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RESEARCH ARTICLE

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Parameterized continuous models of fuzzy reasoning

Abstract In this paper, the ideas of universal logic is introduced into fuzzy systems. After giving the definitions of the softened fuzzy reasoning models based on Schweizer-Sklar t-norms and Schweizer-Sklar implications, i.e., α-models and β-models, we give the sufficient and necessary conditions for these models to be continuous, and discuss the continuity of some commonly used models. We also prove that when an α-model or a β-model is used as a fuzzy controller, it has universal property with respect to function approximation. The results we obtained show that α-models and β-models are more flexible than the existing models in applications.

Keywords Schweizer-Sklar t-norms, Schweizer-Sklar implications, continuous fuzzy reasoning models

1 Introduction

Fuzzy reasoning is the basis of fuzzy information process, fuzzy decision, fuzzy systems, and fuzzy control. As shown in Ref. [1], a system of fuzzy IF-THEN rules being modeled correctly works as a partially given (fuzzy) function. Its behavior is determined by a chosen structure for IF-THEN rules which assigns meaning to fuzzy sets in the IF and THEN parts as well as to basic connectives. This partial function can be extended to a whole domain of fuzzy sets with the help of compositional rule of inference, which plays a role of a mechanism for computing the dependent functional values. We call this extended partial function a fuzzy reasoning model. The choice of fuzzy reasoning model can directly impact the effect of inference.

Let {IF x is A_i, THEN y is B_i, $i = 1,2,...,n$} be a group of fuzzy rules. In general, we hope a fuzzy reasoning model based on these rules should satisfy the following conditions:

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1) Consistency (or the reproduction of the given fuzzy rules), i.e., when the input is A_i, the output is exactly B_i;

2) Continuity, i.e., the model is consistent and the closer the input is to A_i, the closer the output is to B_i. The continuity of fuzzy reasoning models with Archimedean t-norms have been studied in Ref. [1]. There is much work to do on this subject.

Just like we know very well, the basic operations of fuzzy sets, such as negation, intersection, and union, usually are computed by applying the one-complement, minimum, and maximum operators to the membership functions of fuzzy sets. However, generally fuzzy sets express humanity's knowledge and experience whose main character is flexibility. Many decisions must be made under imprecision and uncertainty, which cannot be represented easily by the crisp intersection, union, and negation of fuzzy sets [2]. References [2–6] have taken notice of this flaw of the basic operations of fuzzy sets and proposed many schemes to soften these operations. One typical scheme is to use Schweizer-Sklar t-norms to soften crisp intersection $\wedge$; the other soft basic operations can be induced by this softened intersection. This scheme is called SS-soft-method. Reference [5] studied Schweizer-Sklar t-norms and residual implications induced by them, explained the relation between the strength of interaction of two words or two concepts and the value of the parameter p in Schweizer-Sklar t-norms. In the universal logics proposed in Ref. [7], Schweizer-Sklar t-norms and implications induced by them are used as the most important flexible logical operators. Reference [8] studied the fuzzy logic system UL^* based on the Schweizer-Sklar t-norms, proved that UL^* is reliable and complete. Therefore, SS-soft-method has a reliable logical basis and has become a successful pattern of the soften operations of fuzzy sets.

Naturally, when the operations of fuzzy sets are softened, the corresponding fuzzy reasoning models should also be softened and become more flexible. This paper should advance a theory of soft continuous models of fuzzy reasoning based on SS-soft-method and concentrate on the necessary and sufficient conditions of two kinds of typical models to be continuous models. Because the soft continuous fuzzy reasoning models can change with the value of a parameter, in applications they can be tuned according to actual situations. This is significant for optimization of fuzzy systems.

2 Preliminaries

In this section we introduce the Schweizer-Sklar operators. The original definitions of these operators appear in Refs. [9,10], more detailed discussion is in Ref. [5].

Definition 1 Let P be a poset, the binary operations $\otimes$ and $\rightarrow$ defined on P form an adjoint couple, if they satisfy the following conditions:

(M0) $\otimes$: $P\times P\rightarrow P$ is isotonic in both arguments;

(R0) $\rightarrow$: $P\times P\rightarrow P$ is isotonic in the second and antitonic in the first argument;

(A) $a\otimes b\leqslant c$ if and only if $a\leqslant b\rightarrow c$, $a,b,c\in P$.

Definition 2 Let m be any given real number, the binary operation $T(\cdot,\cdot,m):[0,1]^2\rightarrow[0,1]$ defined by $T(x,y,m)=[\max(0,\ x^m+y^m-1)]^{1/m}$ is called Schweizer-Sklar t-norm. This t-norm is briefly written as $*_m$ in this paper.

