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# Analysis of stability and robust stability for stochastic hybrid systems with impulsive effects

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**Abstract** In this article, the problems of stability and robust stability analysis are investigated for a class of Markovian switching stochastic systems, which has impulses at switching instants. The switching parameters considered form a continuous-time discrete-state homogeneous Markov process. Multiple Lyapunov techniques are used to derive sufficient conditions for stability in probability of the overall system. The conditions are in linear matrix inequalities form, and can be used to solve stabilization synthesis problems. The results are extended to the design of a robust-stabilized state-feedback controller as well. A numerical example shows the effectiveness of the proposed approach.

**Keywords** impulsive system, Markovian switching, stable in probability, linear matrix inequality (LMI), robust stability

## 1 Introduction

In recent years, stochastic systems have been widely used in practical life, especially in control and communication science. Therefore, the study of stochastic systems has become a hot spot in the control community. Additionally, its stability analysis has always been the focus of scholars' attention [1–3]. Methods of research are mainly the extensions of classical Lyapunov function method [4,5]. As a special class of stochastic systems, the Markovian switching stochastic system is constituted by subsystems perturbed by Wiener process and a switching signal governed by a Markov process. There has been an increasing interest for Markovian switching systems

since 1960. Many problems such as stability and stabilizability, have been addressed and results reported in Refs. [6–8]. Compared with the deterministic system, the research on the Markovian switching stochastic system is just at the beginning [9,10]. By using the Lyapunov function method, Ref. [9] gives a systematic study on the existence uniqueness of the solution and the exponential stability for the nonlinear Markovian switching stochastic system. Furthermore, in Ref. [9], the  $M$  matrix method is introduced to give several sufficient conditions for the stability of the system. Reference [10] discusses the problem of mean square exponential stabilization for the nonlinear Markovian switching stochastic system.

Because of the influence of modeling error, it is difficult to avoid uncertain factors in the model. However, the existence of such factors may lead to performance degradation, or even unstability of the system. Many researchers discussed the problems of robust stability for uncertain linear systems. The major methods are a further study based on the Riccati equation or linear matrix inequality (LMI) [11,12]. However, these references did not take noise into account. On the other hand, as is well known, many practical systems in physics, biology, engineering, and information science exhibit impulsive dynamical behaviors because of abrupt changes at certain instants during the dynamical processes. Some Lyapunov functions of switching systems are essentially unsuitable to analyze this kind of situation and will lead to large deviations. For this reason, research toward the system with impulsive effect has gained more attention. There are references for deterministic systems considering the impulsive effects [13–15]. However, for stochastic hybrid systems, there are few studies concerning impulsive phenomena.

In this article, for a class of Markovian switching stochastic systems having impulses at switching instants, the problems of stability and robust stability analysis are investigated. First, multiple Lyapunov techniques are used to derive sufficient conditions for stability in probability of the overall system. Furthermore, linear matrix inequalities method is used to solve the synthesis problem of stabilization and robust stability. At the same time, the corresponding state feedback gain matrices and impulsive gain matrices are computed.

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## 2 Problem statement and preliminaries

Let  $(\Omega, F, \{F_t\}_{t \geq 0}, P)$  be a complete probability space with a filtration  $\{F_t\}_{t \geq 0}$ . Let  $\{r(t), t \geq 0\}$  be a Markov chain on the probability space taking values in a finite state space  $S = \{1, 2, \dots, N\}$  with generator given by

$$P\{r(t + \Delta t) = j | r(t) = i\} = \begin{cases} \lambda_{ij}\Delta t + o(\Delta t), & i \neq j, \\ 1 + \lambda_{ii}\Delta t + o(\Delta t), & i = j, \end{cases}$$

where  $\Delta t > 0$ ,  $\lim_{\Delta t \rightarrow 0} o(\Delta t)/\Delta t = 0$ . Here,  $\lambda_{ij}$  is the transition rate from  $i$  to  $j$ , and  $\lambda_{ij} \geq 0$ ,  $\lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}$  are satisfied

for any value of  $i, j$  ( $i \neq j$ ).

