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Closed-form solutions to fractional-order linear differential equations

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Abstract The definitions and properties of widely used fractional-order derivatives are summarized in this paper. The characteristic polynomials of the fractional-order systems are pseudo-polynomials whose powers of the complex variable are non-integers. This kind of systems can be approximated by high-order integer-order systems, and can be analyzed and designed by the sophisticated integer-order systems methodology. A new closed-form algorithm for fractional-order linear differential equations is proposed based on the definitions of fractional-order derivatives, and the effectiveness of the algorithm is illustrated through examples.

Keywords fractional-order differentiator, linear systems, numerical solutions, calculus, simulation, differential equations, integer-order approximations

1 Introduction

The so-called fractional-order calculus whose orders of the derivatives and integrals are non-integers is an extension to the well-established calculus. It expands the scopes of the integer-order calculus. Fractional-order calculus used in mathematical models of the systems can improve the design, properties and controlling abilities in dynamical system. In many newly emergent research subjects such as bioengineering [1], limitations exist in the dynamical modeling by using classical integer-order ordinary differential equations, then the fractional-order calculus is introduced. Recently, fractional-order calculus is used to solve problems in spectral analysis, electrolyte, and modulus of elasticity as well as control systems [2,3].

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To analyze and design a practical system, an effective method is needed in the fractional-order calculus modeling. This paper introduced the definitions and basic properties of fractional-order derivatives and integrals. The characteristic polynomials of the fractional-order systems are pseudo-polynomials whose powers of the complex variable are non-integers. This kind of systems can be approximated by high-order integer-order systems, and can be analyzed and designed by the sophisticated integer-order systems methodology [4]. A new closed-form algorithm for fractional-order linear differential equations is proposed based on the definitions of fractional-order derivatives, with the effectiveness of the algorithm illustrated.

2 Definitions and properties of fractional-order calculus

The fractional-order differentiator is often denoted by ${}_aD_t^\alpha$, where a and t are respectively the lower- and upper-bounds, and α is the order of derivative or integrals, which can be non-integers, or even complex numbers [5]. The definition of fractional-order operator is:

$${}_aD_t^\alpha = \begin{cases} d^\alpha/dt^\alpha, & R(\alpha) > 0 \\ 1, & R(\alpha) = 0, \\ \int_a^t (d\tau)^{-\alpha}, & R(\alpha) < 0 \end{cases} \quad (1)$$

where $R(\alpha)$ is the real part of α . The most commonly used definitions are the Grünwald-Letnikov (GL) definitions and the Riemann-Liouville (RL) definitions. The RL definition is

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt} \right)^m \left(\int_a^t \frac{f(\tau)}{(t-\tau)^{1-(m-\alpha)}} d\tau \right), \quad (2)$$

where $m-1 < \alpha < m$, and $\Gamma(x)$ is the well-known Euler Gamma function. The GL definition is:

$${}_a D_x^\alpha [f(t)] = \lim_{h \rightarrow 0} \frac{1}{\Gamma(\alpha)h^\alpha} \sum_{k=0}^{\lfloor x-a/h \rfloor} \frac{\Gamma(\alpha+k)}{\Gamma(k+1)} f(t-kh). \quad (3)$$

It is shown that [6] for a great variety of functions, the two definitions are identical. Fractional-order calculus is an extension to the traditional integer-order calculus, with the following properties [7]:

1) ${}_0 D_t^\alpha f(t)$ is analytic for an analytic function $f(t)$.

2) For integers $\alpha = n$, the definition of fractional-order derivative is identical to the integer-order derivatives, further ${}_0 D_t^0 f(t) = f(t)$.

