol{W}+\bar{\tau}\tilde{\boldsymbol{A}}_1^{\mathrm{T}}\tilde{\boldsymbol{Z}}\boldsymbol{A} & \bar{\tau}\tilde{\boldsymbol{A}}_1^{\mathrm{T}}\tilde{\boldsymbol{Z}}\tilde{\boldsymbol{A}}_1-\boldsymbol{Q}_2-\boldsymbol{W}-\boldsymbol{W}^{\mathrm{T}} & \bar{\tau}\boldsymbol{W} \\ \bar{\tau}\boldsymbol{Y}^{\mathrm{T}} & \bar{\tau}\boldsymbol{W}^{\mathrm{T}} & -\bar{\tau}\boldsymbol{Z} \end{bmatrix} < 0. \tag{17}$$

Therefore, we can see that Eq. (10) implies Eq. (12).

Second, in the following we show Eq. (12) implies Eq. (10). We choose two scalars $\alpha > 0$ and $\varepsilon > 0$ such that $\tilde{\boldsymbol{A}}_1 = \boldsymbol{A}_1 + \varepsilon\boldsymbol{I}$ is nonsingular and inequalities shown in Eqs. (17) and

$$\begin{bmatrix} \alpha\boldsymbol{I} & \varepsilon\tilde{\boldsymbol{P}}_1^{\mathrm{T}} & 0 \\ \varepsilon\tilde{\boldsymbol{P}}_1 & \alpha\boldsymbol{I} & \varepsilon\tilde{\boldsymbol{P}}_2 \\ 0 & \varepsilon\tilde{\boldsymbol{P}}_2^{\mathrm{T}} & \bar{\tau}\alpha\boldsymbol{I} \end{bmatrix} \geqslant 0 \tag{18}$$

hold simultaneously, where $\boldsymbol{Q}_1=\boldsymbol{Q}+\alpha\boldsymbol{I}$, $\boldsymbol{Q}_2=\boldsymbol{Q}-\alpha\boldsymbol{I}$, $\tilde{\boldsymbol{Z}}=\boldsymbol{Z}+\alpha\boldsymbol{I}$.

From Eq. (17), it is easy to see that the following inequality is true:

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{Y}+\boldsymbol{Y}^{\mathrm{T}}+\boldsymbol{Q}_1+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{Z}}\boldsymbol{A} & \boldsymbol{P}\tilde{\boldsymbol{A}}_1-\boldsymbol{Y}+\boldsymbol{W}^{\mathrm{T}}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{Z}}\tilde{\boldsymbol{A}}_1 & \bar{\tau}\boldsymbol{Y} & 0 \\ \tilde{\boldsymbol{A}}_1^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{Y}^{\mathrm{T}}+\boldsymbol{W}+\bar{\tau}\tilde{\boldsymbol{A}}_1^{\mathrm{T}}\tilde{\boldsymbol{Z}}\boldsymbol{A} & \bar{\tau}\tilde{\boldsymbol{A}}_1^{\mathrm{T}}\tilde{\boldsymbol{Z}}\tilde{\boldsymbol{A}}_1-\boldsymbol{Q}_2-\boldsymbol{W}-\boldsymbol{W}^{\mathrm{T}} & \bar{\tau}\boldsymbol{W} & 0 \\ \bar{\tau}\boldsymbol{Y}^{\mathrm{T}} & \bar{\tau}\boldsymbol{W}^{\mathrm{T}} & -\bar{\tau}\boldsymbol{Z} & 0 \\ 0 & 0 & 0 & -\bar{\tau}\tilde{\boldsymbol{A}}_1^{\mathrm{T}}\tilde{\boldsymbol{Z}}\tilde{\boldsymbol{A}}_1 \end{bmatrix} < 0. \tag{19}$$

Then, pre- and post-multiplying Eq. (19) by

$$\begin{bmatrix} \boldsymbol{I} & 0 & 0 & -\boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{A}}_1^{-\mathrm{T}} \\ 0 & \boldsymbol{I} & 0 & -\boldsymbol{I} \\ 0 & 0 & 0 & \tilde{\boldsymbol{A}}_1^{-\mathrm{T}} \\ 0 & 0 & \boldsymbol{I} & 0 \end{bmatrix}$$

and its transpose, respectively, we can get

$$\begin{bmatrix} PA+A^{\mathrm{T}}P+Y+Y^{\mathrm{T}}+Q_1 & P\tilde{A}_1-Y+W^{\mathrm{T}} & \bar{\tau}A^{\mathrm{T}}\tilde{Z} & \bar{\tau}Y \\ \tilde{A}_1^{\mathrm{T}}P-Y^{\mathrm{T}}+W & -Q_2-W-W^{\mathrm{T}} & \bar{\tau}\tilde{A}_1^{\mathrm{T}}Z & \bar{\tau}W \\ -\bar{\tau}\tilde{Z}A & -\bar{\tau}\tilde{Z}\tilde{A}_1 & -\bar{\tau}\tilde{Z} & 0 \\ \bar{\tau}Y^{\mathrm{T}} & \bar{\tau}W^{\mathrm{T}} & 0 & -\bar{\tau}Z \end{bmatrix}<0. \tag{20}$$

