Front. Electr. Electron. Eng. China 2007, 2(3): 293–297
DOI 10.1007/s11460-007-0054-3

RESEARCH ARTICLE

LU Xiaoxu, ZHONG Liyun, ZHANG Yimo

The holographic reconstructing algorithm and its error analysis about phase-shifting phase measurement

Abstract Phase-shifting measurement and its error estimation method were studied according to the holographic principle. A function of synchronous superposition of object complex amplitude reconstructed from N-step phase-shifting through one integral period (N-step phase-shifting function for short) was proposed. In N-step phase-shifting measurement, the interferograms are seen as a series of in-line holograms and the reference beam is an ideal parallel-plane wave. So the N-step phase-shifting function can be obtained by multiplying the interferogram by the original reference wave. In ideal conditions, the proposed method is a kind of synchronous superposition algorithm in which the complex amplitude is separated, measured and superposed. When error exists in measurement, the result of the N-step phase-shifting function is the optimal expected value of the least-squares fitting method. In the above method, the $N+1$-step phase-shifting function can be obtained from the N-step phase-shifting function. It shows that the N-step phase-shifting function can be separated into two parts: the ideal N-step phase-shifting function and its errors. The phase-shifting errors in N-steps phase-shifting phase measurement can be treated the same as the relative errors of amplitude and intensity under the understanding of the N+1-step phase-shifting function. The difficulties of the error estimation in phase-shifting phase measurement were restricted by this error estimation method. Meanwhile, the maximum error estimation method of phase-shifting phase measurement and its formula were proposed.

Translated from *Acta Optica Sinica*, 2006, 26(9): 1 367–1 371 [译自: 光学学报]

LU Xiaoxu (✉)
School of Information and Optoelectronic Science and Engineering, South China Normal University, Guangzhou 510631, China
E-mail: xiaoxulu48@sina.com

ZHONG Liyun
College of Life Science and Technology, Jinan University, Guangzhou 510632, China

ZHANG Yimo
School of Precision Instrument and Optoelectronic Engineering, Tianjin University, Tianjin 300072, China

Keywords optical measurement, information optics, phase-shifting phase measurement, N-step phase-shifting function, $N+1$-step phase-shifting algorithm, error

1 Introduction

Phase-shifting phase measurement is a well-known technique for accurate measurement of phase distributions in optical interferometry techniques [1], which has been broadly used in some related scientific researches and measurements [2–4]. Because of the high accuracy of phase-shifting phase measurement and the cosine characteristics of the intensity of the phase-shifting interferogram, the error analysis of this phase measurement method has been an important and difficult problem. With the changing of the phase distribution of the target, the method of phase-shifting phase measurement introduces more than two designed phase shifts, and correspondingly, grabs more than three interference fringe images, and then the phase map could be calculated. With further researches, the factors causing phase error have been well studied [5–7]. Factors, such as accuracy of the phase steps, nonlinear property and the limitation of the dynamic range of the image sensor, A/D conversion accuracy, air turbulence, optical elements and systems, the property of the object, the frequency stability and the output power fluctuation of the laser, all would introduce errors in the result. Accordingly, through many years of efforts, many methods have been developed to compensate for the influences of these factors on the measured result [8–12]. But having so many possible error-causing factors, together with the error randomness, periodicity and nonlinearity of the phase expression function, the actual error is often the integrated result of these factors and it is hard to differentiate the effects of these factors from each other. The four, five and N-step phase-shifting algorithms obtained from the least-squares method, are the means widely used. However, the evaluation of the integrated error using these algorithms still has not been solved. In order to apply and greatly improve the phase measurement technology, it is necessary to find new ways to analyze the error from the optical aspect.

In this paper, N-step and $N+1$-step phase-shifting algorithms are derived from the holographic principle and method. Based on this, a new error evaluation method, which gives an analysis and a formula of the potential maximum error, is presented.

2 A function of synchronous superposition of object complex amplitude reconstructed from *N*-steps phase-shifting in one integral period

In holographic terms, the optical wave modulated by the object is called the object wave, and the other one that interferes with it is the reference wave. Suppose both of their propagation directions are normal to the recording plane and there are N phase-shifts in each period, and the original phase distribution of the reference wave is zero, then the ith phase-shifting interferogram can be expressed as

$$\begin{aligned} I_i(x,y) &= |A_O(x,y)|^2 + |A_R(x,y)|^2 \\ &+ A_O(x,y)A_R(x,y)\exp\left\{j\left[\varphi_O(x,y)-\varphi_{PSi}\right]\right\} \\ &+ A_O(x,y)A_R(x,y)\exp\left\{-j\left[\varphi_O(x,y)-\varphi_{PSi}\right]\right\} \end{aligned} \quad (1)$$

where $A_O(x,y)$ and $A_R(x,y)$ are the amplitudes of the object wave and the reference wave respectively. Both of them are nonnegative, $\varphi_O(x,y)$ is the phase distribution of the object, and φ_{PSi} is the phase shift of the ith interferogram, it can be written as

$$\varphi_{PSi} = \frac{2\pi}{N} i, \qquad (i = 0, 1, 2, \ldots, M-1) \quad (2)$$

