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RESEARCH ARTICLE

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Design of controllers for a class of switched nonlinear systems based on backstepping method

Abstract The backstepping method is applied to a certain class of switched nonlinear systems to design state feedback controllers and a switching law based on multi-Lyapunov functions. The state feedback controllers and the switching law that can stabilize the system are developed. The switched nonlinear systems with uncertainties can be stabilized robustly by using the proposed method. Finally, simulation results show the effectiveness of the method.

Keywords switched nonlinear systems, backstepping method, multi-Lyapunov functions, robust stabilization

1 Introduction

Switched systems are a class of important hybrid systems consisting of continuous or discrete subsystems by rule of a switching law [1]. It is important to develop stability and stabilization of switched systems and many works in this direction have appeared [2–4]. Recently, Astrom et al. proposed a minimum switching law approach for a class of switched systems, where there exists a Lyapunov function for each subsystem [5]. Pettersson proposed the projection approach to solve the stabilization for switched nonlinear systems [6]. The concept of uniform norm form for single-input single-output (SISO) switched nonlinear systems is proposed, and stabilization for this kind of switched system is developed [7]. The feedback control law and switching rule is designed for switched nonlinear systems based on common Lyapunov function [8].

Feedback control law can be designed for a class of strict-feedback nonlinear systems based on backstepping technique, which consists of n iterative steps. The virtual control inputs are designed from the first step to the $(n-1)$th step, and the control input is designed in the nth step [9]. In this paper, the backstepping technique will be utilized to design controllers for a class of switched nonlinear systems consisting of strict-feedback subsystems. Finally, the simulation result shows the effectiveness of the backstepping technique.

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2 Problem formulation and preliminaries

Consider the switched nonlinear system as follows:

$$\dot{x} = f_i(x), \tag{1}$$

where i: $[0,+\infty) \to I = \{1,2,\ldots,N\}$ is a switching signal, $x \in R^n$ and the mapping $f_i(\cdot)$: $R^n \to R^n$ is smooth enough.

Lemma [1] For Eq. (1), if there exist continuous differentiable positive function V_i, $i \in I$, $\dot{V}_i < 0$, and $V_i(x(\tau_{i,k})) \leqslant V_i(x(\tau_{i,k-1}))$, where $\tau_{i,k}$ denotes the time when the ith subsystem is activated at the kth time, then the switched system is stable.

Consider the following pure feedback switched nonlinear system:

$$\begin{cases} \dot{x} = f_{i0}(x) + g_{i0}(x)z_1, \\ \dot{z}_1 = f_{i1}(x,z_1) + g_{i1}(x,z_1)z_2, \\ \cdots \\ \dot{z}_{k-1} = f_{ik-1}(x,z_1,\ldots,z_{k-1}) + g_{ik-1}(x,z_1,\ldots,z_{k-1})z_k, \\ \dot{z}_k = f_{ik}(x,z_1,\ldots,z_k) + g_{ik}(x,z_1,\ldots,z_k)u_i, \end{cases} \tag{2}$$

where $x \in R^n$, $z_1,z_2,\ldots,z_k$ are scalars, and $u_i \in R$ are control input. $f_{i0},f_{i1},\ldots,f_{ik}$ are smooth mappings that satisfy $f_{ij}(0) = 0$, $i \in I$, $0 \leqslant j \leqslant k$. $g_{i0},g_{i1},\ldots,g_{ik}$ are smooth enough and satisfy $g_{ij}(x,z_1,\ldots,z_j) \neq 0$, $i \in I$, $1 \leqslant j \leqslant k$.