Now, choose $Q_1=Q+\alpha I$, $\tilde{A}_1=A_1+\alpha I$, $\tilde{Z}=Z+\alpha I$, $\tilde{P}_2=\bar{\tau}\tilde{Z}$, $Q=\tilde{Q}$, $Q_2=Q-\alpha I$, $\tilde{P}=\tilde{P}_1=P$, we can get

$$\begin{bmatrix} A^{\mathrm{T}}\tilde{P}_1+\tilde{P}_1^{\mathrm{T}}A+Y+Y^{\mathrm{T}}+Q & \tilde{P}_1^{\mathrm{T}}A_1-Y+W^{\mathrm{T}} & \bar{\tau}A^{\mathrm{T}}\tilde{Z} & \bar{\tau}Y \\ A_1^{\mathrm{T}}\tilde{P}_1-Y^{\mathrm{T}}+W & -\tilde{Q}-W-W^{\mathrm{T}} & A_1^{\mathrm{T}}\tilde{P}_2 & \bar{\tau}W \\ \bar{\tau}\tilde{Z}A & \tilde{P}_2^{\mathrm{T}}A_1 & -\tilde{P}_2 & 0 \\ \bar{\tau}Y^{\mathrm{T}} & \bar{\tau}W^{\mathrm{T}} & 0 & -\bar{\tau}Z \end{bmatrix}+\begin{bmatrix} \alpha I & \varepsilon\tilde{P}_1^{\mathrm{T}} & 0 & 0 \\ \varepsilon\tilde{P}_1 & \alpha I & \varepsilon\tilde{P}_2 & 0 \\ 0 & \varepsilon\tilde{P}_2^{\mathrm{T}} & \bar{\tau}\alpha I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}<0. \tag{21}$$

From Eq. (21), it is easy to see that the following inequality is true:

$$\begin{bmatrix} A^{\mathrm{T}}\tilde{P}_1+\tilde{P}_1^{\mathrm{T}}A+Y+Y^{\mathrm{T}}+Q & \tilde{P}_1^{\mathrm{T}}A_1-Y+W^{\mathrm{T}} & \bar{\tau}A^{\mathrm{T}}\tilde{Z} & \bar{\tau}Y \\ A_1^{\mathrm{T}}\tilde{P}_1-Y^{\mathrm{T}}+W & -\tilde{Q}-W-W^{\mathrm{T}} & \bar{\tau}A_1^{\mathrm{T}}\tilde{Z} & \bar{\tau}W \\ \bar{\tau}\tilde{Z}A & \bar{\tau}\tilde{Z}A_1 & -\bar{\tau}\tilde{Z} & 0 \\ \bar{\tau}Y^{\mathrm{T}} & \bar{\tau}W^{\mathrm{T}} & 0 & -\bar{\tau}Z \end{bmatrix}<0. \tag{22}$$

Then, choose $P=\tilde{P}$, $Q=\tilde{Q}$, $P_1=\tilde{P}_1$, $P_2=\tilde{P}_2$, $N_1=Y$, $N_2=W$, $W=Z$, we can get Eq. (10). Thus, Eq. (12) implies Eq. (10).

From above, we can see that Theorem 1 is true.

Theorem 2 There exist matrices $P>0$, $Q_1>0$, N_1, N_2, X_{ij} $(i,j=1,2)$, W and scalar $\varepsilon>0$ such that both Eq. (8) and Eq. (9) hold if and only if there exist matrices $P>0$, $Q>0$, $Z>0$, Y, W and scalar $\varepsilon>0$ such that Eq. (7) holds.

Proof From Eqs. (8) and (9), we can easily get

$$\begin{bmatrix} PA+A^{\mathrm{T}}P+Q_1+N_1+N_1^{\mathrm{T}}+\varepsilon E^{\mathrm{T}}E & PA_1-N_1+N_2^{\mathrm{T}}+\varepsilon E^{\mathrm{T}}E_1 & \bar{\tau}A^{\mathrm{T}}W_1 & PD & \bar{\tau}N_1 \\ A_1^{\mathrm{T}}P-N_1^{\mathrm{T}}+N_2+\varepsilon E_1^{\mathrm{T}}E & -Q_1-N_2-N_2^{\mathrm{T}}+\varepsilon E_1^{\mathrm{T}}E_1 & \bar{\tau}A_1^{\mathrm{T}}W_1 & 0 & \bar{\tau}N_2 \\ \bar{\tau}W_1A & \bar{\tau}W_1A_1 & -\bar{\tau}W_1 & \bar{\tau}W_1D & 0 \\ D^{\mathrm{T}}P & 0 & \bar{\tau}W_1D^{\mathrm{T}} & -\varepsilon I & 0 \\ \bar{\tau}N_1^{\mathrm{T}} & \bar{\tau}N_2^{\mathrm{T}} & 0 & 0 & -\bar{\tau}W_1 \end{bmatrix}<0. \tag{23}$$