Consider an Ito type Markovian switching system with stochastic perturbation of the form

$$\begin{cases} \begin{aligned} d\mathbf{x}(t) &= \mathbf{A}_{r(t)}\mathbf{x}(t)dt + \mathbf{B}_{r(t)}\mathbf{u}(t)dt \\ &\quad + \mathbf{C}_{r(t)}\mathbf{x}(t)dw(t), \quad r(t^+) = r(t), \end{aligned} \\ \mathbf{x}(t^+) &= \mathbf{E}_{r(t^+), r(t)}\mathbf{x}(t^-) + \mathbf{F}_{r(t^+), r(t)}\mathbf{u}(t^-), \quad r(t^+) \neq r(t), \\ \mathbf{x}(t_0^+) &= \mathbf{x}_0, \end{cases} \quad (1)$$

where  $\mathbf{x}(t) \in \mathbb{R}^n$  is the state vector,  $\mathbf{u}(t) \in \mathbb{R}^m$  is the control input,  $\mathbf{x}(t^+) := \lim_{h \rightarrow 0^+} \mathbf{x}(t+h)$ ,  $\mathbf{x}(t^-) := \lim_{h \rightarrow 0^+} \mathbf{x}(t-h)$ ,  $\mathbf{x}(t^-) = \mathbf{x}(t)$ . Let  $w(t) \in \mathbb{R}$  be a standard Wiener process defined on  $(\Omega, F, \{F_t\}_{t \geq 0}, P)$ .  $r(t) = i$  means that the subsystem  $\{\mathbf{A}_i, \mathbf{B}_i, \mathbf{C}_i\}$  is active;  $r(t) = i$  and  $r(t^+) = j$  mean that the system is switched from the  $i$ th subsystem to the  $j$ th subsystem at time instant  $t$ . At the switching times, there exists an impulse described by the second equation of Eq. (1).  $\mathbf{A}_i, \mathbf{B}_i, \mathbf{C}_i, \mathbf{E}_{j,i}, \mathbf{F}_{j,i}$ ,  $i, j \in \{1, 2, \dots, N\}$  are constant matrices with appropriate dimensions. Particularly,  $\mathbf{E}_{i,i} = \mathbf{I}_n$ ,  $\mathbf{F}_{i,i} = \mathbf{0}$ ,  $i \in \{1, 2, \dots, N\}$ , which means that the switching between the same subsystem is smooth without impulse, in other words, there is no impulse when a subsystem is remaining active.

For ease of research, the following assumptions are made for systems in this article:

**Assumption 1** There exists a unique stochastic process satisfying system (1).

**Assumption 2**  $r(t)$  is independent of  $w(t)$ , and it ensures that the switchings are finite in any finite time interval.

## 3 Main results

### 3.1 Stability analysis

For convenience, when  $r(t) = i$ ,  $r(t^+) = j$ ,  $\mathbf{A}_{r(t)}$ ,  $\mathbf{B}_{r(t)}$ ,  $\mathbf{C}_{r(t)}$ ,  $\mathbf{E}_{r(t^+), r(t)}$ ,  $\mathbf{F}_{r(t^+), r(t)}$  are denoted by  $\mathbf{A}_i$ ,  $\mathbf{B}_i$ ,  $\mathbf{C}_i$ ,  $\mathbf{E}_{j,i}$ ,  $\mathbf{F}_{j,i}$ ,  $i, j \in \{1, 2, \dots, N\}$ , respectively. To study the

stability of system (1), its autonomous situation is considered first, that is,

$$\begin{cases} d\mathbf{x}(t) = \mathbf{A}_i\mathbf{x}(t)dt + \mathbf{C}_i\mathbf{x}(t)dw(t), & r(t^+) = r(t), \\ \mathbf{x}(t^+) = \mathbf{E}_{j,i}\mathbf{x}(t^-), & r(t^+) \neq r(t), \\ \mathbf{x}(t_0^+) = \mathbf{x}_0. \end{cases} \quad (2)$$

Let  $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$  denote the family of all non-negative functions on  $\mathbb{R}^n \times \mathbb{R}_+ \times S$ , which are continuously twice differentiable in  $\mathbf{x}$  and once differentiable in  $t$ .