If there are uncertainties in Eq. (2), then it can be described as follows:

$$\begin{cases} \dot{x} = f_{i0}(x) + g_{i0}(x)z_1 + \delta_{ix}(x,z), \\ \dot{z}_1 = f_{i1}(x,z_1) + g_{i1}(x,z_1)z_2 + \delta_{iz_1}(x,z), \\ \cdots \\ \dot{z}_{k-1} = f_{ik-1}(x,z_1,\ldots,z_{k-1}) + g_{ik-1}(x,z_1,\ldots,z_{k-1})z_k \\ \qquad + \delta_{iz_{k-1}}(x,z), \\ \dot{z}_k = f_{ik}(x,z_1,\ldots,z_k) + g_{ik}(x,z_1,\ldots,z_k)u_i + \delta_{iz_k}(x,z), \end{cases} \tag{3}$$

where $\delta_{ix},\delta_{iz_1},\ldots,\delta_{iz_k}$ are uncertainties and satisfy the inequalities

$$\|\delta_{ix}\| \leqslant a_i\|x\|_2, \tag{4}$$

$$|\delta_{iz_m}| \leqslant b_{im}\|x\|_2 + \sum_{n=1}^{m} c_{in}|z_n|, \quad m=1,2,\ldots,k, \tag{5}$$

where a_i, b_{im}, c_{in} are positive real values and the other symbols are the same to those of Eq. (2).

3 Main result

3.1 Controller design for switched nonlinear systems

Consider the following switched nonlinear system:

$$\dot{\eta} = f_i(\eta) + g_i(\eta)\xi, \tag{6}$$

$$\dot{\xi} = f_{ia}(\eta,\xi) + g_{ia}(\eta,\xi)u_i, \tag{7}$$

where $\eta \in R^n$, $\xi \in R$, $u_i \in R$. As $g_{ia}(\eta,\xi) \neq 0$, let

$$u_i = (u_{ia} - f_{ia}(\eta,\xi))/g_{ia}(\eta,\xi).$$

Then Eqs. (6) and (7) can be described as

$$\dot{\eta} = f_i(\eta) + g_i(\eta)\xi, \tag{8}$$

$$\dot{\xi} = u_{ia}. \tag{9}$$

It is assumed that there exist a state feedback control law $\xi = \phi_i(\eta)$, $\phi_i(0) = 0$ and positive functions $V_{i0}(\eta)$ for Eq. (8), where $V_{i0}(\eta)$ satisfies

$$\dot{V}_{i0} = \frac{\partial V_{i0}[f_i(\eta) + g_i(\eta)\phi_i(\eta)]}{\partial \eta} \leqslant -\alpha_i(\eta), \tag{10}$$

where $\alpha_i(\eta)$ is positive. Let

$$V_i(\eta,\xi) = V_{i0}(\eta) + [\xi - \phi_i(\eta)]^2/2,$$

where $i \in I$ are Lyapunov functions for each subsystem of the switched Eqs. (8) and (9). When the ith subsystem is operating, we have

$$\dot{V}_i(\eta,\xi) \leqslant -a_i(\eta) + \frac{\partial V_{i0}}{\partial \eta} g_i(\eta)[\xi - \phi_i(\eta)] + [\xi - \phi_i(\eta)]\left[u_{ia} - \dot{\phi}_i(\eta)\right].$$

Let

$$u_{ia} = -\frac{\partial V_{i0}}{\partial \eta} g_i(\eta) - k_i[\xi - \phi_i(\eta)] + \dot{\phi}_i(\eta),\ k_i > 0.$$

We have

$$\dot{V}_i = -a_i(\eta) - k_i[\xi - \phi_i(\eta)]^2 < 0.$$

Substituting u_{ia} to u_i yields

$$u_i = \frac{1}{g_{ia}(\eta,\xi)}\left\{\frac{\partial \phi_i}{\partial \eta}[f_i(\eta) + g_i(\eta)\xi] - \frac{\partial V_{i0}}{\partial \eta} g_i(\eta) - k_i[\xi - \phi_i(\eta)] - f_{ia}(\eta,\xi)\right\}. \tag{11}$$

If the ith subsystem is switched to the jth subsystem at switching time τ_n, $n = 1,2,\ldots$, which satisfies the following inequalities:

$$V_i(\eta(\tau_n),\xi(\tau_n)) \geqslant V_j(\eta(\tau_n),\xi(\tau_n)), \tag{12}$$

then $V_i(x(\tau_{i,k})) \leqslant V_i(x(\tau_{i,k-1}))$ can be obtained. According to Lemma 1, when input u_i is Eq. (11) and the switched law satisfies Eq. (12), systems Eqs. (6) and (7) are stable.