When $m<0$, t-norm may be meaningless, we assign $x*_m y=0$.

Definition 3 The adjoint of Schweizer-Sklar t-norm $T(\cdot,\cdot,m):[0,1]^2\rightarrow[0,1]$ which is defined by $I(x,y,m)=[\min(1,1-x^m+y^m)]^{1/m}$ is called a Schweizer-Sklar implication. This implication is briefly written as $\rightarrow_m$ in this paper.

In order to avoid ambiguity, we make the following provisions (when $m<0$): $I(0,0,m)=1$; if $x>0$, then $I(x,0,m)=0$; If $1-x^m+y^m$ is nonpositive, then $I(x,y,m)=1$ (in fact, now it must hold $x<y$).

It is easy to verify that $([0,1],\vee,\wedge,*_m,\rightarrow_m)$ is a basic logic algebra (BL-algebra), i.e., the following conditions are satisfied:

(P1) $([0,1],*_m)$ is a commutative monoid with the unit 1;

(P2) $(*_m,\rightarrow_m)$ is an adjoint couple;

(P3) $\forall\ x,y\in[0,1]$, it holds that $x*_m(x\rightarrow_m y)=x\wedge y$;

(P4) $\forall\ x,y\in[0,1]$, it holds that $(x\rightarrow_m y)\wedge(y\rightarrow_m x)=1$.

Definition 4 The parameterized binary function $Q(x,y,m)=I(x,y,m)\wedge I(y,x,m)$, which can be represented as

$$Q(x,y,m)=\begin{cases}(1+|x^m-y^m|)^{1/m}, & m<0,\\ \min\left(\dfrac{y}{x},\dfrac{x}{y}\right), & m=0,\\ (1-|x^m-y^m|)^{1/m}, & m>0,\end{cases}$$

is called Schweizer-Sklar equivalence. The Schweizer-Sklar equivalence $Q(x,y,m)$ is written as $\leftrightarrow_m$ in this paper.

3 Continuity of α-models and β-models

For convenience, we denote the set $\{1,2,\ldots,n\}$ as $\underline{n}$.

Definition 5 Let X and Y be the universes of the input and output variables, respectively. All fuzzy subsets of X and Y are denoted as $F(X)$ and $F(Y)$, respectively. Let $A_i\in F(X)$, $B_i\in F(Y)$, $i=1,2,\ldots,n$, and $R\in F(X\times Y)$ be a fuzzy relation from X to Y, assume that a system of fuzzy rules

$$\text{IF } x \text{ is } A_i,\ \text{THEN } y \text{ is } B_i,\quad i=1,2,...,n, \tag{$*$}$$

are given, we define a mapping $f_R:F(X)\rightarrow F(Y)$ as

$$f_R(A)=A\circ R, \tag{1}$$

where $\circ$ is a compositional operation, then we call f_R a fuzzy reasoning model based on the system of rules ($*$) (briefly, a model). When $\circ$ is chosen as $\wedge-\rightarrow_m$ compositional operation, i.e.,

$$f_R(A)=A\odot_m R=\wedge_{x\in X}(A(x)\rightarrow_m R(x,y)),$$

we call f_R an α-model; when $\circ$ is chosen as $\vee-*_m$ compositional operation, i.e.,

$$f_R(A)=A\circ_m R=\vee_{x\in X}(A(x)*_m R(x,y)),$$

we call f_R a β-model.

Definition 6 Assume that system ($*$) is given, if for every $i\in\underline{n}$, the equality $f_R(A_i)=B_i$ holds, then we call f_R a model with reproduction of the given rules, or call f_R a consistent model.

Definition 7 Let $A,B\in F(X)$, $m\neq 0$ define the mapping $D_m:F(X)\times F(X)\rightarrow[0,\infty)$ as $D_m(A,B)=\vee_{x\in X}|A^m(x)-B^m(x)|$, then we call $D_m(A,B)$ the distance between fuzzy sets A and B.

We agreed that in the discussion below, parameter m is not equal to 0, because when $m=0$, we cannot define the distance function as Definition 7.

Definition 8 Let $(F(X),\rho_X)$ and $(F(Y),\rho_Y)$ be both metric spaces, $T:F(X)\rightarrow F(Y)$ be a mapping, and $\{A_n\}$ be a sequence of fuzzy sets in $F(X)$, $A\in F(X)$, if when $A_n\rightarrow A\ (n\rightarrow\infty)$ we always have $T(A_n)\rightarrow T(A)\ (n\rightarrow\infty)$, then we say the mapping T is continuous.