Equation (1) can be seen as in-line holography which uses ideal plane wave as the reference wave, and according to the holography reconstruction theory, by calculating the sum of the products of the interferograms in Eq. (1) with the corresponding phase-shifted unit reference waves, we obtain

$$\begin{aligned} &\sum_{i=0}^{M-1} I_i(x,y)\exp\left(j\frac{2\pi}{N}i\right) \\ &= \left[|A_O(x,y)|^2 + |A_R(x,y)|^2\right]\sum_{i=0}^{M-1}\exp\left(j\frac{2\pi}{N}i\right) \\ &\quad + MA_O(x,y)A_R(x,y)\exp\left[j\varphi_O(x,y)\right] \\ &\quad + A_O(x,y)A_R(x,y)\exp\left[-j\varphi_O(x,y)\right]\sum_{i=0}^{M-1}\exp\left(j2\frac{2\pi}{N}i\right) \end{aligned} \quad (3)$$

The left of Eq. (3) is a known value derived from the intensity distribution of the interferograms, and when the first and third terms at the right of Eq. (3) are zero, we can get $\varphi_O(x,y)$ directly. Based on the assumption of the phase-shift step and periodicity of the complex function, the sufficient condition that makes the first and third terms zero, is that the step number M is an integral multiple of the steps N in every period. Ordinarily, we choose $M = N$, then Eq. (3) can be rewritten as

$$\sum_{i=0}^{N-1} I_i(x,y)\exp\left(j\frac{2\pi}{N}i\right) = NA_O(x,y)A_R(x,y)\exp[j\varphi_O(x,y)] \quad (4)$$

This is the N-step phase-shifting algorithm. Equation (4) can be explained as the following two aspects.

First, the N-step phase-shifting algorithm is a method in which the complex amplitude is separated and measured. Different from the complex amplitude separation method in Off-Axis Holography, it is a separation method in the time domain. It eliminates the time independent components, the first and the second terms in Eq. (1), and the conjugation part of the phase, the fourth terms in Eq. (1), only leaving the object wave that contains the phase information. Therefore, phase-shifting phase measurement can be seen as a spatial filter method using temporal signal.

Second, in this algorithm, it changes the recorded object complex amplitudes, which have different phases, into synchronous superposition of the object waves. Thus, the phase information is preserved with its intensity of N times as strong as that of a single interferogram, and the time dependent random noise is suppressed. For simplicity, Eq. (4) is called the function of synchronous superposition of object complex amplitude reconstructed from N-step phase-shifting through one integral period, briefly called N-step phase-shifting function, which is

$$A_{NSPS}(x,y) = \sum_{i=0}^{N-1} I_i(x,y)\exp\left(j\frac{2\pi}{N}i\right) \quad (5)$$

From Eqs. (4) and (5), noticing that $I_i(x,y)$ is a positive real function, we get the same result as the N-step phase-shifting algorithm

$$\tan\left[\varphi_O(x,y)\right] = \frac{\mathrm{Im}[A_{NSPS}(x,y)]}{\mathrm{Re}[A_{NSPS}(x,y)]} = \frac{\sum_{i=0}^{N-1} I_i(x,y)\sin\left(j\frac{2\pi}{N}i\right)}{\sum_{i=0}^{N-1} I_i(x,y)\cos\left(j\frac{2\pi}{N}i\right)} \quad (6)$$

It is obvious that the results from N-step phase-shifting function is the same as from the optimal expected value of the least-squares fitting algorithm.

