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RESEARCH ARTICLE

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Image restoration using total variation and anisotropic diffusion equation

Abstract This paper proposes a new model for the image restoration which combines the total variation minimization with the "pure" anisotropic diffusion equation of Alvarez and Morel. According to the introduction of new diffusion term, this model can not only remove noise but also enhance edges and keep their locality. And it can also keep textures and large-scale fine features that are not characterized by edges. Due to these favorable characteristics, the processed images turn much clearer and smoother, meanwhile, their significant details are kept, which results in appealing vision.

Keywords total variation, anisotropic diffusion equation, noise removal, detail subject classification

1 Introduction

Digital image processing has been widely applied in science and engineering fields. And image quality is usually chosen as the external measure for these applications. However, due to the faultiness of equipment and the limit of external conditions, images are usually destroyed in the process of transmitting. Thus, it is necessary to restore the image. During the last decade, mathematical image processing has become one of the active fields due to high pursuit of visual effect of image. Especially, much attention has been paid to partial differential equations (PDE) and variational methods.

There are two PDE-based approaches to image denoising. They are either based on the axiomatic approach of nonlinear scale space (nonlinear diffusions), or on the variational approach of energy functional minimization. And the interaction and close relationship among these approaches can be found in Refs. [1–3]. Inspired from these relationships, a new image restoration model, which combines total variation with diffusion term of "pure" anisotropic diffusion equation [6], was proposed on the basis of Refs. [4] and [5]. And the model shows excellent denoising effect.

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2 Preliminaries

2.1 Total variation model

Assume a noisy image is in the following form

$$I_0 = I + n \tag{1}$$

where I is the true image; n is additive Gaussian whiter noise of standard deviation σ and I_0 is the observed noisy image.

In order to recover I, some information about noise is needed. In principle, this problem can be understood as an inverse problem. Consequently, the restoration can be done by means of regularization techniques and minimization of related variaitonal functionals. One classical model for the recovery of I was proposed by Rudin, Osher and Fatemi in Ref. [4]. The objective is to minimize the following functional with respect to I

$$\min_I E(I) = \int_\Omega \left(|\nabla I| + \frac{1}{2}\lambda (I - I_0)^2 \right) \mathrm{d}x\mathrm{d}y \tag{2}$$

where $\lambda > 0$ is a tuning parameter. The first term in the energy is a regularizing term called the total variation (TV) of I, while the second term is a fidelity term.

It is well known that the implicit assumption that underlies rudin-osher-fatemi (ROF) model is the approximation of images by piecewise constant functions in TV space. Thus, ROF model performs very well for the oscillatory noise and a repeated pattern of small scale details [7] while preserving edges. However, it has several obvious drawbacks as follows: textures are excluded, significant small details may be left out, and even large-scale fine features which are not characterized by edges are disregarded. To overcome these, Gilboa [5] proposed a variational method with adaptive fidelity term

$$\min_I E(I) = \int_\Omega \left(\phi(|\nabla I|) + \frac{1}{2}\lambda(x, y) P_z(x, y) \right) \mathrm{d}x\mathrm{d}y \tag{3}$$

where $P_z(x,y)=\frac{1}{|\Omega|}\int_\Omega (I_z-\eta[I_z])^2\omega(|\tilde{x}-x|,|\tilde{y}-y|)\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}$ is the local power, C is a constant, ω is a window function and $\eta[\cdot]$ is the expected value.

2.2 "Pure" anisotropic diffusion equation

Alvarez and Morel gave the "pure" anisotropic diffusion equation which improves the edge detection theory proposed by Malik and Perona [6]

$$\begin{cases} I_t=|\nabla I|\operatorname{div}\left(\frac{\nabla I}{|\nabla I|}\right), & (x,y)\in\Omega, t>0 \\ I(x,y,0)=I_0(x,y), & (x,y)\in\Omega \end{cases} \tag{4}$$

where $I_0(x, y)$ is a original gray image and $I(x, y, t)$ is a smooth version of $I_0(x, y)$ at scale t.

The diffusion term of the right side in Eq. (4) represents a degenerating diffusion term which diffuses I in the direction ξ orthogonal to its gradient ∇I, i.e.

$$|\nabla I|\operatorname{div}\left(\frac{\nabla I}{|\nabla I|}\right)=|\nabla I|\left(\Delta I-\frac{1}{|\nabla I|^2}\frac{\nabla^2 I(\nabla I,\nabla I)}{|\nabla I|}\right)$$
$$=\Delta I-\frac{\nabla^2 I(\nabla I,\nabla I)}{|\nabla I|^2}=I_{\xi\xi} \tag{5}$$

where $I_{\xi\xi}=\frac{I_{xx}I_y^2-2I_{xy}I_xI_y+I_{yy}I_x^2}{I_x^2+I_y^2}$, $\xi=\frac{[-I_y,I_x]}{\sqrt{I_x^2+I_y^2}}$.