Definition 9 Let $R\in F(X\times Y)$, we say the model f_R is k-continuous if and only if there exists a positive real number k, such that for every $i\in\underline{n}$ and for any $A\in F(X)$, the following inequality holds:

$$D_m(B_i,\ A\circ R)\leqslant kD_m(A_i,\ A). \tag{2}$$

Usually, a 1-continuous model is also briefly called a continuous model.

Obviously, when $f_R:(F(X),D_m)\rightarrow(F(Y),D_m)$ is a k-continuous model, it is actually continuous at the points $A_i, i=1,2,...,n$. This is a kind of partial continuity. The condition (2) is partially similar to Lipschitz condition in analytics.

Theorem 1 Let $R \in F(X \times Y)$, $m \neq 0$, the model f_R is k-continuous if and only if for every $i \in \underline{n}$ and for any $A \in F(X)$, the following inequality always holds:

$$\wedge_{y \in Y}(B_i(y) \leftrightarrow_m (A \circ R)(y))$$
$$\geqslant \wedge_{x \in X}(k^{1/m}A_i(x) \leftrightarrow_m k^{1/m}A(x)).$$

Proof Let $R \in F(X \times Y)$, $m \neq 0$. If the model f_R is k-continuous, then for any $i \in \underline{n}$ and for any $A \in F(X)$, we have $D_m(B_i, A \circ R) \leqslant kD_m(A_i, A)$, i.e.,

$$\vee_{y \in Y}|B_i^m(y) - (A \circ R)^m(y)| \leqslant \vee_{x \in X} k|A_i^m(x) - A^m(x)|.$$

In the case $m < 0$, we have

$$\wedge_{y \in Y}[1 + |B_i^m(y) - (A \circ R)^m(y)|]^{1/m}$$
$$\geqslant \wedge_{x \in X}[1 + k|A_i^m(x) - A^m(x)|]^{1/m}$$
$$= \wedge_{x \in X}\left[1 + \left|(k^{1/m}A_i)^m(x) - (k^{1/m}A)^m(x)\right|\right]^{1/m}.$$

In the case $m > 0$, we have

$$\wedge_{y \in Y}[1 - |B_i^m(y) - (A \circ R)^m(y)|]^{1/m}$$
$$\geqslant \wedge_{x \in X}[1 - k|A_i^m(x) - A^m(x)|]^{1/m}$$
$$= \wedge_{x \in X}\left[1 - \left|(k^{1/m}A_i)^m(x) - (k^{1/m}A)^m(x)\right|\right]^{1/m}.$$

So we always have

$$\wedge_{y \in Y}(B_i(y) \leftrightarrow_m (A \circ R)(y))$$
$$\geqslant \wedge_{x \in X}(k^{1/m}A_i(x) \leftrightarrow_m k^{1/m}A(x)).$$

Obviously, the steps above are reversible. So we complete the proof.

From Definition 9 and Theorem 1, we immediately draw the following corollary.

Corollary 1 Let $R \in F(X \times Y)$, $m \neq 0$, then the model f_R is 1-continuous if and only if for every $i \in \underline{n}$ and any $A \in F(X)$, one of the following conditions is satisfied:

$$D_m(B_i, A \circ R) \leqslant D_m(A_i, A), \quad (3)$$

$$\wedge_{y \in Y}(B_i(y) \leftrightarrow_m f_R(y)) \geqslant \wedge_{x \in X}(A_i(x) \leftrightarrow_m A(x)). \quad (4)$$

Lemma 1 Let $([0,1], *, \rightarrow, \leqslant)$ be a complete residuated lattice, $a_i, b_i \in [0,1]$, $i \in I$, then the following inequalities always hold:

1) $\wedge_{i \in I}(a_i \leftrightarrow b_i) \leqslant (\wedge_{i \in I} a_i) \leftrightarrow (\wedge_{i \in I} b_i)$,
2) $\wedge_{i \in I}(a_i \leftrightarrow b_i) \leqslant (\vee_{i \in I} a_i) \leftrightarrow (\vee_{i \in I} b_i)$.

Lemma 2 Let $([0,1], *, \rightarrow, \leqslant)$ be a complete residuated lattice, $a,b,c,d \in [0,1]$, then the following inequalities always hold:

1) $a \leftrightarrow b \leqslant (a \rightarrow c) \leftrightarrow (b \rightarrow c)$,
2) $(a \leftrightarrow b) * (c \leftrightarrow d) \leqslant (a * c) \leftrightarrow (b * d)$.

Noticing that $([0,1], *_m, \rightarrow_m, \leqslant)$ is a complete residuated lattice, so when $* = *_m$, $\rightarrow = \rightarrow_m$, Lemmas 1 and 2 are still true.