First, we show that Eq. (23) implies Eq. (7). Pre- and post-multiplying Eq. (23) by

$$\begin{bmatrix} I & 0 & A^{\mathrm{T}} & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$$

and its transpose, respectively, we can get

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{Q}_1+\boldsymbol{N}_1+\boldsymbol{N}_1^{\mathrm{T}}+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{W}_1\boldsymbol{A} & \boldsymbol{PA}_1-\boldsymbol{N}_1+\boldsymbol{N}_2^{\mathrm{T}}+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}_1+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{W}_1\boldsymbol{A}_1 & 0 & \boldsymbol{PD}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{W}_1\boldsymbol{D} & \bar{\tau}\boldsymbol{N}_1 \\ \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{N}_1^{\mathrm{T}}+\boldsymbol{N}_2+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}+\bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{W}_1\boldsymbol{A} & -\boldsymbol{Q}_1-\boldsymbol{N}_2-\boldsymbol{N}_2^{\mathrm{T}}+\varepsilon\boldsymbol{E}_1^{\mathrm{T}}\boldsymbol{E}_1 & \bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{W}_1 & 0 & \bar{\tau}\boldsymbol{N}_2 \\ 0 & \bar{\tau}\boldsymbol{W}_1\boldsymbol{A}_1 & -\bar{\tau}\boldsymbol{W}_1 & \bar{\tau}\boldsymbol{W}_1\boldsymbol{D} & 0 \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}+\bar{\tau}\boldsymbol{W}_1\boldsymbol{D}^{\mathrm{T}}\boldsymbol{A} & 0 & \bar{\tau}\boldsymbol{W}_1\boldsymbol{D}^{\mathrm{T}} & -\varepsilon\boldsymbol{I} & 0 \\ \bar{\tau}\boldsymbol{N}_1^{\mathrm{T}} & \bar{\tau}\boldsymbol{N}_2^{\mathrm{T}} & 0 & 0 & -\bar{\tau}\boldsymbol{W}_1 \end{bmatrix} < 0. \tag{24}$$

From Eq. (24), it is easy to see that there exists a scalar $\alpha > 0$ such that the following inequality holds:

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{Q}_1+\boldsymbol{N}_1+\boldsymbol{N}_1^{\mathrm{T}}+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{W}_1\boldsymbol{A}+\alpha\boldsymbol{I} & 0 & \boldsymbol{PA}_1-\boldsymbol{N}_1+\boldsymbol{N}_2^{\mathrm{T}}+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}_1+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{W}_1\boldsymbol{A}_1 & 0 & \boldsymbol{PD}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{W}_1\boldsymbol{D} & \bar{\tau}\boldsymbol{N}_1 \\ 0 & -\alpha\boldsymbol{I} & 0 & 0 & 0 & 0 \\ \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{N}_1^{\mathrm{T}}+\boldsymbol{N}_2+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}+\bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{W}_1\boldsymbol{A} & 0 & -\boldsymbol{Q}_1-\boldsymbol{N}_2-\boldsymbol{N}_2^{\mathrm{T}}+\varepsilon\boldsymbol{E}_1^{\mathrm{T}}\boldsymbol{E}_1 & \bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{W}_1 & 0 & \bar{\tau}\boldsymbol{N}_2 \\ 0 & 0 & \bar{\tau}\boldsymbol{W}_1\boldsymbol{A}_1 & -\bar{\tau}\boldsymbol{W}_1 & \bar{\tau}\boldsymbol{W}_1\boldsymbol{D} & 0 \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}+\bar{\tau}\boldsymbol{W}_1\boldsymbol{D}^{\mathrm{T}}\boldsymbol{A} & 0 & 0 & \bar{\tau}\boldsymbol{W}_1\boldsymbol{D}^{\mathrm{T}} & -\varepsilon\boldsymbol{I} & 0 \\ \bar{\tau}\boldsymbol{N}_1^{\mathrm{T}} & 0 & \bar{\tau}\boldsymbol{N}_2^{\mathrm{T}} & 0 & 0 & -\bar{\tau}\boldsymbol{W}_1 \end{bmatrix} < 0. \tag{25}$$

