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RESEARCH ARTICLE

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Five precision point-path synthesis of planar four-bar linkage using algebraic method

Abstract The problem of synthesizing a planar four-bar linkage with two given fixed pivots such that the coupler curve passes through five given points is considered with the Groebner-Sylvester hybrid approach. First, closed-form equations of a single point are constructed. The reduced Groebner basis in degree lexicographic ordering for the closed-form equations is then obtained using computer algebra. A 23×23 Sylvester's matrix can be constructed by selecting 23 out of 89 Groebner bases. A 36th degree univariate equation is obtained directly from the determinate of the matrix. The same result can be obtained with a continuation method. A numerical example is given and verifies that the problem has at most 36 solutions in the complex field.

Keywords path synthesis, mechanism synthesis, Groebner basis, Sylvester resultant

1 Introduction

The path synthesis of a planar four-bar linkage determines the size of the planar four-bar based on a given path curve or given discrete points. The path curve of the planar four-bar linkage is described as a complex sextic curve, whose path synthesis via the specification of precision points on the desired curve is still unsolved in kinematics. Solutions to the problem can be divided into two classes: the direct-synthesis method and indirect-synthesis method. The former adopts the "expected path → mechanical structure" approach and deduces the parameters of the expected path (generally described by discrete points) according to the kinematics principle. The latter, which is the primary method of path synthesis of the planar linkage, can be further classified into the graphical method [1] and algebraic method. The planar four-bar linkage designed by a graphical method approximately satisfies the requirements of design, while that designed by an algebraic method, although with high precision, progresses slowly because of the difficulty.

However, the indirect-synthesis method does not generate the expected mechanism by calculation. It implements the "expected path → matching path → mechanism" pattern, which first searches for some characteristics or similar existing paths from the established atlas database of four-bar coupler curves, then obtains the parameters of the known path. This is the focus of the current research of path synthesis of a planar four-bar linkage. The essence of the method is to establish the electronic atlas database of four-bar coupler curves and effectively search for the matching path. The significant mathematical methods for such a search are extracting the geometric relationship based on curve characteristics [2], extracting shape spectrum by mathematical morphological analysis [3], extracting moment invariant using the theory of algebraic moment invariant [4] , extracting the main harmonic composition and Fourier components adopting a fast Fourier transform [5,6] and extracting low frequency wavelet coefficients of the coupler curve using the wavelet multi-resolution analysis [7]. Rapid development of computer technology enhances performance of the indirect-synthesis method in the analysis of path synthesis of a planar four-bar linkage.

With the development of sophisticated mechanisms, algebraic methods have made corresponding progress. Reference [8] provides a uniform model for synthesis optimization. In Ref. [9], the problem of synthesizing a four-bar linkage with given fixed pivots to generate a path through five precision points is accomplished. Similar to a rigid body guide, this relatively simple problem obtains 4 groups of solutions. In Ref. [10], the 9 precision points of a four-bar mechanism on the fixed pivots to synthesize the

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curve path are analyzed using the continuation method to obtain 4326 solutions. However, this method needs to utilize a huge computer and costs nearly 332 machine hours of computing. In Ref. [11], the 5 precision points of a four-bar mechanism on the fixed pivots to synthesize the curve path are analyzed using the continuation method. It is proven that the problem has 36 groups of solutions, but no numerical samples are provided by this method. Consequently, current researches of 5 precision points curve path of four-bar linkage are still in the stage of quantitative analysis that yields numerical solutions, rather than in a qualitative phase where equations of a higher order are deduced with one variable. Thus, an analytical method will contribute both in theory and practice to five precision point-path synthesis of a planar four-bar linkage.

The problem of synthesizing a planar four-bar linkage with two given fixed pivots and where the coupler curve passes through five given points is considered with the Groebner-Sylvester hybrid approach. A 36th degree univariate equation is obtained and agrees with the result in Ref. [11]. All computation is performed in a combination of signed and rational number operations. Therefore, no numerical error will occur in calculation.

2 Mathematical model

A planar four-bar linkage is shown in Fig. 1. A_0 and B_0 are given fixed pivots. A_1 and B_1 are dynamic pivots. P is a point on the planar linkage. Five precision point path synthesis aims to get the size of the linkage mechanism and ensure that point P of the four-bar linkage will pass the given five points $P_i(p_{ix},p_{iy})$, $i = 1,2,\ldots,5$, on the plane during motion. To simplify, set A_0 and P_1 as the origin points and separately make the coordinates $O_1x_1y_1$ fixed to the rack and the coordinates $O_2x_2y_2$ fixed to the linkage.