Lemma 3 Let $x,y,z \in [0,1]$, $m \neq 0$, then the following inequality holds:

$$(x \leftrightarrow_m y) *_m (y \leftrightarrow_m z) \leqslant x \leftrightarrow_m z,$$

and only when $x = y = z$, we have

$$(x \leftrightarrow_m y) *_m (y \leftrightarrow_m z) = x \leftrightarrow_m z = 1.$$

Lemma 4 Let $R \in F(X \times Y)$, then for any $i = 1,2,\ldots,n$ and $A \in F(X)$, we have

$$B_i(y) \leftrightarrow_m (A \odot_m R)(y) \geqslant \theta_R^i(y) *_m \wedge_{x \in X}(A_i(x) \leftrightarrow_m A(x)), \quad \forall y \in Y, \quad (5)$$

where $\theta_R^i(y) = B_i(y) \leftrightarrow_m (A \odot_m R)(y)$.

Proof Let $B = A \odot_m R$, by Lemma 3, for any $y \in Y$ we have

$$B(y) \leftrightarrow_m B_i(y)$$
$$\geqslant (B(y) \leftrightarrow_m (A \odot_m R)(y))$$
$$*_m ((A \odot_m R)(y) \leftrightarrow_m (A_i \odot_m R)(y))$$
$$*_m ((A_i \odot_m R)(y) \leftrightarrow_m B_i(y)), \quad i = 1,2,\ldots,n. \quad (6)$$

Note that

1) $B(y) \leftrightarrow_m (A \odot_m R)(y) = 1$;
2) $\theta_R^i(y) = (A \odot_m R)(y) \leftrightarrow_m B_i(y)$;
3) By Lemma 1 1) and Lemma 2 1), we have

$$(A \odot_m R)(y) \leftrightarrow_m (A_i \odot_m R)(y)$$
$$= \wedge_{x \in X}(A(x) \rightarrow_m R(x,y)) \leftrightarrow_m \wedge_{x \in X}(A_i(x) \rightarrow_m R(x,y))$$
$$\geqslant \wedge_{x \in X}[(A(x) \rightarrow_m R(x,y)) \leftrightarrow_m (A_i(x) \rightarrow_m R(x,y))]$$
$$\geqslant \wedge_{x \in X}(A_i(x) \rightarrow_m A(x)).$$

Substituting 1), 2) and 3) into Eq. (6), then it is trivial that Eq. (5) is true.

Theorem 2 Let $R \in F(X \times Y)$, then f_R is a consistent α-model if and only if f_R is a continuous α-model.

Proof "$\Rightarrow$" Let f_R be a consistent α-model, then R is a solution of the system of fuzzy relation equations $A_i \odot_m R = B_i, i = 1,2,\ldots,n$, so for any $i = 1,2,\ldots,n$ and $y \in Y$, we have $\theta_R^i(y) = 1$, by the Definition 9, Corollary 1 and Lemma 4, f_R is a continuous α-model.

"$\Leftarrow$" Suppose that f_R is a continuous α-model, then for any $i = 1,2,\ldots,n$ and $A \in F(X)$, Eq. (4) holds. For arbitrary $i = 1,2,\ldots,n$, when $A = A_i$, by Eq. (4) we have

$$\wedge_{y \in Y}(B_i(y) \leftrightarrow_m f_R(A_i)(y)) \geqslant 1,$$

hence

$$B_i(y) \leftrightarrow_m f_R(A_i)(y)$$
$$= B_i(y) \leftrightarrow_m (A_i \odot_m R)$$
$$= 1, \quad \forall y \in Y.$$

This shows that R is a solution of the system $A_i \odot_m R = B_i, i = 1,2,\ldots,n$, so f_R is a consistent model.

Corollary 2 Let $R \in F(X \times Y)$, then f_R is a continuous α-model if and only if for any $i = 1,2,...,n$ and $A \in F(X)$,

$$\wedge_{y \in Y}(B_i(y) \leftrightarrow_m (A_i \odot_m R)) \geqslant \wedge_{x \in X}(A_i(x) \leftrightarrow_m A(x)) \quad (7)$$

holds.

Lemma 5 Let $R \in F(X \times Y)$, then for any $i = 1,2,...,n$ and $A \in F(X)$,

$$B_i(y) \leftrightarrow_m (A \circ_m R)(y) \geqslant \delta_R^i(y) *_m [\wedge_{x \in X}(A_i(x) \leftrightarrow_m A(x))], \quad \forall y \in Y, \quad (8)$$

where $\delta_R^i(y) = B_i(y) \leftrightarrow_m (A \circ_m R)(y)$.