Then, pre- and post-multiplying Eq. (25) by

$$\begin{bmatrix} \boldsymbol{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{I} & \boldsymbol{I} & \boldsymbol{A}_1^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & \boldsymbol{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{I} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{I} & 0 \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{I} \end{bmatrix}$$

and its transpose, respectively, we can get

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{Q}_1+\boldsymbol{N}_1+\boldsymbol{N}_1^{\mathrm{T}}+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{W}_1\boldsymbol{A}+\alpha\boldsymbol{I} & \boldsymbol{PA}_1-\boldsymbol{N}_1+\boldsymbol{N}_2^{\mathrm{T}}+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}_1+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{W}_1\boldsymbol{A}_1 & \boldsymbol{PA}_1-\boldsymbol{N}_1+\boldsymbol{N}_2^{\mathrm{T}}+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}_1+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{W}_1\boldsymbol{A} & 0 & \boldsymbol{PD}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{W}_1\boldsymbol{D} & \bar{\tau}\boldsymbol{N}_1 \\ \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{N}_1^{\mathrm{T}}+\boldsymbol{N}_2+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}+\bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{W}_1\boldsymbol{A} & -\alpha\boldsymbol{I}-\boldsymbol{Q}_1-\boldsymbol{N}_2-\boldsymbol{N}_2^{\mathrm{T}}+\varepsilon\boldsymbol{E}_1^{\mathrm{T}}\boldsymbol{E}_1+\bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{W}_1\boldsymbol{A}_1 & -\boldsymbol{Q}_1-\boldsymbol{N}_2-\boldsymbol{N}_2^{\mathrm{T}}+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}_1+\bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{W}_1\boldsymbol{A}_1 & 0 & \bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{W}_1\boldsymbol{D} & 0 \\ \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{N}_1^{\mathrm{T}}+\boldsymbol{N}_2+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}+\bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{W}_1\boldsymbol{A} & -\boldsymbol{Q}_1-\boldsymbol{N}_2-\boldsymbol{N}_2^{\mathrm{T}}+\varepsilon\boldsymbol{E}_1^{\mathrm{T}}\boldsymbol{E}_1+\bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{W}_1\boldsymbol{A} & -\boldsymbol{Q}_1-\boldsymbol{N}_2-\boldsymbol{N}_2^{\mathrm{T}}+\varepsilon\boldsymbol{E}_1^{\mathrm{T}}\boldsymbol{E}_1 & \bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{W}_1 & 0 & \bar{\tau}\boldsymbol{N}_2 \\ 0 & \bar{\tau}\boldsymbol{W}_1\boldsymbol{A}_1 & \bar{\tau}\boldsymbol{W}\boldsymbol{A}_1 & -\bar{\tau}\boldsymbol{W}_1 & \bar{\tau}\boldsymbol{W}_1\boldsymbol{D} & 0 \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}+\bar{\tau}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{W}_1\boldsymbol{A} & \bar{\tau}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{W}_1\boldsymbol{A}_1 & 0 & \bar{\tau}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{W}_1 & -\varepsilon\boldsymbol{I} & 0 \\ \bar{\tau}\boldsymbol{N}_1^{\mathrm{T}} & \bar{\tau}\boldsymbol{N}_2^{\mathrm{T}} & \bar{\tau}\boldsymbol{N}_2^{\mathrm{T}} & 0 & 0 & -\bar{\tau}\boldsymbol{W}_1 \end{bmatrix} < 0. \tag{26}$$

Then, pre- and post-multiplying Eq. (26) by

$$\begin{bmatrix} \boldsymbol{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\boldsymbol{I} \\ 0 & 0 & 0 & 0 & \boldsymbol{I} & 0 \end{bmatrix}$$

and its transpose, respectively, we can get

$$\begin{bmatrix} PA + A^{T}P + Q_1 + N_1 + N_1^{T} + \varepsilon E^{T}E + \bar{\tau}A^{T}W_1A + \alpha I & PA_1 - N_1 + N_2^{T} + \varepsilon E^{T}E_1 + \bar{\tau}A^{T}W_1A_1 & -\bar{\tau}N_1 & PD + \bar{\tau}A^{T}W_1D \\ A_1^{T}P - N_1^{T} + N_2 + \varepsilon E^{T}E + \bar{\tau}A_1^{T}W_1A & -\alpha I - Q_1 - N_2 - N_2^{T} + \varepsilon E_1^{T}E_1 + \bar{\tau}A_1^{T}W_1A_1 & -\bar{\tau}N_2 & \bar{\tau}A_1^{T}W_1D \\ -\bar{\tau}N_1^{T} & -\bar{\tau}N_2^{T} & -\bar{\tau}W_1 & 0 \\ D^{T}P + \bar{\tau}D^{T}W_1A & \bar{\tau}D^{T}W_1A_1 & 0 & -\varepsilon I \end{bmatrix} < 0. \quad (27)$$