3 Error factors of phase-shifting measurement and *N*+1-step phase-shifting algorithm

The above analysis is deduced under the ideal conditions. However, the error-causing factors are inevitable. In summary, in the case of two-wave interference, these factors can form into two kinds—the coherent reasons and the incoherent reasons. The amplitude and phase noises belong to the former, while the power noise constitutes the latter. For example, phase-shifting error, the flow and temperature fluctuation of the air and vibrations are the source of phase noise, and the output power fluctuation mainly affects the amplitudes of the interfering waves, while the image sensor's non-linearity, A/D conversion accuracy, and the electronic noise mostly have an effect on the recorded images in the image-grabbing time. In order to discuss the effect that these factors have on the measurement, we change Eq. (1) to

$$\begin{aligned} I_i(x,y) = {} & [A_O(x,y)+\Delta A_{Oi}(x,y)]^2 \\ & +[A_R(x,y)+\Delta A_{Ri}(x,y)]^2+\Delta n_i(x,y) \\ & +[A_O(x,y)+\Delta A_{Oi}(x,y)][A_R(x,y) \\ & +\Delta A_{Ri}(x,y)]\exp\{j[\varphi_O(x,y)-\varphi_{PSi}-\Delta\varphi_i(x,y)]\} \\ & +[A_O(x,y)+\Delta A_{Oi}(x,y)][A_R(x,y)+\Delta A_{Ri}(x,y)] \\ & \quad \exp\{-j[\varphi_O(x,y)-\varphi_{PSi}-\Delta\varphi_i(x,y)]\} \\ = {} & \tilde{A}_{Oi}{}^2(x,y)+\tilde{A}_{Ri}{}^2(x,y)+\Delta n_i(x,y) \\ & +\tilde{A}_{Oi}(x,y)\tilde{A}_{Ri}(x,y)\exp\{j[\varphi_O(x,y) \\ & -\varphi_{PSi}-\Delta\varphi_i(x,y)]\} \\ & +\tilde{A}_{Oi}(x,y)\tilde{A}_{Ri}(x,y)\exp\{-j[\varphi_O(x,y) \\ & -\varphi_{PSi}-\Delta\varphi_i(x,y)]\} \end{aligned} \tag{7}$$

where $\Delta A_{Oi}(x,y)$ and $\Delta A_{Ri}(x,y)$ are the amplitude change of the object beam and the reference beam, respectively. $\Delta\varphi_i(x,y)$ is the integrated phase change, which are all real functions; $\Delta n_i(x,y)$ is the influence of incoherent factors on image intensity, which is a non-negative real function. Putting Eq. (7) into Eq. (5), we can get an N-step phase-shifting function containing the error factors. In order to compare the difference between it and Eq. (4), then get the influences on the phase measurement, we should discuss the algorithm of $N+1$-step phase-shifting in an integral period and its significance.

According to the above explanations we have done on the phase-shifting measurement in an integral period, we perform $N+1$-steps phase-shifting to the interferential field, namely from $\varphi_{PS0}=0$ to $\varphi_{PSN}=2\pi$ with the shifted phase interval as $2\pi/N$, record $N+1$ phase-shifting interferograms aggregately. Then the interferograms from the first to the Nth consist the first group of phase-shifting interferograms in an integral period; those from the second to the $(N+1)$th consist the second group of phase-shifting interferograms also in an integral period. Notice that in the ideal conditions the phase difference between the first phase-shifting interferogram and the $(N+1)$th phase-shifting interferogram is 2π. Using the reverse calculation in the second group of phase-shifting interferograms, we can calculate the phase of the $(N+1)$th phase-shifting interferogram and get the reverse N-step phase-shifting function

$$\begin{aligned} A^{-}{}_{NSPS}(x,y) &= \sum_{k=0}^{N-1} I_k(x,y)\exp\left(j\frac{2\pi}{N}k\right) \\ &= \sum_{i=1}^{N} I_i(x,y)\exp\left[j\left(2\pi-\frac{2\pi}{N}i\right)\right] \end{aligned} \tag{8}$$

where the relation between k and i is $k=N-i$, $i=1,2,\ldots,N$. If there is no error, the result is the same as Eq. (4).

While errors exist in phase-shifting process, the relations between the amplitude and the intensity in the obverse and the reverse calculations are

$$\begin{cases} \Delta A_{Oi}(x,y)=\Delta A_{O(N-k)}(x,y) \\ \Delta A_{Ri}(x,y)=\Delta A_{R(N-k)}(x,y) \\ \Delta n_i(x,y)=\Delta n_{(N-k)}(x,y) \end{cases} \tag{9}$$

whereas the phase difference between them is

$$\begin{aligned} \varphi_{PSi}+\Delta\varphi_i(x,y) &= 2\pi-[\varphi_{PS(N-k)}+\Delta\varphi_{(N-k)}(x,y)] \\ &= \varphi_{PSi}-\Delta\varphi_{(N-k)}(x,y) \end{aligned} \tag{10}$$

adding Eq. (5) to Eq. (8) and averaging it, we get the average N-step phase-shifting function. The algorithm of the $N+1$-step phase-shifting function is

$$\begin{aligned} \bar{A}_{NSPS}(x,y) = {} & \frac{1}{2}[I_0(x,y)+I_N(x,y)] \\ & +\sum_{i=1}^{N-1} I_i(x,y)\exp\left(j\frac{2}{N}i\right) \end{aligned} \tag{11}$$