Proof Let $B = A \circ_m R$, by Lemma 3, for any $y \in Y$, we have

$$\begin{aligned} B(y) \leftrightarrow_m B_i(y) &\geqslant (B(y) \leftrightarrow_m (A \circ_m R)(y)) \\ &*_m ((A \circ_m R)(y) \leftrightarrow_m (A_i \circ_m R)(y)) \\ &*_m ((A_i \circ_m R)(y) \leftrightarrow_m B_i(y)), \quad i = 1,2,...,n. \end{aligned} \quad (9)$$

Note that

1) $B(y) \leftrightarrow_m (A \circ_m R)(y) = 1$;

2) $\delta_R^i(y) = (A \circ_m R)(y) \leftrightarrow_m B_i(y)$;

3) By the facts shown in Lemma 1 2) and Lemma 2 2), we have

$$\begin{aligned} (A \circ_m R)(y) \leftrightarrow_m (A_i \circ_m R)(y) &= \vee_{x \in X}(A(x) *_m R(x,y)) \leftrightarrow_m \vee_{x \in X}(A_i(x) *_m R(x,y)) \\ &\geqslant \wedge_{x \in X}[(A(x) *_m R(x,y)) \leftrightarrow_m (A_i(x) *_m R(x,y))] \\ &\geqslant \wedge_{x \in X}(A_i(x) \leftrightarrow_m A(x)). \end{aligned}$$

Substituting 1), 2) and 3) into Eq. (9), it is easy to verify that Eq. (8) is true. The proof is completed.

Theorem 3 Let $R \in F(X \times Y)$, then f_R is a consistent β-model if and only if f_R is a continuous β-model.

Proof Using Lemma 5 and Corollary 1, the proof is similar to the proof of Theorem 2, thus it is omitted here.

Corollary 3 Let $R \in F(X \times Y)$, then f_R is a continuous β-model if and only if for any $i = 1,2,...,n$ and $A \in F(X)$,

$$\wedge_{y \in Y}(B_i(y) \leftrightarrow_m (A_i \circ_m R)(y)) \geqslant \wedge_{x \in X}(A_i(x) \leftrightarrow_m A_i(x)) \quad (10)$$

holds.

As shown in Ref. [11], while $\mathcal{A} = \{A_i\}_{1 \leqslant i \leqslant n}$ and $\mathcal{B} = \{B_i\}_{1 \leqslant i \leqslant n}$ are the fuzzy partitions of the universes $X = [a, b]$ and $Y = [c,d]$ respectively, the system $A_i \odot_m R = B_i, i = 1, 2,...,n$, is solvable and its least solution is $R_{\min} = \vee_{i=1}^n (A_i *_m B_i)$; the system $A_i \odot_m R = B_i, i = 1,2,...,n$, is also solvable and the greatest solution is $R_{\max} = \wedge_{i=1}^n (A_i \rightarrow_m B_i)$. Consider the cases of $m = 1$ and $m = -\infty$ we have the following corollary.

Corollary 4 Let $\mathcal{A} = \{A_i\}_{1 \leqslant i \leqslant n}$ and $\mathcal{B} = \{B_i\}_{1 \leqslant i \leqslant n}$ be fuzzy partitions of the universes of $X = [a,b]$ and $Y = [c, d]$, respectively, $(*)$ is a system of fuzzy rules, then all of the models $f_{R_G}, f_{R_L}, g_{R_G}$, and g_{R_L} which are defined as

$$f_{R_G}(A) = \wedge_{x \in X} I_3[A(x), \vee_{i=1}^n (A_i(x) \wedge B_i(y))],$$

$$f_{R_L}(A) = \wedge_{x \in X} I_1[A(x), \vee_{i=1}^n T_1(A_i(x), B_i(y))],$$

$$g_{R_G}(A) = \vee_{x \in X}\{A(x) \wedge [\wedge_{i=1}^n I_3(A_i(x), B_i(y))]\},$$

and

$$g_{R_L}(A) = \vee_{x \in X}\{T_1[A(x), \wedge_{i=1}^n I_1(A_i(x), B_i(y))]\},$$

respectively are continuous, where

$$T_1(x,y) = \max(0, x + y - 1),$$

$$T_3(x,y) = x \wedge y,$$

$$I_1(x,y) = \min(1, 1 + y - x)(\text{Lukasiewicz}),$$

$$I_3(x,y) = \begin{cases} 1, & 0 \leqslant x \leqslant y \leqslant 1, \\ y, & 0 \leqslant y \leqslant x \leqslant 1 \end{cases} (\text{Gödel implication}).$$

It should be pointed out that Theorem 3 and Corollary 3 are similar to the major findings of Ref. [1], but there is a great difference between the definitions of Schweizer-Sklar t-norm $*_m$, Schweizer-Sklar equivalence operator $\leftrightarrow_m$ and the t-norm $*$, equivalence operator $\leftrightarrow$ in Ref. [1].