We choose $Q = Q_1 + \alpha I$, $N_1 = Y$, $N_2 = W$, $W_1 = Z$, Eq. (27) can be rewritten as

$$\begin{bmatrix} PA + A^{T}P + Q + Y + Y^{T} + \varepsilon E^{T}E + \bar{\tau}A^{T}ZA & PA_1 - Y + W^{T} + \varepsilon E^{T}E_1 + \bar{\tau}A^{T}ZA_1 & -\bar{\tau}Y & PD + \bar{\tau}A^{T}ZD \\ A_1^{T}P - Y^{T} + W + \varepsilon E^{T}E + \bar{\tau}A_1^{T}ZA & -Q - W - W^{T} + \varepsilon E_1^{T}E_1 + \bar{\tau}A_1^{T}ZA_1 & -\bar{\tau}W & \bar{\tau}A_1^{T}ZD \\ -\bar{\tau}Y^{T} & -\bar{\tau}W^{T} & -\bar{\tau}Z & 0 \\ D^{T}P + \bar{\tau}D^{T}ZA & \bar{\tau}D^{T}ZA_1 & 0 & -\varepsilon I \end{bmatrix} < 0. \quad (28)$$

From Eq. (28), we can easily have

$$\begin{bmatrix} PA + A^{T}P + Q + Y + Y^{T} + \varepsilon E^{T}E + \bar{\tau}A^{T}ZA & PA_1 - Y + W^{T} + \varepsilon E^{T}E_1 + \bar{\tau}A^{T}ZA_1 & -\bar{\tau}Y & 0 & PD + \bar{\tau}A^{T}ZD \\ A_1^{T}P - Y^{T} + W + \varepsilon E^{T}E + \bar{\tau}A_1^{T}ZA & -Q - W - W^{T} + \varepsilon E_1^{T}E_1 + \bar{\tau}A_1^{T}ZA_1 & -\bar{\tau}W & 0 & \bar{\tau}A_1^{T}ZD \\ -\bar{\tau}Y^{T} & -\bar{\tau}W^{T} & -\bar{\tau}Z & 0 & 0 \\ 0 & 0 & 0 & -\bar{\tau}D^{T}ZD & 0 \\ D^{T}P + \bar{\tau}D^{T}ZA & \bar{\tau}D^{T}ZA_1 & 0 & 0 & -\varepsilon I + \bar{\tau}D^{T}ZD \end{bmatrix} < 0. \quad (29)$$

Then pre- and post-multiplying Eq. (29) by

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$$

and its transpose, respectively, we can provide

$$\begin{bmatrix} PA + A^{T}P + Q + Y + Y^{T} + \varepsilon E^{T}E + \bar{\tau}A^{T}ZA & PA_1 - Y + W^{T} + \varepsilon E^{T}E_1 + \bar{\tau}A^{T}ZA_1 & -\bar{\tau}Y & PD + \bar{\tau}A^{T}ZD \\ A_1^{T}P - Y^{T} + W + \varepsilon E^{T}E + \bar{\tau}A_1^{T}ZA & -Q - W - W^{T} + \varepsilon E_1^{T}E_1 + \bar{\tau}A_1^{T}ZA_1 & -\bar{\tau}W & \bar{\tau}A_1^{T}ZD \\ -\bar{\tau}Y^{T} & -\bar{\tau}W^{T} & -\bar{\tau}Z & 0 \\ D^{T}P + \bar{\tau}D^{T}ZA & \bar{\tau}D^{T}ZA_1 & 0 & -\varepsilon I + \bar{\tau}D^{T}ZD \end{bmatrix} < 0. \quad (30)$$

Using Schur complement, we get

$$\begin{bmatrix} PA + A^{T}P + Q + Y + Y^{T} + \varepsilon E^{T}E & PA_1 - Y + W^{T} + \varepsilon E^{T}E_1 & -\bar{\tau}Y & PD & \bar{\tau}A^{T}Z \\ A_1^{T}P - Y^{T} + W + \varepsilon E^{T}E & -Q - W - W^{T} + \varepsilon E_1^{T}E_1 & -\bar{\tau}W & 0 & \bar{\tau}A_1^{T}Z \\ -\bar{\tau}Y^{T} & -\bar{\tau}W^{T} & -\bar{\tau}Z & 0 & 0 \\ D^{T}P & 0 & 0 & -\varepsilon I & \bar{\tau}D^{T}Z \\ \bar{\tau}ZA & \bar{\tau}ZA_1 & 0 & \bar{\tau}ZD & -\bar{\tau}Z \end{bmatrix} < 0. \quad (31)$$

Exchanging row 4 and row 5 in Eq. (31), at the same time exchanging column 4 and column 5, we have Eq. (23), so Eq. (23) implies Eq. (7).

Second, we show Eq. (7) implies Eq. (23).