For system (2), the switched Lyapunov function is constructed as follows:

$$V(\mathbf{x}(t), i) = \mathbf{x}^T(t)\mathbf{P}_i\mathbf{x}(t), \quad (3)$$

where  $V(\mathbf{x}(t), i) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ , and  $\mathbf{P}_i$  are positive-definite matrices,  $1 \leq i \leq N$ . for  $V(\mathbf{x}(t), i)$ , an operator is defined by

$$\begin{aligned} LV(\mathbf{x}(t), i) &:= V_{\mathbf{x}}(\mathbf{x}(t), i)(\mathbf{A}_i\mathbf{x}(t)) \\ &\quad + 2^{-1}\text{tr}((\mathbf{C}_i\mathbf{x}(t))^T V_{\mathbf{xx}}(\mathbf{x}(t), i)\mathbf{C}_i\mathbf{x}(t)) \\ &\quad + \sum_{j=1}^N \lambda_{ij}V(\mathbf{x}(t), j), \end{aligned}$$

where

$$V_{\mathbf{x}}(\mathbf{x}(t), i) = \left( \frac{\partial V(\mathbf{x}(t), i)}{\partial x_1}, \frac{\partial V(\mathbf{x}(t), i)}{\partial x_2}, \dots, \frac{\partial V(\mathbf{x}(t), i)}{\partial x_n} \right),$$

$$V_{\mathbf{xx}}(\mathbf{x}(t), i) = \left( \frac{\partial^2 V(\mathbf{x}(t), i)}{\partial x_l \partial x_m} \right)_{n \times n},$$

$$l, m = 1, 2, \dots, n.$$

To design the controller for system (1), the definition and some useful lemmas are needed.

**Definition 1** [5] The solution to system (2) is assumed stable in probability if for any  $s \geq 0$  and  $\varepsilon > 0$ ,

$\lim_{x_0 \rightarrow 0} P \left\{ \sup_{t > s} \|\mathbf{x}^{s, x_0}(t)\| > \varepsilon \right\} = 0$ . Here,  $\mathbf{x}^{s, x_0}(t)$  denotes the sample path of the solution to system (2) starting from  $x_0$  at time  $s$ .

**Lemma 1** (Schur complement) The following three inequalities are equivalent:

$$1) \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} < \mathbf{0};$$

$$2) \mathbf{A} < \mathbf{0} \text{ and } \mathbf{C} - \mathbf{B}^T \mathbf{A}^{-1} \mathbf{B} < \mathbf{0};$$

$$3) \mathbf{C} < \mathbf{0} \text{ and } \mathbf{A} - \mathbf{B} \mathbf{C}^{-1} \mathbf{B}^T < \mathbf{0}.$$

**Lemma 2** [16] Let  $\mathbf{M}, \mathbf{N}, \mathbf{\Gamma}$  be given matrices with appropriate dimensions and  $\mathbf{\Gamma}$  satisfy  $\mathbf{\Gamma}^T \mathbf{\Gamma} \leq \mathbf{I}$ , then for any  $\mu > 0$ ,  $\mathbf{M}\mathbf{\Gamma}\mathbf{N} + (\mathbf{M}\mathbf{\Gamma}\mathbf{N})^T \leq \mu^2 \mathbf{M}\mathbf{M}^T + \mu^{-2} \mathbf{N}^T \mathbf{N}$  can be obtained.

**Lemma 3** If there exists switched Lyapunov function

(3) satisfying

- 1)  $LV(\mathbf{x}(t), i) \leq 0, r(t^+) = r(t);$
- 2)  $EV(\mathbf{x}(t^+), j) \leq EV(\mathbf{x}(t), i), r(t^+) \neq r(t).$

Then system (2) is stable in probability.