4 Continuity of some commonly used models

Let $(*)$ be a system of fuzzy rules, $\mathcal{A} = \{A_i\}_{1 \leqslant i \leqslant n}$ and $\mathcal{B} = \{B_i\}_{1 \leqslant i \leqslant n}$ be the fuzzy partitions of the universes $X = [a,b]$ and $Y = [c,d]$, respectively. Now we consider the continuity of some commonly used models (only in the case of single-input single-output models).

1) Continuity of Mamdanian model

According to Mamdanian method, the inference relation of the ith rule is a fuzzy relation from X to Y, which is represented by $R_i(x,y) = A_i(x) \wedge B_i(y)$. Since the n inference rules should be joined by "or" (corresponding to set-theoretical operator "$\cup$"), the overall inference relation will be $R_M = \cup_{i=1}^n R_i$, i.e.,

$$R_M(x,y) = \vee_{i=1}^n (A_i(x) \wedge B_i(y)),$$

and the model is represented as

$$\begin{aligned} f_{R_M}(A)(y) &= \vee_{x \in X}(A(x) \wedge R_M(x,y)) \\ &= \vee_{x \in X} A(x) \wedge [\vee_{i=1}^n (A_i(x) \wedge B_i(y))]. \end{aligned}$$

It is easy to verify that, in general, the equality $f_{R_M}(A_i) = B_i, i = 1,2,...,n$, is not true. By Theorem 3, the Mamdanian model is not consistent, thus it is not continuous. If $\vee_{x \in X}$ in the model above is replaced by $\wedge_{x \in X}$, and the operation "$\wedge$" is replaced by Gödel implication, then the model f_{R_M} will become the model f_{R_G} in Corollary 4, which is continuous. If $R_M(x,y)$ is replaced by $\wedge_{i=1}^{n} I_3(A_i(x), B_i(y))$, then the model f_{R_M} will become the model g_{R_G} in Corollary 4, which is also continuous.

2) Continuity of the models constructed with compositional rules of inference (CRI) method

In such a fuzzy reasoning model, we usually link the left-hand side and right-hand side of the rules by an implication operator θ, i.e., the ith rule "IF x is A_i, THEN y is B_i" is represented as $\theta(A_i(x), B_i(y))$, join n rules by "or", then the overall inference relation is

$$R_C(x,y) = \vee_{i=1}^{n} \theta(A_i(x), B_i(y)),$$

and the model is represented as

$$\begin{aligned} f_{R_C}(A)(y) &= \vee_{x\in X}(A(x) \wedge R_C(x,y)) \\ &= \vee_{x\in X} A(x) \wedge [\vee_{i=1}^{n} \theta(A_i(x), B_i(y))], \end{aligned}$$

where the θ is often chosen as Gödel implication, Goguen implication, Lukasiewicz implication, and so on. It is easy to verify that the commonly used models constructed with CRI method are not consistent, of course they are not continuous. However, if we give up the above-mentioned using implication operators to generate the fuzzy inference relation R_C, but take R_C for the solution of the system of fuzzy relation equations

$$\vee_{x\in X}(A_i(x) \wedge R(x,y)) = B_i(y), \quad i = 1,2,\ldots,n,$$

then by Theorem 3, f_{R_C} is a continuous model.

3) Continuity of the $(+,\cdot)$ model and the weighted $(+,\cdot)$ model

In Mamdanian model, to replace $\vee$ and $\wedge$ by the bounded addition $+$ and the ordinary multiplication $\cdot$ respectively, we will obtain the $(+,\cdot)$ model which is expressed as

$$\begin{aligned} f_{(+,\cdot)}(A)(y) &= \vee_{x\in X}(A(x)R_{(+,\cdot)}(x,y)) \\ &= \vee_{x\in X} A(x) \left[\sum_{i=1}^{n} (A_i(x)\cdot B_i(y))\right], \end{aligned}$$

its weighted form is

$$\begin{aligned} f_{(+,\cdot)}(A)(y) &= \vee_{x\in X}(A(x)R_{(+,\cdot)}(x,y)) \\ &= \vee_{x\in X} A(x) \left[\sum_{i=1}^{n} \omega_i(A_i(x)\cdot B_i(y))\right], \end{aligned}$$

where every $\omega_i \geqslant 0$ and $\sum_{i=1}^{n} \omega_i = 1$. It is easy to verify that these two models are not continuous.