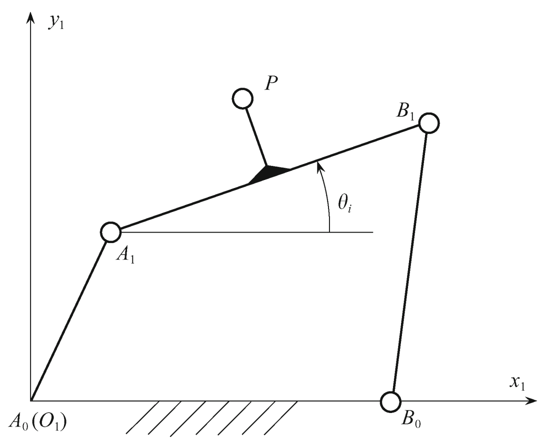

Fig. 1 Planar four-bar linkage

In Fig. 1, the coordinates of A_0 and B_0 in $O_1x_1y_1$ are (a_{0x},a_{0y}), (b_{0x},b_{0y}). The coordinates of A_1 and B_1 in $O_2x_2y_2$ are (a_{1x},a_{1y}), (b_{1x},b_{1y}). The matrix of the planar rigid body movement is

$$\boldsymbol{D}_{1i}=\begin{bmatrix}\cos\theta_{1i} & -\sin\theta_{1i} & p_{ix}-p_{1x}\cos\theta_{1i}+p_{1y}\sin\theta_{1i}\\ \sin\theta_{1i} & \cos\theta_{1i} & p_{iy}-p_{1x}\sin\theta_{1i}-p_{1y}\cos\theta_{1i}\\ 0 & 0 & 1\end{bmatrix}, \quad (1)$$

where $\theta_{1i} = \theta_i - \theta_1$, $i = 2,3,4,5$, denoting the relative angle of rotation by which the rigid body turns from position 1 to position i.

With given linkage lengths, the constraint equations corresponding to the conditions of constraint length of each linkage are as follows:

$$(\boldsymbol{D}_{1i}\boldsymbol{A}_1-\boldsymbol{A}_0)^{\mathrm{T}}(\boldsymbol{D}_{1i}\boldsymbol{A}_1-\boldsymbol{A}_0)-(\boldsymbol{A}_1-\boldsymbol{A}_0)^{\mathrm{T}}(\boldsymbol{A}_1-\boldsymbol{A}_0)=0, \quad (2)$$

$$(\boldsymbol{D}_{1i}\boldsymbol{B}_1-\boldsymbol{B}_0)^{\mathrm{T}}(\boldsymbol{D}_{1i}\boldsymbol{B}_1-\boldsymbol{B}_0)-(\boldsymbol{B}_1-\boldsymbol{B}_0)^{\mathrm{T}}(\boldsymbol{B}_1-\boldsymbol{B}_0)=0, \quad (3)$$

where $\boldsymbol{A}_i = [a_{ix}, a_{iy},1]^{\mathrm{T}}$, $\boldsymbol{B}_i = [b_{ix}, b_{iy}, 1]^{\mathrm{T}}$, $i = 0,1$.

Substituting Eq. (1) and the coordinates of points into Eqs. (2) and (3), we obtain

$$c_i^2+s_i^2=1, \quad i=2,3,4,5, \quad (4)$$

$$\begin{aligned}&2a_{1x}a_{0x}+2a_{1y}a_{0y}+2c_ih_{aci}-2a_{1x}p_{1x}+p_{1x}^2-2a_{1y}p_{1y}\\&\quad+p_{1y}^2-2a_{0x}p_{ix}+p_{ix}^2-2a_{0y}p_{iy}+p_{iy}^2+2h_{asi}s_i=0,\\&\qquad i=2,3,4,5,\end{aligned} \quad (5)$$

$$\begin{aligned}&2b_{1x}b_{0x}+2b_{1y}b_{0y}+2c_ih_{bci}-2b_{1x}p_{1x}+p_{1x}^2-2b_{1y}p_{1y}\\&\quad+p_{1y}^2-2b_{0x}p_{ix}+p_{ix}^2-2b_{0y}p_{iy}+p_{iy}^2+2h_{bsi}s_i=0,\\&\qquad i=2,3,4,5,\end{aligned} \quad (6)$$