**Proof** It is assumed the switching instants are  $t_1, t_2, \dots$ , where  $t_1 < t_2 < \dots$ , for  $\forall k \in N$  in  $(t_{k-1}, t_k]$ , general Ito's formula results in

$$\begin{aligned} &EV(\mathbf{x}(t_k), i_{k-1}) \\ &= EV(\mathbf{x}(t_{k-1}^+), i_{k-1}) + E \int_{t_{k-1}^+}^{t_k} LV(\mathbf{x}(s), i_{k-1}) ds. \end{aligned}$$

From condition 1), the following can be obtained:

$$EV(\mathbf{x}(t_k), i_{k-1}) \leq EV(\mathbf{x}(t_{k-1}^+), i_{k-1}). \quad (4)$$

Use the fundamental result in Ref. [5] that the process  $V(\mathbf{x}(t), i_{k-1})$  is a supermartingale. By the supermartingale inequality, for  $\forall \varepsilon_1 > 0$ , the following can be obtained:

$$\begin{aligned} &P \left\{ \sup_{t_{k-1}^+ \leq t \leq t_k} V(\mathbf{x}(t), i_{k-1}) \geq \varepsilon_1 \right\} \\ &\leq \varepsilon_1^{-1} EV(\mathbf{x}(t_{k-1}^+), i_{k-1}). \end{aligned} \quad (5)$$

On the other hand, combining Eq. (4) and condition 2), the following can be obtained:

$$\begin{aligned} &EV(\mathbf{x}(t_{k-1}^+), i_{k-1}) \leq EV(\mathbf{x}(t_{k-1}), i_{k-2}) \\ &\leq EV(\mathbf{x}(t_{k-2}^+), i_{k-2}) \leq \dots \leq EV(\mathbf{x}(t_0^+), i_0) \\ &= EV(\mathbf{x}_0) = V(\mathbf{x}_0). \end{aligned} \quad (6)$$

From Eqs. (5) and (6), it is derived that

$$\begin{aligned} &P \left\{ \sup_{t_{k-1}^+ \leq t \leq t_k} V(\mathbf{x}(t), i_{k-1}) \geq \varepsilon_1 \right\} \leq \varepsilon_1^{-1} EV(\mathbf{x}_0) \\ &= \varepsilon_1^{-1} V(\mathbf{x}_0). \end{aligned}$$

Considering the arbitrariness of  $k$ , the following can be obtained:

$$P \left\{ \sup_{t \geq 0} V(\mathbf{x}(t), i) \geq \varepsilon_1 \right\} \leq \varepsilon_1^{-1} V(\mathbf{x}_0). \quad (7)$$

The radial unboundedness of  $V$  implies that  $\forall \varepsilon > 0, \exists \varepsilon_1(\varepsilon) > 0$ , s.t.  $V(\mathbf{x}(t), i) \geq \varepsilon_1$  whenever  $\|\mathbf{x}(t)\| \geq \varepsilon$ . The positive definiteness and continuity of  $V$  imply that  $\forall \varepsilon_2 > 0, \exists \delta(\varepsilon_2) > 0$ , s.t.  $\varepsilon_1^{-1} V(\mathbf{x}_0) \leq \varepsilon_2$  whenever  $\|\mathbf{x}_0\| \leq \delta$ . Therefore, for  $\forall \mathbf{x}_0$ , Eq. (7) is equivalent to  $P \left\{ \sup_{t \geq 0} \|\mathbf{x}(t)\| \geq \varepsilon \right\} \leq \varepsilon_2$ , whenever  $\|\mathbf{x}_0\| \leq \delta$ . Let  $\mathbf{x}_0$  tend to zero and the desired result can be derived.

Next, according to Lemma 3, sufficient conditions ensuring the system (2) is stable in probability are given.

**Theorem 1** For system (2), if there exist positive-definite matrices  $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_N$  satisfying

$$\mathbf{A}_i^T \mathbf{P}_i + \mathbf{P}_i \mathbf{A}_i + \mathbf{C}_i^T \mathbf{P}_i \mathbf{C}_i + \sum_{j=1}^N \lambda_{ij} \mathbf{P}_j < \mathbf{0}, \quad (8)$$

$$\begin{bmatrix} \mathbf{P}_j & \mathbf{P}_j \mathbf{E}_{j,i}^T \\ \mathbf{E}_{j,i} \mathbf{P}_j & \mathbf{P}_i \end{bmatrix} \geq \mathbf{0}, \quad i, j = 1, 2, \dots, N, \quad (9)$$

then system (2) is stable in probability.