Theorem 1 For Eqs. (6) and (7), $\phi_i(\eta)$, $\phi_i(0) = 0$, $i \in I$ are the state feedback control law for Eq. (6), $\alpha_i(\eta)$, $i \in I$, $V_{i0}(\eta)$, $i \in I$ are continuous positive functions satisfying Eq. (10), then

$$V_i(\eta,\xi) = V_{i0}(\eta) + [\xi - \phi_i(\eta)]^2/2,\ i \in I$$

are Lyapunov functions for each subsystem. If the switched law guarantees that when the ith subsystem is switched to the jth subsystem at the switching time τ_n, $n = 1,2,\ldots$, the Lyapunov function satisfies Eq. (12), then the feedback control law (11) can guarantee the Eqs. (6) and (7) asymptotically stable.

Remark 1 If $L_{g_i}V_{i0}(\eta) \neq 0$, $i \in I$, and the control law $\phi_i(\eta)$ of Eq. (6) can be designed as

$$\phi_i(\eta) = -\frac{L_{f_i}V_{i0}(\eta) + \beta_i(\eta)}{L_{g_i}V_{i0}(\eta)},\ i \in I,$$

where $\beta_i(\eta)$ is positive, then we have

$$\dot{V}_{i0} = L_{f_i}V_{i0}(\eta) + L_{g_i}V_{i0}(\eta)\phi_i(\eta) \leqslant -\beta_i(\eta),\ i \in I.$$

The designed state feedback controller satisfies Eq. (10).

The design procedure of controller for Eq. (2) can be described as follows:

Step 1 Take z_1 as virtual control input. If there exists the state feedback control law $z_1 = \phi_{i0}(x)$, $\phi_{i0}(0) = 0$, $i \in I$ and continuous positive functions $V_{i0}(x)$, $i \in I$ satisfying

$$\dot{V}_{i0} \leqslant -\alpha_{i0}(x),\ i \in I, \tag{13}$$

where $\alpha_{i0}(x)$, $i \in I$ are positive, by Theorem 1 the state feedback control law of the following system:

$$\begin{cases} \dot{x} = f_{i0}(x) + g_{i0}(x)z_1, \\ \dot{z}_1 = f_{i1}(x,z_1) + g_{i1}(x,z_1)z_2, \end{cases} \quad i \in I, \tag{14}$$

is

$$\begin{aligned} z_2 &= \phi_{i1}(x,z_1) \\ &= \frac{1}{g_{iz_1}}\left[\frac{\partial\phi_{i0}}{\partial x}(f_{i0}+g_{i0}z_1) - \frac{\partial V_{i0}}{\partial x}g_{i0} - k_{i1}(z_1-\phi_{i0}) - f_{iz_1}\right], \end{aligned}$$

and the Lyapunov function for each subsystem is

$$V_{i1}(x,z_1) = V_{i0}(x) + \frac{1}{2}[z_1-\phi_{i0}(x)]^2,\ k_{i1}>0.$$

If the switching law from the ith subsystem to the jth subsystem is a subset of the following set:

$$\Omega_1 = \{x,z_1 | V_{i1}(x,z_1) \geqslant V_{j1}(x,z_1),\ i,j\in I \text{ and } i\neq j\},$$

the Eq. (14) in set Ω_1 then satisfies $V_{i1}(\tau_n) \geqslant V_{j1}(\tau_n)$ when the ith subsystem is switched to the jth subsystem. According to Theorem 1, $z_2 = \phi_{i1}(x,z_1)$ can stabilize the Eq. (14) under corresponding switching law.