where $c_i = \cos\theta_{1i}$, $s_i = \sin\theta_{1i}$,

$$\begin{aligned}h_{aci}=&-a_{1x}a_{0x}-a_{1y}a_{0y}+a_{0x}p_{1x}+a_{0y}p_{1y}+a_{1x}p_{ix}\\&-p_{1x}p_{ix}+a_{1y}p_{iy}-p_{1y}p_{iy},\end{aligned} \quad (7)$$

$$\begin{aligned}h_{asi}=&a_{1y}a_{0x}-a_{1x}a_{0y}+a_{0y}p_{1x}-a_{0x}p_{1y}-a_{1y}p_{ix}\\&+p_{1y}p_{ix}+a_{1x}p_{iy}-p_{1x}p_{iy},\end{aligned} \quad (8)$$

$$\begin{aligned}h_{bci}=&-b_{1x}b_{0x}-b_{1y}b_{0y}+b_{0x}p_{1x}+b_{0y}p_{1y}+b_{1x}p_{ix}\\&-p_{1x}p_{ix}+b_{1y}p_{iy}-p_{1y}p_{iy},\end{aligned} \quad (9)$$

$$\begin{aligned}h_{bsi}=&b_{1y}b_{0x}-b_{1x}b_{0y}+b_{0y}p_{1x}-b_{0x}p_{1y}-b_{1y}p_{ix}\\&+p_{1y}p_{ix}+b_{1x}p_{iy}-p_{1x}p_{iy}.\end{aligned} \quad (10)$$

Equations (4)–(6) are considered as the mathematical model of the five precision point-path synthesis of a planar four-bar linkage. There is a total of 12 variants to be solved, i.e., a_{1x}, a_{1y}, b_{1x}, b_{1y}, c_i, s_i in the 12 equations and the remainder are structural parameters of the planar four-bar linkage.

3 Elimination process

Equations (4)–(6) are 12 two-degree nonlinear polynomial equations, where the total degree (product of the degrees of the polynomials) of the mathematical model equals $2^{12} = 4096$. It is difficult to obtain solutions by the traditional algebraic methods such as the Dixon resultant method, the Sylvester resultant method and Wu's method. We solve the problem using a two-step elimination method. First, we reduce the Groebner basis in lexicographic ordering of Eqs. (4)–(6). We then analyze the variants of the basis and the terms of the degrees of different variants and construct the Sylvester resultant by hiding proper variants such that the number of polynomials is equal to that of monomials. Meanwhile, to reduce time for symbol computing, we utilize related principles in linear algebra to analyze the Sylvester resultant, minimize the degrees of the resultant and obtain a 36th univariate equation.

3.1 Groebner basis in lexicographic ordering

The Groebner basis in lexicographic ordering is depicted as follows: in the polynomial circle constructed from the original polynomial, the *S*-polynomial is deduced through proper degree lexicographic term ordering of its monomials and reducing the polynomial to obtain a normal form of reduced Groebner basis that is equivalent to the original system. The Groebner package available in Methematica-5.2 is utilized to yield a reduced Groebner basis of Eqs. (4)–(10) in the degree lexicographic term ordering with $c_2 > s_2 > c_3 > s_3 > c_4 > s_4 > c_5 > s_5 > a_{1x} > a_{1y} > b_{1x} > b_{1y}$. The reduced Groebner basis with 89 polynomials in 125 unknown monomials is not reported herein due to space limitations.

Suppressing the unknown a_{1y}, the reduced Groebner basis in degree lexicographic ordering can be viewed as a linear system of 89 equations in 89 unknown monomials and expressed in the form of a matrix as

$$\boldsymbol{M}'_{89\times 89}\boldsymbol{t} = 0, \tag{11}$$

where $\boldsymbol{M}'_{89\times 89}$ is the 89 × 89 coefficient matrix and $\boldsymbol{t}$ is the 89 × 1 column matrix of the unknown variables (the 89 monomials).

According to the algebra, Eqs. (4)–(6) have the solutions under the condition that the determinant of the matrix equates to zero, i.e.,

$$\det\left(\boldsymbol{M}'_{89\times 89}\right) = 0. \tag{12}$$

Solving Eq. (12) will cost much computing time and computer memory. Therefore, it is necessary to reduce the degree of the coefficient matrix to improve efficiency.