**Proof** For system (2), choose the switched Lyapunov function Eq. (3), the following can be obtained:

$$\begin{aligned} LV(\mathbf{x}(t), i) &= \mathbf{x}^T(t) \left( \mathbf{A}_i^T \mathbf{P}_i + \mathbf{P}_i \mathbf{A}_i + \mathbf{C}_i^T \mathbf{P}_i \mathbf{C}_i \right. \\ &\quad \left. + \sum_{j=1}^N \lambda_{ij} \mathbf{P}_j \right) \mathbf{x}(t), \quad r(t^+) = r(t), \end{aligned}$$

$$\begin{aligned} &EV(\mathbf{x}(t^+), j) - EV(\mathbf{x}(t), i) \\ &= E \{ \mathbf{x}^T(t) (\mathbf{E}_{j,i}^T \mathbf{P}_j \mathbf{E}_{j,i} - \mathbf{P}_i) \mathbf{x}(t) \}, \quad r(t^+) \neq r(t). \end{aligned}$$

From Eq. (8), it is derived that  $LV(\mathbf{x}(t), i) \leq 0$ . Combining Eq. (9) and Lemma 1,  $\mathbf{E}_{j,i} \mathbf{P}_j \mathbf{E}_{j,i}^T - \mathbf{P}_i \leq \mathbf{0}$ , thus  $\mathbf{E}_{j,i}^T \mathbf{P}_j \mathbf{E}_{j,i} - \mathbf{P}_i \leq \mathbf{0}$ , therefore,  $EV(\mathbf{x}(t^+), j) \leq EV(\mathbf{x}(t), i)$ . From Lemma 3 it is known that system (2) is stable in probability.

Based on Theorem 1, the problem of stabilization of system (1) is further considered. By using the method of LMI, the controller of the subsystem is designed to ensure the stability of the closed-loop system. For this, the switched state feedback controller is chosen as

$$\begin{cases} \mathbf{u}(t) = \mathbf{K}_i \mathbf{x}(t), & r(t^+) = r(t), \\ \mathbf{u}(t^-) = \mathbf{L}_{j,i} \mathbf{x}(t^-), & r(t^+) \neq r(t), \end{cases} \quad (10)$$

where  $\mathbf{K}_i$  is the state feedback gain matrix and  $\mathbf{L}_{j,i}$  is the impulsive control gain matrix,  $i, j \in \{1, 2, \dots, N\}$ .

**Theorem 2** For system (1), if there exist  $\rho > 0$ , positive-definite matrices  $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_N$ , appropriate matrices  $\mathbf{Y}_{1,1}, \dots, \mathbf{Y}_{1,N}, \dots, \mathbf{Y}_{N,N}$  satisfying

$$\mathbf{A}_i^T \mathbf{P}_i + \mathbf{P}_i \mathbf{A}_i + 2\rho \mathbf{P}_i \mathbf{B}_i \mathbf{B}_i^T \mathbf{P}_i + \mathbf{C}_i^T \mathbf{P}_i \mathbf{C}_i + \sum_{j=1}^N \lambda_{ij} \mathbf{P}_j < \mathbf{0},$$

$$\begin{bmatrix} \mathbf{P}_j & \mathbf{P}_j \mathbf{E}_{j,i}^T + \mathbf{Y}_{j,i}^T \mathbf{F}_{j,i}^T \\ \mathbf{E}_{j,i} \mathbf{P}_j + \mathbf{F}_{j,i} \mathbf{Y}_{j,i} & \mathbf{P}_i \end{bmatrix} \geq \mathbf{0},$$

$$i, j = 1, 2, \dots, N,$$

then there exists a state feedback controller (10) ensuring that the closed-loop system is stable in probability. The corresponding state feedback gain matrices and the impulsive control gain matrices are given as

$$\mathbf{K}_i = \rho \mathbf{B}_i^T \mathbf{P}_i, \quad \mathbf{L}_{j,i} = \mathbf{Y}_{j,i} \mathbf{P}_j^{-1}, \quad i, j = 1, 2, \dots, N. \quad (11)$$

**Proof** By adding controller Eqs. (10) and (11) into system (1), the results are easily proved to hold with the similar processes of proof in Theorem 1.