3) The fractional-order operator ${}_a D_t^\alpha$ is linear, i.e., for any a and b ,

$${}_0 D_t^\alpha [af(t) + bg(t)] = a{}_0 D_t^\alpha f(t) + b{}_0 D_t^\alpha g(t). \quad (4)$$

4) The fractional-order operator conforms to the commutative law and superposition principle

$${}_0 D_t^\alpha [{}_0 D_t^\beta f(t)] = {}_0 D_t^\beta [{}_0 D_t^\alpha f(t)] = {}_0 D_t^{\alpha+\beta} f(t). \quad (5)$$

Laplace transform is an effective tool in studying integer-order systems, which can also be used in fractional-order calculus. For a non-integer n ($n \in \mathbb{R}_+$), the Laplace transform of the n th order derivative of a signal $x(t)$ can be written as [8]:

$$\begin{aligned} \mathcal{L}\{D^n x(t)\} &= \int_0^\infty e^{-st} {}_0 D_t^n x(t) dt \\ &= s^n X(s) - \sum_{k=0}^{m-1} s^k {}_0 D_t^{n-k-1} x(t)|_{t=0}, \end{aligned} \quad (6)$$

where $m-1 < n < m$, and m is an integer.

3 Approximate fitting of fractional-order differentiators

Fractional-order systems are the mathematical models established based on the definitions of fractional-order calculus. That the integer-order approximating to the fractional-order system is an important approach in dealing with fractional-order systems [9]. By means of approximation techniques, the fractional-order transfer functions can be converted into ordinary integer-order transfer functions, and the problems can be converted into ordinary control problems.

Stable optimal fitting methods can be used in the approximating process. Optimal fitting methods have many advantages in dealing with fractional-order differentiators for unknown signals. The integer-order filter proposed by Oustaloup is among the satisfactory filters [10]. An improved filter is presented in the paper to approximate

fractional-order differentiators. If the frequency range of interest is (ω_b, ω_h) , a filter $C_0(1+s/\omega_b)/(1+s/\omega_h)$ can be used to approximate the Laplace operator s . Thus the fractional-order differentiator can be approximated by

$$H(s) = C \left(\frac{1 + \frac{s}{\omega_b}}{1 + \frac{s}{\omega_h}} \right)^\alpha, \quad (7)$$

where $C = C_0^\alpha$ and the pole-zero form of the filter can be written as:

$$H(s) = \lim_{N \rightarrow \infty} H_N(s) = \lim_{N \rightarrow \infty} \left(\frac{1}{\omega_h} \right)^\alpha \prod_{k=-N}^N \frac{1 + s/\omega'_k}{1 + s/\omega_k}, \quad (8)$$

where the poles and zeros of the filter can respectively be evaluated from:

$$\omega'_k = \omega_b \left(\frac{\omega_h}{\omega_b} \right)^{\frac{k+N+(1-\alpha)/2}{2N+1}}, \quad \omega_k = \omega_b \left(\frac{\omega_h}{\omega_b} \right)^{\frac{k+N+(1+\alpha)/2}{2N+1}}, \quad (9)$$

and the integer-order filter to the fractional-order differentiator can be expressed as:

$$G(s) = K \prod_{k=-N}^N \frac{1 + \frac{s}{\omega'_k}}{1 + \frac{s}{\omega_k}} \quad (10)$$

with the gain

$$K = \omega_h^{-\alpha} \prod_{k=-N}^N \frac{\omega_k}{\omega'_k}. \quad (11)$$

The procedures of the approximation are as follows:

1) Select suitable frequency range (ω_b, ω_h) and order N .
2) Calculate the parameters ω_k and ω'_k from Eq. (9) and gain K from Eq. (11).

3) Calculate the linear integer-order filter from Eq. (10)

For discrete-time systems, the above algorithms cannot be applied to directly. Finite impulse response (FIR) [7] or infinite impulse response (IIR) filters [11] should be used for the differentiators.