3.2 Reduce degree of resultant

Only one element of the columns in the matrix $\boldsymbol{M}'_{89\times 89}$ is constant (e.g., *j*th column) and others are zero, i.e., only the *i*th equation is related with *j*th unknown variable, and removing the *i*th equation will not affect solving the equations.

Similarly, there are only 23 polynomial equations in the Groebner basis in degree lexicographic ordering, i.e., number 1, 2, 4, 7, 8, 9, 10, 14, 17, 20, 23, 27, 30, 35, 38, 44, 47, 54, 57, 65, 68, 77, and 80, with the matrix form:

$$\boldsymbol{M}_{23\times 23}\boldsymbol{T} = 0, \tag{13}$$

where $\boldsymbol{M}_{23\times 23}$ is the 23 × 23 coefficient matrix of a_{1y} and $\boldsymbol{T}$ is the 23 × 1 column matrix in 23 unknown variables (the 23 monomials).

$$\boldsymbol{T} = \left[1, a_{1x}, b_{1x}, b_{1y}, a_{1x}b_{1y}, b_{1x}b_{1y}, b_{1y}^2, c_2, b_{1y}c_2, c_3, b_{1y}c_3, c_4, b_{1y}c_4, c_5, b_{1y}c_5, s_2, b_{1y}s_2, s_3, b_{1y}s_3, s_4, b_{1y}s_4, s_5, b_{1y}s_5\right]^{\mathrm{T}}.$$

3.3 Deriving univariate polynomial

According to the algebra, the condition of Eq. (13) having solutions is that the determinant of the coefficient is zero, i.e.,

$$\det(\boldsymbol{M}_{23\times 23}) = 0. \tag{14}$$

Limited by the computer memory, it is impossible to expand the determinant of matrix $\boldsymbol{M}_{23\times 23}$ in symbols. Numerical computation must be carried out. It is important to determine the degrees of the equation to obtain the exact number of solutions. The maximum degree of the unknown variant of each column of matrix $\boldsymbol{M}_{23\times 23}$ are: 3, 2, 2, 2, 1, 1, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, making a total of 36. Based on the calculation principle of the determinant, the highest degree of the polynomials with only variant a_{1y} is no more than 36 after expanding the determinant.

Expanding Eq. (14) directly, we can obtain the following 36th degree univariate polynomial without factoring out or deriving any common divisor:

$$\sum_{i=0}^{36} s_i a_{1y}^i = 0, \tag{15}$$

where s_i, $i = 0,1,\ldots,36$, are real constants depending on input parameters only.

According to the analysis above, the maximum degree of the constructed univariate coefficient determinant and the univariate polynomial derived from expanding the determinant agree symbolically with each other. 36 complex solutions will be obtained by solving Eq. (15).

3.4 Back substitution for other unknowns

By solving the linear system obtained by removing any row from the matrix $\boldsymbol{M}_{23\times 23}$ in Eq. (14) with a_{1y} replaced by a_{1yi}, $i = 1,2,\ldots,36$, solutions of a_{1x}, a_{1y}, b_{1x}, b_{1y}, c_i, and s_i can be easily computed in the complex domain. For one solution of a_{1y}, there will be one solution of a_{1x}, a_{1y}, b_{1x}, b_{1y}, c_i, and s_i.

4 Continuation method

In the last decade, homotopy continuation has developed into a convenient, reliable tool for solving nonlinear polynomial systems. It is a numerical process that finds all isolated roots of a polynomial system. Starting at the root of a suitable start system, the method tracks the solution paths as the start system is continuously transformed into the target system. When the start system and the transformation procedure, called a homotopy, are properly chosen, the endpoints of these solution paths are guaranteed to include all isolated solutions of the target system. Some publicly available software for polynomial continuation are available [12,13]. In this section, we solve the mathematical model of the five precision point-path synthesis of a planar four-bar linkage by using homotopy continuation.

Combining Eqs. (4)–(6), we get the nonlinear algebraic system. With a regular personal computer running on Intel Pentium III 2.93 GHz and RAM 256 M, we solve the nonlinear algebraic system by using PHCpack [13], which is a kind of homotopy continuation software exploited by Illinois University. The result shows that there are 36 roots in total, which is consistent with the previous results.