### 3.2 Analysis of robust stability

Based on the results achieved above, the authors further extend them to uncertain systems, and let the subsystem be stable in probability for all permitted uncertainty. Consider the system as follows:

$$\left\{ \begin{array}{l} \mathrm{d}\mathbf{x}(t) = (\mathbf{A}_i + \Delta\mathbf{A}_i)\mathbf{x}(t)\mathrm{d}t + \mathbf{B}_i\mathbf{u}(t)\mathrm{d}t \\ \quad + \mathbf{C}_i\mathbf{x}(t)\mathrm{d}w(t), \quad r(t^+) = r(t), \\ \mathbf{x}(t^+) = (\mathbf{E}_{j,i} + \Delta\mathbf{E}_{j,i})\mathbf{x}(t^-) + \mathbf{F}_{j,i}\mathbf{u}(t^-), \\ \quad r(t^+) \neq r(t), \\ \mathbf{x}(t_0^+) = \mathbf{x}_0, \end{array} \right. \quad (12)$$

where the uncertain parameter matrices have the following structures:

$$[\Delta\mathbf{A}_i, \Delta\mathbf{E}_{j,i}] = \mathbf{H}_i\Gamma_i[\mathbf{D}_i, \mathbf{W}_{j,i}]. \quad (13)$$

Here,  $\mathbf{H}_i$ ,  $\mathbf{D}_i$ ,  $\mathbf{W}_{j,i}$  are given constant matrices of appropriate dimensions,  $\Gamma_i$  are unknown matrices satisfying  $\Gamma_i^T\Gamma_i \leq \mathbf{I}$ ,  $i, j = 1, 2, \dots, N$ .

**Theorem 3** For system (12), if there exist  $\rho > 0$ , positive-definite matrices  $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_N$ , appropriate matrices  $\mathbf{Y}_{1,1}, \dots, \mathbf{Y}_{1,N}, \dots, \mathbf{Y}_{N,N}$  and positive constants  $\mu_1, \mu_2, \dots, \mu_N, \eta_1, \eta_2, \dots, \eta_N$  satisfying

$$\begin{bmatrix} \Omega_i & 2\rho\mathbf{P}_i\mathbf{B}_i & \mu_i\mathbf{P}_i\mathbf{H}_i & \mu_i^{-1}\mathbf{D}_i^T \\ 2\rho\mathbf{B}_i^T\mathbf{P}_i & -2\rho\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mu_i\mathbf{H}_i^T\mathbf{P}_i & \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mu_i^{-1}\mathbf{D}_i & \mathbf{0} & \mathbf{0} & -\mathbf{I} \end{bmatrix} < \mathbf{0}, \quad (14)$$

$$\begin{bmatrix} \mathbf{P}_j & \mathbf{P}_j\mathbf{E}_{j,i}^T + \mathbf{Y}_{j,i}^T\mathbf{F}_{j,i}^T & \mathbf{0} & \mathbf{P}_j\mathbf{W}_{j,i}^T \\ \mathbf{E}_{j,i}\mathbf{P}_j + \mathbf{F}_{j,i}\mathbf{Y}_{j,i} & \mathbf{P}_i & \eta_j^2\mathbf{H}_i & \mathbf{0} \\ \mathbf{0} & \eta_j^2\mathbf{H}_i^T & \eta_j^2\mathbf{I} & \mathbf{0} \\ \mathbf{W}_{j,i}\mathbf{P}_j & \mathbf{0} & \mathbf{0} & \eta_j^2\mathbf{I} \end{bmatrix} \geq \mathbf{0}, \quad (15)$$

$i, j = 1, 2, \dots, N,$

where  $\Omega_i = \mathbf{A}_i^T\mathbf{P}_i + \mathbf{P}_i\mathbf{A}_i + \mathbf{C}_i^T\mathbf{P}_i\mathbf{C}_i + \sum_{j=1}^N \lambda_{ij}\mathbf{P}_j$ . Then the state feedback controller Eq. (10) ensures that the closed-loop system is stable in probability. The corresponding state feedback gain matrices and the impulsive control gain matrices are given as

$$\mathbf{K}_i = \rho\mathbf{B}_i^T\mathbf{P}_i, \quad \mathbf{L}_{j,i} = \mathbf{Y}_{j,i}\mathbf{P}_j^{-1}, \quad i, j = 1, 2, \dots, N.$$