5 Interpolation mechanism of continuous fuzzy reasoning models

An important conclusion in Ref. [12] tells us that every fuzzy controller based on CRI algorithm in common use can be regarded as an interpolation method that is a kind of approximation to a certain response function. Reference [13] pointed out that if a fuzzy controller is without universal property with respect to function approximation but only of step response ability, then it can hardly be used in any practical fuzzy control systems. Thus, it is very important to verify whether a fuzzy controller constructed by a continuous α- or β-model is of universal property with respect to function approximation. In this section, we will discuss this problem.

Let S be a system, $X = [a,b]$ and $Y = [c,d]$ be the universe of input variable and the universe of output variable respectively, and $(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$, where $a < x_1 < x_2 < \cdots < x_n < b$, $c < y_1 < y_2 < \cdots < y_n < d$, are a family of input-output data. Based on these data, we make the fuzzy partitions of X and Y, namely, $\mathcal{A} = \{A_i\}_{1\leqslant i\leqslant n}$ and $\mathcal{B} = \{B_i\}_{1\leqslant i\leqslant n}$, where x_i and y_i are the peak-points of A_i and B_i, respectively. In order to obtain a fuzzy inference relation, we build a system of fuzzy relation equations:

$$A_i \odot_m R = B_i, \ i = 1,2,\ldots,n, \tag{11}$$

or

$$A_i \circ_m R = B_i, \ i = 1,2,\ldots,n. \tag{12}$$

With any solution R of Eq. (11), we can build a continuous α-model:

$$f_R : F(X) \to F(Y), \ A \mapsto A \odot_m R.$$

With any solution R of Eq. (12), we can build a continuous β-model:

$$f_R : F(X) \to F(Y), \ A \mapsto A \circ_m R.$$

According to the finding of Ref. [14], the direct solution set of Eq. (11) is

$$\left[\overset{\vee}{R}, \hat{R}\right] = \left\{R \middle| \overset{\vee}{R} \leqslant R \leqslant \hat{R}\right\},$$

where

$$\overset{\vee}{R}(x,y) = \vee_{i=1}^{n}[A_i(x) *_m B_i(y)], \ (x,y) \in X \times Y,$$

$$\hat{R}(x,y) = \begin{cases} B_1(y), & x = x_1, \\ \cdots & \\ B_n(y), & x = x_n, \\ 1, & \text{else}, \end{cases}$$

and the attainable solution set of Eq. (12) is

$$[\underline{R}, \overline{R}] = \left\{R \middle| \underline{R} \leqslant R \leqslant \overline{R}\right\},$$

where

$$\underline{R}(x,y) = \begin{cases} B_1(y), & x = x_1, \\ \cdots & \\ B_n(y), & x = x_n, \\ 0, & \text{otherwise}, \end{cases}$$

$$\overline{R}(x,y) = \wedge_{i=1}^{n}[A_i(x) \rightarrow_m B_i(y)].$$

Let

$$\mathcal{M}_\alpha = \left\{f_R | f_R : F(X) \rightarrow F(Y), A \mapsto A \odot_m R, R \in \left[\overset{\vee}{R}, \hat{R}\right]\right\},$$

$$\mathcal{M}_\beta = \left\{f_R | f_R : F(X) \rightarrow F(Y), A \mapsto A \circ_m R, R \in [\underline{R}, \overline{R}]\right\}.$$

Then $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ are the cluster of the continuous α-models and the cluster of continuous β-models of system S, respectively.

In the following, we will show that for a practical fuzzy control system, the controller constructed by any element of $\mathcal{M}_\alpha \cup \mathcal{M}_\beta$ is approximate to an interpolation function.

Let R be an attainable solution of Eq. (12), i.e., $R \in \mathcal{M}_\beta$. Then R determines a continuous β-model f_R. For an input fuzzy set $A^* \in F(X)$, the corresponding output fuzzy set $B^* \in F(Y)$ should be computed by

$$B^*(y) = f_R(A^*) = \vee_{x \in X}[A^*(x) *_m R(x,y)]. \tag{13}$$

When the input value is a crisp quantity $x' \in X$, in order to use Eq. (13), we change x' into a fuzzy set by the following singleton fuzzification scheme:

$$A'(x) = \begin{cases} 1, & x = x', \\ 0, & x \neq x'. \end{cases}$$

Substituting it into Eq. (13), we obtain the inference result:

$$B'(y) = R(x', y).$$

To defuzzify B' into a crisp quantity y', the commonly used method of centroid is adopted:

$$y' = \frac{\int_{y \in Y} yB'(y)\mathrm{d}y}{\int_{y \in Y} B'(y)\mathrm{d}y} = \frac{\int_c^d yR(x', y)\mathrm{d}y}{\int_c^d R(x', y)\mathrm{d}y} \triangleq f(x').$$