Using Schur complement, exchanging row 4 and row 5, column 4 and column 5 respectively, Eq. (24) can be rewritten as

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{Q}+\boldsymbol{Y}+\boldsymbol{Y}^{\mathrm{T}}+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{ZA} & \boldsymbol{PA}_1-\boldsymbol{Y}+\boldsymbol{W}^{\mathrm{T}}+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}_1+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{ZA}_1 & \boldsymbol{PD}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{ZD} & \bar{\tau}\boldsymbol{Y} \\ \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{Y}^{\mathrm{T}}+\boldsymbol{W}+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}+\bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{ZA} & -\boldsymbol{Q}-\boldsymbol{W}-\boldsymbol{W}^{\mathrm{T}}+\varepsilon\boldsymbol{E}_1^{\mathrm{T}}\boldsymbol{E}_1+\bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{ZA}_1 & \bar{\tau}\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{ZD} & \bar{\tau}\boldsymbol{W} \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}+\bar{\tau}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{ZA} & \bar{\tau}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{ZA}_1 & -\varepsilon\boldsymbol{I}+\bar{\tau}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{ZD} & 0 \\ \bar{\tau}\boldsymbol{Y}^{\mathrm{T}} & \bar{\tau}\boldsymbol{W}^{\mathrm{T}} & 0 & -\bar{\tau}\boldsymbol{Z} \end{bmatrix} < 0. \quad (32)$$

Choose scalars $\alpha > 0$ and $\sigma > 0$ such that $\tilde{\boldsymbol{A}}_1 = \boldsymbol{A}_1 + \varepsilon\boldsymbol{I}$ is nonsingular and inequalities

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{Q}_1+\boldsymbol{Y}+\boldsymbol{Y}^{\mathrm{T}}+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{Z}}\boldsymbol{A} & \boldsymbol{P}\tilde{\boldsymbol{A}}_1-\boldsymbol{Y}+\boldsymbol{W}^{\mathrm{T}}+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}_1+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{Z}}\tilde{\boldsymbol{A}}_1 & \boldsymbol{PD}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{Z}}\boldsymbol{D} & \bar{\tau}\boldsymbol{Y} \\ \tilde{\boldsymbol{A}}_1^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{Y}^{\mathrm{T}}+\boldsymbol{W}+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}+\bar{\tau}\tilde{\boldsymbol{A}}_1^{\mathrm{T}}\tilde{\boldsymbol{Z}}\boldsymbol{A} & -\boldsymbol{Q}_2-\boldsymbol{W}-\boldsymbol{W}^{\mathrm{T}}+\varepsilon\boldsymbol{E}_1^{\mathrm{T}}\boldsymbol{E}_1+\bar{\tau}\tilde{\boldsymbol{A}}_1^{\mathrm{T}}\tilde{\boldsymbol{Z}}\tilde{\boldsymbol{A}}_1 & \bar{\tau}\tilde{\boldsymbol{A}}_1^{\mathrm{T}}\tilde{\boldsymbol{Z}}\boldsymbol{D} & \bar{\tau}\boldsymbol{W} \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}+\bar{\tau}\boldsymbol{D}^{\mathrm{T}}\tilde{\boldsymbol{Z}}\boldsymbol{A} & \bar{\tau}\boldsymbol{D}^{\mathrm{T}}\tilde{\boldsymbol{Z}}\tilde{\boldsymbol{A}}_1 & -\varepsilon\boldsymbol{I}+\bar{\tau}\boldsymbol{D}^{\mathrm{T}}\tilde{\boldsymbol{Z}}\boldsymbol{D} & 0 \\ \bar{\tau}\boldsymbol{Y}^{\mathrm{T}} & \bar{\tau}\boldsymbol{W}^{\mathrm{T}} & 0 & -\bar{\tau}\boldsymbol{Z} \end{bmatrix} < 0, \quad (33)$$

$$\begin{bmatrix} \alpha\boldsymbol{I} & \sigma\tilde{\boldsymbol{P}}_1^{\mathrm{T}} & 0 \\ \sigma\tilde{\boldsymbol{P}}_1 & \alpha\boldsymbol{I} & \sigma\tilde{\boldsymbol{P}}_2 \\ 0 & \sigma\tilde{\boldsymbol{P}}_2^{\mathrm{T}} & \bar{\tau}\alpha\boldsymbol{I} \end{bmatrix} \geqslant 0 \quad (34)$$

hold simultaneously, where $\boldsymbol{Q} = \boldsymbol{Q}_1 + \alpha\boldsymbol{I}$, $\boldsymbol{Q}_2 = \boldsymbol{Q} - \alpha\boldsymbol{I}$, $\tilde{\boldsymbol{Z}} = \boldsymbol{Z} - \alpha\boldsymbol{I}$.