Because the denominator $|\nabla I|$ in Eq. (4) cannot be zero in the computation, the term is often regularized as $|\nabla I|_\varepsilon=\left(|\nabla I|^2+\varepsilon\right)^{1/2}$, $\varepsilon>0$.

3 New model and numerical discretization

Replacing the smooth term $\phi(|\nabla I|)$ in Eq. (3) by TV norm of Eq. (2), hawse can be obtained $\int_\Omega \phi(|\nabla I|)\mathrm{d}x\mathrm{d}y=\int_\Omega |\nabla I|\mathrm{d}x\mathrm{d}y$.

Here the associated PDE for Gilboa model is

$$I_t=\bar{\lambda}(x,y)(I-I_0-C)-\operatorname{div}\left(\frac{\nabla I}{|\nabla I|_\varepsilon}\right) \tag{6}$$

where $\bar{\lambda}(x,y)=\int_\Omega \lambda(\tilde{x},\tilde{y})\omega_{x,y}(\tilde{x},\tilde{y})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}$.

When Eq. (6) is discretized, the diffusion contributed by the term $\operatorname{div}(\nabla I/|\nabla I|_\varepsilon)$ can be large when $|\nabla I|\approx 0$. That is

$$\frac{1}{|\nabla I|_\varepsilon}=\frac{1}{\left(I_x^2+I_y^2+\varepsilon^2\right)^{1/2}}\approx\frac{1}{\varepsilon}\Rightarrow\operatorname{div}\left(\frac{\nabla I}{|\nabla I|_\varepsilon}\right)\approx\frac{1}{\varepsilon}\Delta I,$$
$$|\nabla I|\approx 0. \tag{7}$$

Here Eq. (7) is the isotropic heat diffusion. This implies that Eq. (6) can enforce a strong diffusion on relatively flat regions and the image can be transformed into locally constant there. This makes image blurry. Therefore, the diffusion term of the "pure" anisotropic diffusion Eq. (4) is incorporated to Gilboa model. This leads to a hybrid model that combines Gilboa model with the "pure" anisotropic diffusion equation. This model has two advantages. On one hand, it avoids strong diffusion in Eq. (6); on the other hand, it diffuses only in the direction orthogonal to its gradient and does not diffuse at all in the direction of gradient, which makes I smooth on both sides of an edge with a minimal smoothing of the edge itself.

3.1 Description of hybrid model

Replacing the diffusion term $\operatorname{div}(\nabla I/|\nabla I|_\varepsilon)$ in Eq. (6) by $|\nabla u|\operatorname{div}(\nabla u/\nabla u|_\varepsilon)$ in Eq. (5), the following new PDE is given as

$$I_t=\bar{\lambda}(x,y)(I_0-I+C)+|\nabla I|\operatorname{div}\left(\frac{\nabla I}{|\nabla I|_\varepsilon}\right) \tag{8}$$

In order to compute the value of λ, PDE is multiplied in Eq. (8) by $(I-I_0-C)$ and integrated over Ω. After the order of integrals in the λ term changes, it is got that

$$\begin{aligned}0&=\int_\Omega\left(\bar{\lambda}(x,y)(I-I_0-C)^2\right.\\&\qquad\left.-(I-I_0-C)|\nabla I|\operatorname{div}\left(\frac{\nabla I}{|\nabla I|_\varepsilon}\right)\right)\mathrm{d}x\mathrm{d}y\\&=\int_\Omega\left(\lambda(x,y)P_z(x,y)\right.\\&\qquad\left.-(I-I_0-C)|\nabla I|\operatorname{div}\left(\frac{\nabla I}{|\nabla I|_\varepsilon}\right)\right)\mathrm{d}x\mathrm{d}y\end{aligned} \tag{9}$$

A sufficient condition for Eq. (9) is

$$\lambda(x,y)=\frac{(I-I_0-C)|\nabla I|\operatorname{div}\left(\frac{\nabla I}{|\nabla I|_\varepsilon}\right)}{P_z(x,y)} \tag{10}$$

Because the noise is an additive white Gaussian process of standard deviation σ, local power $P_z(x, y)\approx\sigma^2$. Finally, the solution $I(x, y)$ is found by a steepest descent method applied to Eq. (8).

3.2 Numerical discretization

To discretize the PDE in Eq. (8) with Neumann boundary conditions, a finite differences scheme and an iterative algorithm are adopted.