5 Numerical sample

The parameters of a planar four-bar linkage are $a_{0x} = a_{0y} = 0$, $b_{0x} = 18$, $b_{0y} = 0$, $p_{1x} = 12$, $p_{1y} = 10$, $p_{2x} = 12$, $p_{2y} = 11$, $p_{3x} = 10$, $p_{3y} = 12$, $p_{4x} = 9$, $p_{4y} = 11$, $p_{5x} = 8$, and $p_{5y} = 10$. Following the steps above, we can get 36 groups of solutions, in which 18 are real solutions as illustrated in Table 1.

6 Conclusions

A planar four-bar linkage design problem is described, and a Groebner-Sylvester hybrid approach for generating all closed-form solution designs is given. Base on the proposed method, we first derive the 36th degree univariate polynomial from the determinant of the 23×23 Sylvester's matrix without factoring out or deriving the greatest common divisor. Simultaneously, the same result can be obtained with the continuation method, illustrating that the problem has 36 groups of solutions in a complex field. The success in solving this problem using the

Table 1 18 real solutions of the numerical sample

No.	a_{1x}	a_{1y}	b_{1x}	b_{1y}	c_2	s_2	c_3	s_3	c_4	s_4	c_5	s_5
1	0.3846	18.5247	34.7513	32.4689	−0.8893	0.4573	−0.9540	0.2999	−0.9769	0.2135	−0.9930	0.1180
2	0.7990	5.3505	17.5910	−10.5578	0.8276	0.5614	0.7082	0.7060	0.6416	0.7670	0.9977	0.0685
3	−4.3494	3.1549	14.0780	−1.2798	0.7953	0.6062	0.9836	0.1803	0.7347	0.6783	0.9374	0.3484
4	18.6438	−3.0870	−22.1418	2.0129	−0.8612	−0.5083	0.9836	0.1803	0.9637	0.2670	0.9334	0.3588
5	17.6645	−2.7473	81.3805	−7.398	0.9999	0.0104	0.9836	0.1803	0.9633	0.2682	0.9329	0.3602
6	2.3144	−2.5476	21.3027	−1.9366	0.9991	0.0415	0.9836	0.1803	0.9736	−0.2283	0.9916	−0.1297
7	15.6218	−2.1346	−95.5933	9.9379	1.0000	0.0088	−0.5481	−0.8364	0.9628	0.2701	0.9322	0.3618
8	0.1834	2.0286	17.7547	−1.6141	0.9868	0.1622	0.9836	0.1803	0.9890	0.1479	0.9176	0.3974
9	2.2016	−1.9482	20.7755	−1.8887	0.9492	−0.3147	0.9836	0.1803	0.9840	−0.178	0.9969	−0.0789
10	14.0452	−1.8277	−161.1622	14.6511	1.0000	0.0081	0.9836	0.1803	−0.4046	−0.9145	0.9331	0.3595
11	−0.0025	1.8163	17.6201	−1.6018	0.9848	0.1738	0.9836	0.1803	0.9869	0.1614	0.9677	0.2522
12	12.6583	−1.7416	−330.2477	30.0225	1.0000	0.0084	0.9836	0.1803	0.9635	0.2676	−0.2483	−0.9687
13	6.2346	1.5050	17.1182	−1.5562	0.9982	−0.0595	0.9836	0.1803	0.9453	0.3262	0.9043	0.4268
14	1.9494	−1.4394	19.6422	−0.8222	0.9697	−0.2443	0.9917	−0.1287	0.9830	0.1838	0.9986	0.0532
15	5.5770	1.3494	21.0720	−2.7348	0.9515	−0.3075	0.9573	−0.2892	0.9472	0.3206	0.9111	0.4122
16	1.9206	−1.134	19.4392	−0.9479	0.9997	0.02464	0.9951	−0.0992	0.9815	0.1913	0.9967	0.0813
17	1.8007	−0.36420	19.7117	−0.2132	1.0000	−0.0127	0.9999	−0.0160	0.9994	−0.0342	0.9983	0.0583
18	2.0617	−0.25514	19.8614	0.04737	0.9998	−0.0219	0.9995	−0.0314	0.9981	−0.0619	0.9794	0.2018

proposed method sheds light on solving other similarly difficult mechanism synthesis problems.

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