4 Closed-form solutions to fractional-order linear differential equations

A fractional-order linear system is a widely studied fractional-order system. In this paper, a new solution algorithm is presented to solve the problems. For the fractional-order linear differential equation:

$$\begin{aligned} b_n D_t^{\beta_n} y(t) + b_{n-1} D_t^{\beta_{n-1}} y(t) + \dots + b_1 D_t^{\beta_1} y(t) + b_0 D_t^{\beta_0} y(t) \\ = u(t), \end{aligned} \quad (12)$$

where $u(t)$ is the linear combination of the fractional-order derivatives of an input function $v(t)$ as:

$$u(t) = a_m D_t^{\alpha_m} v(t) + a_{m-1} D_t^{\alpha_{m-1}} v(t) + \dots + a_1 D_t^{\alpha_1} v(t) + a_0 D_t^{\alpha_0} v(t). \tag{13}$$

For zero initial conditions, Laplace transforms can be taken to both sides of equation, the fractional-order transfer function can be

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{b_0 s^{\beta_0} + b_1 s^{\beta_1} + \dots + b_n s^{\beta_n}}, \tag{14}$$

where $b_m \in \mathbb{R}^2$, $\beta_m \in \mathbb{R}_+^2$, and $\forall(m \in N)$.

In GL definition the closed-form numerical solutions to the fractional-order linear differential equation can be obtained from:

$$y_t = \frac{1}{\sum_{i=0}^n \frac{b_i}{h^{\beta_i}}} \left[u_t - \sum_{i=0}^n \frac{b_i}{h^{\beta_i}} \sum_{j=1}^{\lfloor (t-t_0)/h \rfloor} \omega_j^{(\beta_i)} y_{t-jh} \right], \tag{15}$$

where h is the step-size of computation, and can be obtained recursively from:

$$\omega_0^{(\beta_i)} = 1, \quad \omega_j^{(\beta_i)} = \left(1 - \frac{\beta_i + 1}{j} \right) \omega_{j-1}^{(\beta_i)}, \quad j = 1, 2, \dots \tag{16}$$

and signal $u(t)$ can be evaluated by the direct use of GL definition [12].

The procedures of solving fractional-order linear differential equations are as follows:

- 1) Select a discretization step-size h , and find the discretized samples of the input v_i .
- 2) Calculate the sequence of u_i from v_i using GL definition in Eq. (3).
- 3) Calculate recursively the sequence of $\omega_j^{(\beta_i)}$ from Eq. (16).
- 4) Evaluate the numerical solutions of the differential equation from Eq. (15).

To release computational burden and find accurate solutions, normally a relatively small step-size h should be selected. Normally, if the solutions from two different step-sizes h 's are very close, they can be regarded as accurate.

Example 1 For a fractional-order transfer function [4]:

$$G(s) = \frac{1}{s^{1.5} + 1}.$$

The fractional-order differential equation is $D_t^{1.5} y(t) + y(t) = u(t)$. With the closed-form algorithm in this paper, the step response to the system can be obtained in Fig. 1, which is exactly the same as the one in

Ref. [4]. Also the Bode diagram of the fractional-order system is obtained in Fig. 2.

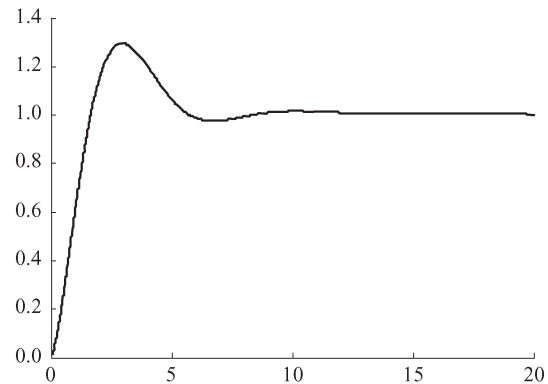


Fig. 1 Step response to fractional-order system

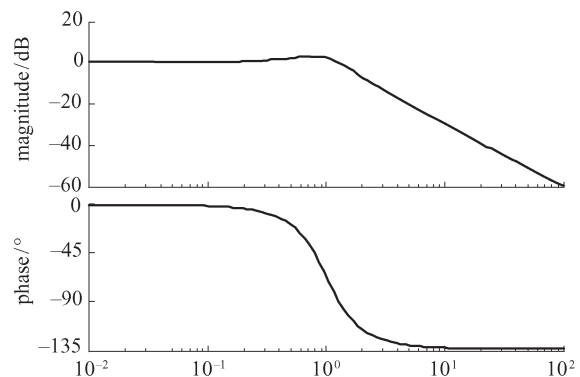


Fig. 2 Bode diagram of fractional-order system

Example 2 A fractional-order linear differential equation is given by

$$2D_t^{3.501} y(t) + 3.8D_t^{2.42} y(t) + 2.6D_t^{1.798} y(t) + 2.5D_t^{1.31} y(t) + 1.5y(t) = -2D_t^{0.63} u(t) - 4u(t).$$