**Proof** By Lemma 1, Eq. (14) is equivalent to

$$\mathbf{A}_i^T\mathbf{P}_i + \mathbf{P}_i\mathbf{A}_i + 2\rho\mathbf{P}_i\mathbf{B}_i\mathbf{B}_i^T\mathbf{P}_i + \mathbf{C}_i^T\mathbf{P}_i\mathbf{C}_i$$

$$+ \sum_{j=1}^N \lambda_{ij}\mathbf{P}_j + \mu_i^2\mathbf{P}_i\mathbf{H}_i\mathbf{H}_i^T\mathbf{P}_i + \mu_i^{-2}\mathbf{D}_i^T\mathbf{D}_i < \mathbf{0}.$$

By combining Lemma 2 and Eq. (13), the following can be obtained:

$$\begin{aligned} & (\mathbf{A}_i + \Delta\mathbf{A}_i)^T\mathbf{P}_i + \mathbf{P}_i(\mathbf{A}_i + \Delta\mathbf{A}_i) + 2\rho\mathbf{P}_i\mathbf{B}_i\mathbf{B}_i^T\mathbf{P}_i \\ & + \mathbf{C}_i^T\mathbf{P}_i\mathbf{C}_i + \sum_{j=1}^N \lambda_{ij}\mathbf{P}_j < \mathbf{0}. \end{aligned}$$

On the other hand, Eq. (15) is equivalent to

$$\begin{bmatrix} -\mathbf{P}_j & -\mathbf{P}_j\mathbf{E}_{j,i}^T - \mathbf{Y}_{j,i}^T\mathbf{F}_{j,i}^T & \mathbf{0} & -\mathbf{P}_j\mathbf{W}_{j,i}^T \\ -\mathbf{E}_{j,i}\mathbf{P}_j - \mathbf{F}_{j,i}\mathbf{Y}_{j,i} & -\mathbf{P}_i & -\eta_j^2\mathbf{H}_i & \mathbf{0} \\ \mathbf{0} & -\eta_j^2\mathbf{H}_i^T & -\eta_j^2\mathbf{I} & \mathbf{0} \\ -\mathbf{W}_{j,i}\mathbf{P}_j & \mathbf{0} & \mathbf{0} & -\eta_j^2\mathbf{I} \end{bmatrix} \leq \mathbf{0}. \quad (16)$$

From Lemma 1, it is known that Eq. (16) is equivalent to

$$\begin{bmatrix} -\mathbf{P}_j & -\mathbf{P}_j\mathbf{E}_{j,i}^T - \mathbf{Y}_{j,i}^T\mathbf{F}_{j,i}^T \\ -\mathbf{E}_{j,i}\mathbf{P}_j - \mathbf{F}_{j,i}\mathbf{Y}_{j,i} & -\mathbf{P}_i \end{bmatrix} + \eta_j^2 \begin{bmatrix} \mathbf{0} \\ \mathbf{H}_i \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{H}_i^T \end{bmatrix} + \eta_j^{-2} \begin{bmatrix} -\mathbf{P}_j\mathbf{W}_{j,i}^T \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} -\mathbf{W}_{j,i}\mathbf{P}_j & \mathbf{0} \end{bmatrix} \leq \mathbf{0}.$$

And by combining Lemma 2 and Eq. (13) again, the following can be obtained:

$$\begin{bmatrix} \mathbf{P}_j & \mathbf{P}_j(\mathbf{E}_{j,i} + \Delta\mathbf{E}_{j,i})^T + \mathbf{Y}_{j,i}^T\mathbf{F}_{j,i}^T \\ (\mathbf{E}_{j,i} + \Delta\mathbf{E}_{j,i})\mathbf{P}_j + \mathbf{F}_{j,i}\mathbf{Y}_{j,i} & \mathbf{P}_i \end{bmatrix} \geq \mathbf{0}.$$