So we obtain the response function of the fuzzy controller constructed by f_R, that is, $y' = f(x')$. Let

$$h_1 = y_1 - c,$$
$$h_i = y_i - y_{i-1},\ i = 2,3,...,n.$$

Because $\mathcal{A}$ and $\mathcal{B}$ are fuzzy partitions, their elements have the Kronecker's property:

$$A_i(x_j) = B_i(y_j) = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

According to the definition of integral, we have

$$\frac{\int_c^d yR(x', y)\mathrm{d}y}{\int_c^d R(x', y)\mathrm{d}y} \approx \frac{\sum_{i=1}^{n} y_i R(x', y_i)h_i}{\sum_{i=1}^{n} R(x', y_i)h_i}$$

$$= \sum_{i=1}^{n} \frac{R(x', y_i)h_i}{\sum_{j=1}^{n} R(x', y_j)h_j} y_i$$

$$\triangleq F(x').$$

Now fixing y_i temporarily, for every $j = 1,2,\ldots,n$, we consider the value of $R(x_j, y_i)$. Because

$$\underline{R}(x_j, y_i) = \begin{cases} B_i(y_i), & j = i, \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1, & j = i, \\ 0, & \text{otherwise}, \end{cases}$$

for every $k = 1,2,\ldots,n$, if $i = j$, we have

$$A_k(x_j) \rightarrow_m B_k(y_i) = A_k(x_i) \rightarrow_m B_k(y_i)$$
$$= \begin{cases} 1 \rightarrow_m 1, & k = i, \\ 0 \rightarrow_m 0, & k \neq i \end{cases}$$
$$= 1.$$

If $i \neq j$, we have

$$A_k(x_j) \rightarrow_m B_k(y_i) = \begin{cases} 0 \rightarrow_m 1, & k = i, \\ 1 \rightarrow_m 0, & k = j, \\ 0 \rightarrow_m 0, & \text{else} \end{cases}$$
$$= \begin{cases} 0, & k = j, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, we have

$$\overline{R}(x_j, y_i) = \wedge_{i=1}^{n} [A_k(x_j) \rightarrow_m B_k(y_i)]$$
$$= \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Because $\underline{R} \leqslant R \leqslant \overline{R}$, we have

$$R(x_j, y_i) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Then for every $k = 1,2,\ldots,n$, we have

$$F(x_k) = \sum_{i=1}^{n} \frac{R(x_k, y_i)h_i}{\sum_{j=1}^{n} R(x_k, y_j)h_j} y_i = y_k.$$

Put

$$A_i(x) = \frac{R(x_k, y_i)h_i}{\sum_{j=1}^{n} R(x_k, y_j)h_j}.$$

Obviously, $\mathcal{A} = \{A_i\}_{1 \leqslant i \leqslant n}$ is a partition of X, and $y = F(x)$ is a univariate piece-wise interpolation function taking $\mathcal{A}$ for its base functions group.

When $R \in \mathcal{M}_\alpha$, we have the similar result. Thereby we have the following theorem.

Theorem 4 Let $f_R \in \mathcal{M}_\alpha \cup \mathcal{M}_\beta$, assume that $y = f(x)$ is the response function of the fuzzy controller constructed by f_R, then there exists a fuzzy partition of X which is denoted as $\mathcal{A} = \{A_i\}_{1 \leqslant i \leqslant n}$, the interpolation function $y = f(x)$ taking $\mathcal{A}$ for its base functions group is approximate to $y = f(x)$.

6 Concluding remarks

In this paper, we introduce the typical operators of UL^*—Schweizer-Sklar operators into fuzzy systems, give the definitions of continuous models of fuzzy reasoning and their equivalent descriptions, prove that α-model and β-model are continuous if and only if they are consistent models (i.e., the models can reproduce the fuzzy rules). Moreover, we discuss the continuity of some commonly used models and give the modified forms of the models which are constructed by Mamdani algorithm and CRI algorithm. At last, we prove that when an α-model or a β-model is used as a fuzzy controller, it is approximate to an interpolation function, so it has universal property with respect to function approximation.

To sum up, when the Schweizer-Sklar operators are introduced into fuzzy reasoning models, the fuzzy reasoning models become flexible and can be tuned by the parameter m. These models have more advantages than others proposed in the past, and we can expect that they will have good application prospects.

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