From Eq. (32), it is easy to see that the following inequality is true:

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{Q}_1+\boldsymbol{Y}+\boldsymbol{Y}^{\mathrm{T}}+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{Z}}\boldsymbol{A} & \boldsymbol{P}\tilde{\boldsymbol{A}}_1-\boldsymbol{Y}+\boldsymbol{W}^{\mathrm{T}}+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}_1+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{Z}}\tilde{\boldsymbol{A}}_1 & \boldsymbol{PD}+\bar{\tau}\boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{Z}}\boldsymbol{D} & \bar{\tau}\boldsymbol{Y} & 0 \\ \tilde{\boldsymbol{A}}_1^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{Y}^{\mathrm{T}}+\boldsymbol{W}+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}+\bar{\tau}\tilde{\boldsymbol{A}}_1^{\mathrm{T}}\tilde{\boldsymbol{Z}}\boldsymbol{A} & -\boldsymbol{Q}_2-\boldsymbol{W}-\boldsymbol{W}^{\mathrm{T}}+\varepsilon\boldsymbol{E}_1^{\mathrm{T}}\boldsymbol{E}_1+\bar{\tau}\tilde{\boldsymbol{A}}_1^{\mathrm{T}}\tilde{\boldsymbol{Z}}\tilde{\boldsymbol{A}}_1 & \bar{\tau}\tilde{\boldsymbol{A}}_1^{\mathrm{T}}\tilde{\boldsymbol{Z}}\boldsymbol{D} & \bar{\tau}\boldsymbol{W} & 0 \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}+\bar{\tau}\boldsymbol{D}^{\mathrm{T}}\tilde{\boldsymbol{Z}}\boldsymbol{A} & \bar{\tau}\boldsymbol{D}^{\mathrm{T}}\tilde{\boldsymbol{Z}}\tilde{\boldsymbol{A}}_1 & -\varepsilon\boldsymbol{I}+\bar{\tau}\boldsymbol{D}^{\mathrm{T}}\tilde{\boldsymbol{Z}}\boldsymbol{D} & 0 & 0 \\ \bar{\tau}\boldsymbol{Y}^{\mathrm{T}} & \bar{\tau}\boldsymbol{W}^{\mathrm{T}} & 0 & -\bar{\tau}\boldsymbol{Z} & 0 \\ 0 & 0 & 0 & 0 & \bar{\tau}\boldsymbol{D}^{\mathrm{T}}\tilde{\boldsymbol{Z}}\boldsymbol{D} \end{bmatrix} < 0. \quad (35)$$

Pre- and post-multiplying Eq. (35) by

$$\begin{bmatrix} \boldsymbol{I} & 0 & 0 & 0 & -\boldsymbol{A}^{\mathrm{T}}\boldsymbol{D}^{-\mathrm{T}} \\ 0 & \boldsymbol{I} & 0 & 0 & -\tilde{\boldsymbol{A}}_1^{\mathrm{T}}\boldsymbol{D}^{-\mathrm{T}} \\ 0 & 0 & 0 & 0 & \boldsymbol{D}^{-\mathrm{T}} \\ 0 & 0 & 0 & \boldsymbol{I} & 0 \\ 0 & 0 & \boldsymbol{I} & 0 & -\boldsymbol{I} \end{bmatrix}$$

and its transpose, respectively, exchanging row 4 and row 5, column 4 and column 5, respectively, then choose $\tilde{\boldsymbol{P}} = \tilde{\boldsymbol{P}}_1 = \boldsymbol{P}$, $\tilde{\boldsymbol{P}}_2 = \bar{\tau}\tilde{\boldsymbol{Z}}$, $\tilde{\boldsymbol{Q}} = \boldsymbol{Q}$, we can obtain

$$\begin{bmatrix} \tilde{\boldsymbol{P}}_1^{\mathrm{T}}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{P}}_1 + \tilde{\boldsymbol{Q}} + \boldsymbol{Y} + \boldsymbol{Y}^{\mathrm{T}} + \varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E} & \tilde{\boldsymbol{P}}_1^{\mathrm{T}}\boldsymbol{A}_1 - \boldsymbol{Y} + \boldsymbol{W}^{\mathrm{T}} + \varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}_1 & \boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{P}}_2 & \boldsymbol{PD} & \bar{\tau}\boldsymbol{Y} \\ \boldsymbol{A}_1^{\mathrm{T}}\tilde{\boldsymbol{P}}_1 - \boldsymbol{Y}^{\mathrm{T}} + \boldsymbol{W} + \varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E} & -\tilde{\boldsymbol{Q}} - \boldsymbol{W} - \boldsymbol{W}^{\mathrm{T}} + \varepsilon\boldsymbol{E}_1^{\mathrm{T}}\boldsymbol{E}_1 & \tilde{\boldsymbol{A}}_1^{\mathrm{T}}\tilde{\boldsymbol{P}}_2 & 0 & \bar{\tau}\boldsymbol{W} \\ \tilde{\boldsymbol{P}}_2^{\mathrm{T}}\boldsymbol{A} & \tilde{\boldsymbol{P}}_2^{\mathrm{T}}\boldsymbol{A}_1 & -\bar{\tau}\tilde{\boldsymbol{Z}} & \bar{\tau}\tilde{\boldsymbol{Z}}\boldsymbol{D} & 0 \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} & 0 & \bar{\tau}\boldsymbol{D}^{\mathrm{T}}\tilde{\boldsymbol{Z}} & -\varepsilon\boldsymbol{I} & 0 \\ \bar{\tau}\boldsymbol{Y}^{\mathrm{T}} & \bar{\tau}\boldsymbol{W}^{\mathrm{T}} & 0 & 0 & -\bar{\tau}\boldsymbol{Z} \end{bmatrix}$$