It is known from Theorem 2 that the closed-loop system is stable in probability, and

$$\mathbf{K}_i = \rho\mathbf{B}_i^T\mathbf{P}_i, \quad \mathbf{L}_{j,i} = \mathbf{Y}_{j,i}\mathbf{P}_j^{-1}, \quad i, j = 1, 2, \dots, N.$$

## 4 Numerical example

Consider the Markovian switching stochastic controlled systems with impulsive effects of the form:

$$\left\{ \begin{array}{l} \mathrm{d}\mathbf{x}(t) = (\mathbf{A}_i + \Delta\mathbf{A}_i)\mathbf{x}(t)\mathrm{d}t + \mathbf{B}_i\mathbf{u}(t)\mathrm{d}t \\ \quad + \mathbf{C}_i\mathbf{x}(t)\mathrm{d}w(t), \quad r(t^+) = r(t), \\ \mathbf{x}(t^+) = (\mathbf{E}_{j,i} + \Delta\mathbf{E}_{j,i})\mathbf{x}(t^-) + \mathbf{F}_{j,i}\mathbf{u}(t^-), \\ \quad r(t^+) \neq r(t), \\ [\Delta\mathbf{A}_i, \Delta\mathbf{E}_{j,i}] = \mathbf{H}_i\Gamma_i[\mathbf{D}_i, \mathbf{W}_{j,i}], \end{array} \right.$$

where the Markov process is given by generator  $\mathbf{\Pi} = (\lambda_{ij}), i, j \in \{1, 2\}$ .

$$A_1 = \begin{bmatrix} -9.3 & -1 \\ 0 & -11 \end{bmatrix}, A_2 = \begin{bmatrix} -10 & 0 \\ 1 & -10 \end{bmatrix},$$

$$B_1 = I_2, B_2 = \begin{bmatrix} 0.9 & 0 \\ 0 & 1.1 \end{bmatrix},$$

$$C_1 = C_2 = 2I_2,$$

$$H_1 = H_2 = 0.1I_2,$$

$$\mathbf{\Pi} = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix},$$

$$F_{12} = I_2, F_{21} = \begin{bmatrix} -1.1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$E_{12} = 2I_2, E_{21} = I_2,$$

$$W_{12} = 0.3I_2, W_{21} = -0.3I_2,$$

$$\rho = 1, \mu_1 = \mu_2 = 0.5, \eta_1 = \eta_2 = 0.1.$$

Compute these with the LMI toolbox of Matlab, and the following results are obtained:

$$P_1 = \begin{bmatrix} 0.0458 & 0.0002 \\ 0.0002 & 0.0469 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.0451 & -0.0003 \\ -0.0003 & 0.0454 \end{bmatrix},$$

$$Y_{12} = \begin{bmatrix} -0.0915 & -0.0005 \\ -0.0005 & -0.0957 \end{bmatrix},$$

$$Y_{21} = \begin{bmatrix} 0.0410 & -0.0003 \\ -0.0003 & 0.0454 \end{bmatrix}.$$

The corresponding state feedback gain matrices and the impulsive control gain matrices are as follows:

$$K_1 = \begin{bmatrix} 0.0458 & 0.0002 \\ 0.0002 & 0.0469 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 0.0405 & -0.0003 \\ -0.0003 & 0.0500 \end{bmatrix},$$

$$L_{12} = \begin{bmatrix} 66.5890 & 0.4216 \\ 0.4216 & 66.0362 \end{bmatrix},$$

$$L_{21} = \begin{bmatrix} 87.4209 & -0.4512 \\ -0.4512 & 85.3702 \end{bmatrix}.$$

In a word, the whole uncertain impulsive hybrid system is stable in probability according to Definition 1. Therefore, the method of controlling is effective.

## 5 Conclusions

For a class of Markovian switching stochastic systems with impulses at switching instants, the authors mainly use the method of multiple Lyapunov techniques and LMI to study their stochastic stability. Furthermore, the problems of stability and robust stability analysis are investigated. At the same time, the controller of subsystems is designed to guarantee the stability of the overall system. Future research directions include extending the results presented here to more general systems such as nonlinear systems and time-delayed systems.

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