Step 2 Consider the following system:

$$\begin{cases} \dot{x} = f_{i0}(x) + g_{i0}(x)z_1, \\ \dot{z}_1 = f_{i1}(x,z_1) + g_{i1}(x,z_1)z_2, \\ \dot{z}_2 = f_{i2}(x,z_1,z_2) + g_{i2}(x,z_1,z_2)z_3, \end{cases} \quad i\in I. \tag{15}$$

Take z_2 as virtual control input. Repeat Step 1 and we have

$$z_3 = \frac{1}{g_{i2}}\left[\frac{\partial\phi_{i1}}{\partial x}(f_{i0}+g_{i0}z_1) + \frac{\partial\phi_{i1}}{\partial z_1}(f_{i1}+g_{i1}z_2) - \frac{\partial V_{i1}}{\partial z_1}g_{i1} - k_{i2}(z_2-\phi_{i1}) - f_{i2}\right],$$

$$V_{i2}(x,z_1,z_2) = V_{i1}(x,z_1) + \frac{1}{2}[z_2-\phi_{i1}(x,z_1)]^2,$$

where $k_{i2}>0$. If the switching law from the ith subsystem to the jth subsystem is the subset of the following set:

$$\Omega_2 = \{x,z_1,z_2 | V_{i2} \geqslant V_{j2}, i,j\in I \text{ and } i\neq j\},$$

$z_3 = \phi_{i1}(x,z_1,z_2)$ can then stabilize the Eq. (15) under corresponding switching law.

Step *k* Iterate the above steps and we can obtain the control law and Lyapunov function

$$\begin{cases} u_i = \dfrac{1}{g_{ik}}\left[\dfrac{\partial\phi_{i1}}{\partial x}(f_{i0}+g_{i0}z_1) + \sum\limits_{m=1}^{k-1}\dfrac{\partial\phi_{im}}{\partial z_m}(f_{im}+g_{im}z_{m+1}) - \dfrac{\partial V_{i(k-1)}}{\partial z_{k-1}}g_{i(k-1)} - k_{ik}\left(z_k-\phi_{i(k-1)}\right) - f_{ik}\right],\ k_{ik}>0, \\ V_{ik}(x,z_1,\ldots,z_k) = V_{i(k-1)} + \dfrac{1}{2}\left[z_k - \phi_{i(k-1)}\right]^2. \end{cases} \tag{16}$$

If the switching law from the ith subsystem to the jth subsystem is a subset of the following set:

$$\Omega_k = \{x,z_1,\ldots,z_k | V_{ik} \geqslant V_{jk},\ i,j\in I \text{ and } i\neq j\}, \tag{17}$$

the feedback control Eq. (16) can then stabilize the Eq. (2) under corresponding switching law.

Theorem 2 Consider Eq. (2). Let $z_1 = \phi_{i0}(x)$, $\phi_{i0}(0) = 0$ be the state feedback control law of $\dot{x} = f_{i0}(x) + g_{i0}(x)z_1$ in Eq. (2), where there exist continuous positive functions $V_{i0}(x)$, $i\in I$ satisfying Eq. (13) for positive functions $\alpha_{i0}(x)$, $i\in I$. ϕ_{im} and $V_{im} = V_{i(m-1)} + [z_m - \phi_{i(m-1)}]^2/2$, $i\in I$, $m = 1,2,\ldots,k$ are the virtual state control law and Lyapunov function for each subsystem, respectively, which are solved in the mth step during the backstepping recurrence process. Let Ω_k satisfy Eq. (17). If the switching law from the ith subsystem to the jth subsystem is $S_{ij}\subseteq\Omega_k$, the feedback control Eq. (16) can guarantee Eq. (2) to be asymptotically stable.

3.2 Controller design for uncertain switched nonlinear systems

Consider the switched nonlinear Eqs. (6) and (7) with uncertainties:

$$\dot{\eta} = f_i(\eta) + g_i(\eta)\xi + \delta_{i\eta}(\eta,\xi), \tag{18}$$

$$\dot{\xi} = f_{ia}(\eta,\xi) + g_{ia}(\eta,\xi)u_i + \delta_{i\xi}(\eta,\xi). \tag{19}$$

Suppose there are state feedback control law $\xi = \phi_i(\eta)$, $\phi_i(0) = 0$ and positive functions $V_{i0}(\eta)$ satisfying

$$\dot{V}_{i0} \leqslant -\gamma_i\|\eta\|_2^2, \tag{20}$$

where γ_i are positive real-values.