Then the fractional-order transfer function can be written as:

$$G(s) = \frac{-2s^{0.63} - 4}{2s^{3.501} + 3.8s^{2.42} + 2.6s^{1.798} + 2.5s^{1.31} + 1.5}.$$

One can then find the numerical solution to the equation in Fig. 3. Also an integer-order approximation technique can be used and a 48th order linear model can be:

$$G(s) = \frac{-159.2s^{45} - 6.223 \times 10^5 s^{44} - \dots - 7.586 \times 10^{11} s - 3.818 \times 10^8}{63.68s^{48} + 2.561 \times 10^5 s^{47} + \dots + 2.822 \times 10^{11} s + 1.423 \times 10^8}.$$

The step response to the integer-order model is also the same as the one given in Fig. 3.

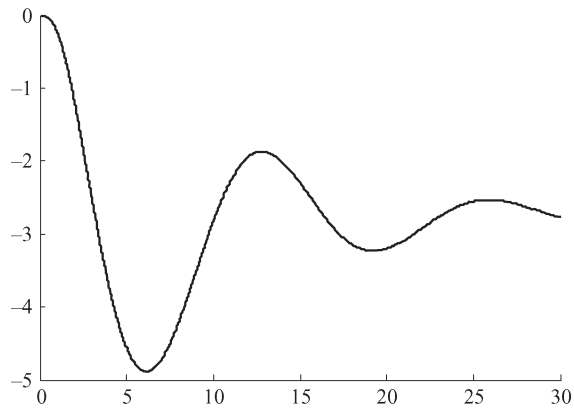


Fig. 3 Step response comparisons

Figure 4 shows the Bode diagrams of both the fractional-order model and the integer-order approximate model. The magnitude plots are identical; though in the phase plot, there is exactly 360° of difference, it can be regarded as identical as well. From the complicated example, it can also be concluded that the closed-form solution algorithm proposed is also effective.

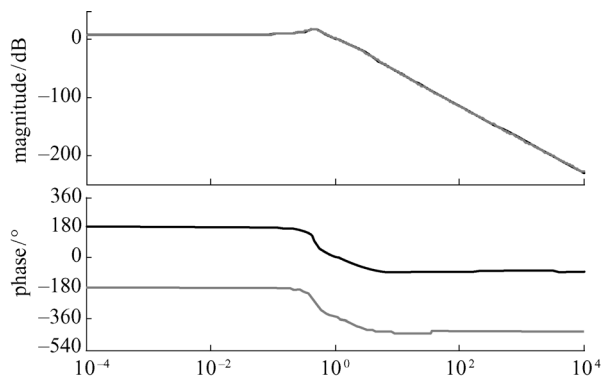


Fig. 4 Bode diagram comparisons

5 Conclusions

With the research and development of fractional-order calculus theories, they are used in more and more areas. Currently, they are applied in physical sciences and engineering, such as electro-magnetic and fractional-order calculus. In contrast, the integer-order calculus shows local information, and fractional-order calculus presents the global behavior via weighting methods. In many applications, fractional-order calculus can be used

to accurately describe the dynamic behaviors of the systems.

Because the conventional integer-order methods cannot be used in dealing with fractional-order models, the integer-order approximation to fractional-order differentiators via filters are introduced to approximate the original-order models by high-order models. The conventional methodologies can be used directly in analyses and designs. The order of the filter and the frequency range of interest may affect the approximation results because of the limitation of the approximation methods. A GL definition based on closed-form solution to the fractional-order linear differential equation is presented in the paper and examples show the effectiveness of the algorithm.

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