$$+\begin{bmatrix} \alpha\boldsymbol{I} & \sigma\tilde{\boldsymbol{P}}_1^{\mathrm{T}} & 0 & 0 & 0 \\ \sigma\tilde{\boldsymbol{P}}_1 & \alpha\boldsymbol{I} & \sigma\tilde{\boldsymbol{P}}_2 & 0 & 0 \\ 0 & \sigma\tilde{\boldsymbol{P}}_2^{\mathrm{T}} & \bar{\tau}\alpha\boldsymbol{I} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0. \tag{36}$$

From Eq. (36), it is easy to get

$$\begin{bmatrix} \tilde{\boldsymbol{P}}_1^{\mathrm{T}}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{P}}_1 + \tilde{\boldsymbol{Q}} + \boldsymbol{Y} + \boldsymbol{Y}^{\mathrm{T}} + \varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E} & \tilde{\boldsymbol{P}}_1^{\mathrm{T}}\boldsymbol{A}_1 - \boldsymbol{Y} + \boldsymbol{W}^{\mathrm{T}} + \varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}_1 & \boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{P}}_2 & \boldsymbol{PD} & \bar{\tau}\boldsymbol{Y} \\ \boldsymbol{A}_1^{\mathrm{T}}\tilde{\boldsymbol{P}}_1 - \boldsymbol{Y}^{\mathrm{T}} + \boldsymbol{W} + \varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E} & -\tilde{\boldsymbol{Q}} - \boldsymbol{W} - \boldsymbol{W}^{\mathrm{T}} + \varepsilon\boldsymbol{E}_1^{\mathrm{T}}\boldsymbol{E}_1 & \tilde{\boldsymbol{A}}_1^{\mathrm{T}}\tilde{\boldsymbol{P}}_2 & 0 & \bar{\tau}\boldsymbol{W} \\ \tilde{\boldsymbol{P}}_2^{\mathrm{T}}\boldsymbol{A} & \tilde{\boldsymbol{P}}_2^{\mathrm{T}}\boldsymbol{A} & -\bar{\tau}\tilde{\boldsymbol{Z}} & \bar{\tau}\tilde{\boldsymbol{Z}}\boldsymbol{D} & 0 \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} & 0 & \bar{\tau}\boldsymbol{D}^{\mathrm{T}}\tilde{\boldsymbol{Z}} & -\varepsilon\boldsymbol{I} & 0 \\ \bar{\tau}\boldsymbol{Y}^{\mathrm{T}} & \bar{\tau}\boldsymbol{W}^{\mathrm{T}} & 0 & 0 & -\bar{\tau}\boldsymbol{Z} \end{bmatrix} < 0. \tag{37}$$

Choose that $\tilde{\boldsymbol{P}}_1 = \tilde{\boldsymbol{P}}_2 = \boldsymbol{P}$, $\tilde{\boldsymbol{Q}} = \boldsymbol{Q}$, $\boldsymbol{Y} = \boldsymbol{N}_1$, $\boldsymbol{W} = \boldsymbol{N}_2$, $\boldsymbol{Z} = \boldsymbol{W}$, then Eq. (37) can be rewritten as Eq. (23), so Eq. (7) implies Eq. (23).

From above, we can see that Theorem 2 is true.

3 Conclusion

In this paper, we first find that the stability criterion given in Refs. [13] and [10], the robust stability criterion given in Refs. [20] and [10], have the same conservativeness through numerical study, respectively.

Acknowledgements This work was supported by the National Natural Science Foundation of China (Grant No. 50707019), Special Fund of the National Fundamental Research (No. 2009CB219701), Tianjin Municipal (No. 09JCZDJC25000), the National Major Project of Scientific and Technical Supporting Programs of China During the 11th Five-Year Plan Period (No. 2006BAJ03A06), a grant from the Ph.D. Programs Foundation of Ministry of Education of China (No. 20090032110064).

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