Let

$$V_i(\eta,\xi) = V_{i0}(\eta) + [\xi - \phi_i(\eta)]^2/2,\ i\in I$$

as Lyapunov functions for each subsystem. The control law is designed as follows:

$$u_i = \frac{1}{g_{ia}}\left[\frac{\partial\phi_i}{\partial\eta}(f_i+g_i\xi) - \frac{\partial V_{i0}}{\partial\eta}g_i - k_i(\xi-\phi_i) - f_{ai}\right],\quad k_i>0. \tag{21}$$

Assume that $\phi_i(\eta)$ satisfies

$$|\phi_i(\eta)| \leqslant d_i\|\eta\|_2,\ \left\|\frac{\partial\phi_i}{\partial\eta}\right\|_2^2 \leqslant e_i, \tag{22}$$

where d_i and e_i are positive real-values. According to Eqs. (4), (5) and (22), it can be proved that for

$$\lambda_i = (b_{i1} + a_ie_i + c_{i1}d_i)/2 > 0,$$

we have

$$\dot{V}_i(\eta,\xi) \leqslant -\begin{bmatrix} \|\eta\|_2 \\ |\xi-\phi_i| \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \gamma_i & -\lambda_i \\ -\lambda_i & k_i - c_{i1} \end{bmatrix} \begin{bmatrix} \|\eta\|_2 \\ |\xi-\phi_i| \end{bmatrix},$$

therefore, in the case of $k_i > c_{i1} + \lambda_i^2/\gamma_i$, $i \in I$, we can derive $\dot{V}_i < 0$.

Similarly to the Theorem 1 in Sect. 3.1, if the switching law from the ith subsystem to the jth subsystem satisfies Eq. (12), and k_i is large enough, Eqs. (18) and (19) are then stable.

Theorem 3 Consider Eqs. (18) and (19). $\xi = \phi_i(\eta)$, $\phi_i(0) = 0$, $i \in I$ is the state feedback control law of Eq. (18) and satisfies Eq. (22). $V_{i0}(\eta)$, $i \in I$ are continuous positive functions satisfying Eq. (22) for positive real-values γ_i, $i \in I$. When the ith subsystem is switched to the jth subsystem at switching time τ_n, $n = 1,2,\ldots$, if the Lyapunov function satisfies Eq. (12) and k_i, $i \in I$ is large enough, the control law (21) can then guarantee Eqs. (18) and (19) to be asymptotically stable.

The robust controller can be obtained from Theorem 3 as follows:

Theorem 4 Consider Eq. (3). Let the control law of $\dot{x} = f_{i0}(x) + g_{i0}(x)z_1 + \delta_{ix}(x,z)$, $i \in I$ in Eq. (3) be $z_1 = \phi_{i0}(x)$, $\phi_{i0}(0) = 0$ and satisfy Eq. (22). For positive real-values γ_i, $i \in I$, there exist positive continuous functions $V_{i0}(x)$, $i \in I$ satisfying $\dot{V}_{i0} \leqslant -\gamma_i \|x\|_2^2$. ϕ_{im} and $V_{im} = V_{i(m-1)} + [z_m - \phi_{i(m-1)}]^2/2$, $i \in I$, $m = 1,2,\ldots,k$ are the virtual state control law and Lyapunov function of each subsystem, respectively, which are solved in the mth step during the backstepping recurrence process. Let Ω_k satisfy Eq. (17). If the switching law from the ith subsystem to the jth subsystem is $S_{ij} \subseteq \Omega_k$, and the feedback control law is designed as

$$u_i = \frac{1}{g_{ik}} \left[\frac{\partial \phi_{i1}}{\partial x}(f_{i0} + g_{i0} z_1) + \sum_{m=1}^{k-1} \frac{\partial \phi_{im}}{\partial z_m}(f_{im} + g_{im} z_{m+1}) - \frac{\partial V_{k-1}}{\partial z_{k-1}} g_{i(k-1)} - k_{ik}\left(z_k - \phi_{i(k-1)}\right) - f_{ik} \right],$$

where k_{ik} is large enough, Eq. (3) is then asymptotically stable.