The details of the numerical algorithm are as follows. The classical notations $I(ih, jh)=I_{i,j}$, $t_n=n\Delta t$ $(n=0, 1, \ldots)$ and

$I(ih, jh, t_n) = I_{i,j}^n$ are employed, where t is the step time and h is the step space for $0 \leqslant i, j \leqslant N$. The discrete form of diffusion term in Eq. (8) is

$$|\nabla I| \operatorname{div}\left(\frac{\nabla I}{|\nabla I|_\varepsilon}\right) = \frac{I_{xx}I_y^2 - 2I_{xy}I_xI_y + I_{yy}I_x^2}{I_x^2 + I_y^2 + \varepsilon^2} \tag{11}$$

where

$$(I_x)_{i,j}^n = \frac{I_{i+1,j}^n - I_{i-1,j}^n}{2h}, (I_y)_{i,j}^n = \frac{I_{i,j+1}^n - I_{i,j-1}^n}{2h}$$

$$(I_{xx})_{i,j}^n = \frac{I_{i+1,j}^n - 2I_{i,j}^n + I_{i-1,j}^n}{h^2}, (I_{yy})_{i,j}^n = \frac{I_{i,j+1}^n - 2I_{i,j}^n + I_{i,j-1}^n}{h^2}$$

$$(I_{xy})_{i,j}^n = \frac{I_{i+1,j+1}^n - I_{i-1,j+1}^n - I_{i+1,j-1}^n + I_{i-1,j-1}^n}{4h^2}$$

Therefore, the iterative form for Eq. (8) is

$$I_{i,j}^{n+1} = I_{i,j}^n + \Delta t[\overline{\lambda}(x,y)(I_{i,j}^n - I_{i,j}^0) + L(I_{i,j}^n)] \tag{12}$$

where $L(I_{i,j}^n) = |\nabla I_{i,j}^n| \operatorname{div}\left(\frac{\nabla I_{i,j}^n}{|\nabla I_{i,j}^n|_\varepsilon}\right), i, j = 1, 2, \ldots, N$.

The boundary conditions are $I_{0,j}^n = I_{1,j}^n, I_{N,j}^n = I_{N-1,j}^n, I_{i,0}^n = I_{i,N}^n = I_{i,N-1}^n$.

4 Simulation experiments and analysis

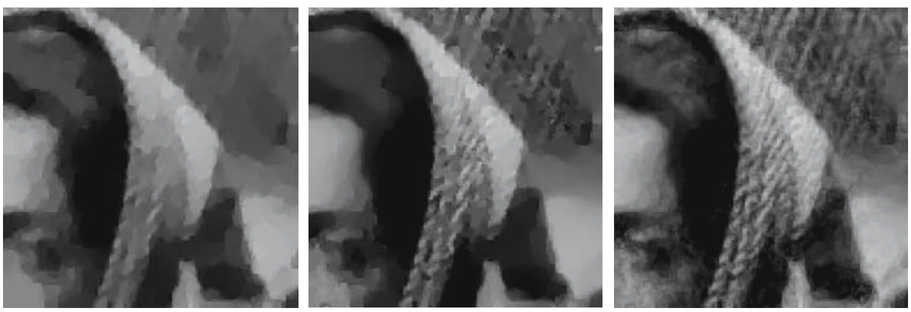
In this section, the numerical results obtained by applying the proposed new method in Eq. (8), ROF model and Gilboa model are presented to image denoising. In all experiments, Gaussian white noise of standard deviation σ is added uniformly to different clean images. And window function $\omega(x, y)$ in $\overline{\lambda}(x,y)$ is chosen as Gaussian function. For the computation convenience, set $C = 0$ and $\varepsilon = 1$.

In Fig. 1, the denoising results are first showed with an intercepting image of Barbara which contains rich textures and non-textured parts. Tables 1 and 2 present respectively comparative results of peak-signal-to- noise-ratio (PSNR) and mean-squared-error (MSE). It can be found that the new method demonstrates results with the best visual effect and data measure. Thus, the new method achieves excellent denoising effect.

In addition, PSNR and MSE are given for Lena image and plane image in Tables 1 and 2. Comparing these data, new method shows improvement over ROF model and G-model.

5 Conclusion

A new hybrid model for image restoration which is based on total variation and "pure" anisotropic diffusion equation was



(a) (b) (c)

Fig. 1 Denoising results for Barbara image from three models
(a) ROF model; (b) Gilboa model; (c) new method

Table 1 PSNR of a few classical images from three models

Image	σ	PSNR	ROF model	Gilboa model	New method
Barbara	20	18.494	25.694	25.713	26.466
Lena	15	24.584	30.065	30.182	31.090
Plane	30	18.563	27.029	27.039	27.438

Table 2 MSE of a few classical images from three models

Image	σ	MSE	ROF model	Gilboa model	New method
Barbara	20	919.76	175.27	174.49	156.71
Lena	15	226.30	64.065	62.358	50.588
Plane	30	915.21	128.87	128.57	117.29

presented. According to the introduction of new diffusion term, this model can not only remove noise but also enhance edges and keep their locality. And it can also keep textures and large-scale fine features that are not characterized by edges. The numerical results showed that the new method is suitable for textured image denoising.

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