4 Numerical example

Consider the following switched system:

$$f_1 = \begin{bmatrix} x_1^2 - x_1^3 + x_2 \\ 0 \end{bmatrix}, \ g_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$f_2 = \begin{bmatrix} 2x_1^2 \sin 10x_1 - x_1 + 2x_2 \\ 0 \end{bmatrix}, \ g_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For $\dot{x} = f_{i0}(x_1) + g_{i0}(x_1)x_2$, there exist $\phi_{11}(x_1) = -x_1^2$ and $\phi_{12}(x_1) = -x_1^2 \sin 10x_1$, such that $V_{10} = V_{20} = x_1^2/2$ satisfies Eq. (13). Then we can obtain the control law and Lyapunov function for each subsystem.

$$u_1 = -x_1 - 2x_1^3 + 2x_1^4 - 2x_1 x_2 - x_1^2 - x_2,$$

$$u_2 = -2x_1 - x_2 - x_1^2 \sin 10x_1 - \left(2x_2 + 2x_1^2 \sin 10x_1 - x_1\right) \times \left(10x_1^2 \cos 10x_1 + 2x_1 \sin 10x_1\right),$$

$$V_1 = \frac{1}{2}x_1^2 + \frac{(x_2 + 5\sin x_1)^2}{2},$$

$$V_2(x_1,x_2) = \frac{1}{2}x_1^2 + \frac{(x_2 + 5\sin x_1)^2}{2}.$$

According to Eq. (12), the switching law can be designed as $i = i(x) = \arg\min\{V_i(x_1,x_2)\}$. The simulation results of the state responses of closed-loop system and switching strategy are shown in Figs. 1 and 2 respectively, where the initial value of system $x(0) = [3,-7]^{\mathrm{T}}$.

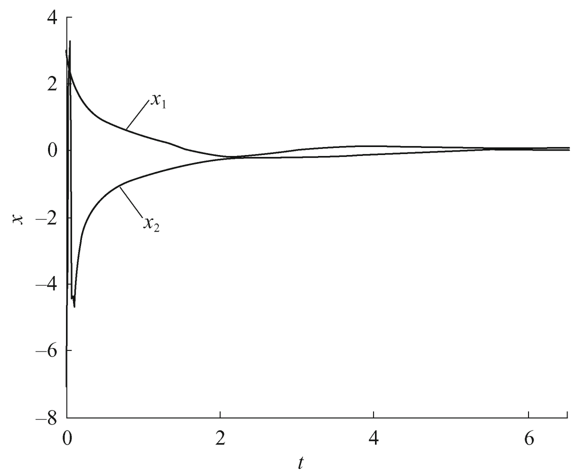

Fig. 1 State response of closed-loop system

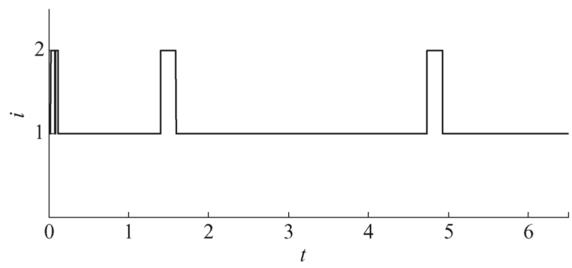

Fig. 2 Switching strategy

5 Conclusions

The design of state feedback controllers is studied for a class of switched nonlinear systems. The backstepping method is applied to design the state feedback controllers and switching law based on multi-Lyapunov functions.

Furthermore, robust controllers for this class of switched nonlinear systems with uncertainties are designed by using the proposed method. Finally, the simulation result shows the effectiveness of the proposed method.

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