Similar to Eq. (4), the significance of the $N+1$-step phase-shifting function can be denoted as

$$\bar{A}_{NSPS}(x,y)\approx NA_O(x,y)A_R(x,y)\exp[j\tilde{\varphi}_O(x,y)] \tag{12}$$

It is the N-steps phase-shifting function which contains the error; the approximate equal sign shows that the actual amplitude is minutely different from the equation (12), $\tilde{\varphi}_O(x,y)$ is the measuring phase with errors being calculated from the actual phase-shifting interferogram. In the following part, we will analyze the maximum error that might exist.

4 The error estimation of *N*-step phase-shifting algorithm

In order to analyze the errors of N-step phase-shifting algorithm, we put Eqs. (7), (9) and (10) into Eq. (11) and replace the $(N+1)$th phase-shifting interferogram by the first and then get

$$\begin{aligned} & \bar{A}_{NSPS}(x,y) \\ & = \sum_{i=0}^{N-1}[\tilde{A}_{Oi}{}^2(x,y)+\tilde{A}_{Ri}{}^2(x,y)+\Delta n_i(x,y)]\exp\left(j\frac{2\pi}{N}i\right) \\ & +\frac{\exp[j\varphi_O(x,y)]}{2}\sum_{i=0}^{N-1}\tilde{A}_{Oi}(x,y)\tilde{A}_{Ri}(x,y)\{\exp[j\Delta\varphi_i(x,y)] \\ & +\exp[-j\Delta\varphi_i(x,y)]\} \\ & +\frac{\exp[-j\varphi_O(x,y)]}{2}\sum_{i=0}^{N-1}\tilde{A}_{Oi}(x,y)\tilde{A}_{Ri}(x,y)\exp\left(j2\frac{2\pi}{N}i\right) \\ & \times\{\exp[j\Delta\varphi_i(x,y)]+\exp[-j\Delta\varphi_i(x,y)]\} \end{aligned} \tag{13}$$

Using Euler formula and Cosine function to deploy and simplify all terms in Eq. (13) and omit the higher order but quite weak terms, we could get

$$\begin{aligned}
&\bar{A}_{\mathrm{NSPS}}(x,y)\\
&\approx A_{\mathrm{O}}(x,y)A_{\mathrm{R}}(x,y)\Bigg\{N\exp\left[\mathrm{j}\varphi_{\mathrm{O}}(x,y)\right]\\
&+\frac{\exp\left[\mathrm{j}\varphi_{\mathrm{O}}(x,y)\right]}{2}\sum_{i=0}^{N-1}\Bigg[2\frac{\Delta A_{\mathrm{O}i}(x,y)}{A_{\mathrm{O}}(x,y)}+2\frac{\Delta A_{\mathrm{R}i}(x,y)}{A_{\mathrm{R}}(x,y)}\\
&-\Delta\varphi_i^2(x,y)\Bigg]+\sum_{i=0}^{N-1}\Bigg[2\frac{\Delta A_{\mathrm{O}i}(x,y)}{A_{\mathrm{R}}(x,y)}+2\frac{\Delta A_{\mathrm{R}i}(x,y)}{A_{\mathrm{O}}(x,y)}\\
&+\frac{\Delta n_i(x,y)}{A_{\mathrm{O}}(x,y)A_{\mathrm{R}}(x,y)}\Bigg]\exp\left(\mathrm{j}\frac{2\pi}{N}i\right)+\frac{\exp\left[-\mathrm{j}\varphi_{\mathrm{O}}(x,y)\right]}{2}\\
&\times\sum_{i=0}^{N-1}\Bigg[2\frac{\Delta A_{\mathrm{O}i}(x,y)}{A_{\mathrm{O}}(x,y)}+2\frac{\Delta A_{\mathrm{R}i}(x,y)}{A_{\mathrm{R}}(x,y)}-\Delta\varphi_i^2(x,y)\Bigg]\\
&\times\exp\left(\mathrm{j}2\frac{2\pi}{N}i\right)\Bigg\}
\end{aligned}\tag{14}$$

where the first section in Eq. (14) is the ideal N-step phase-shifting function, which is denoted in Eq. (4). From the second to the fourth sections are the error functions compared to the ideal N-step phase-shifting function. The second section is the synchronous error, which has the same phase as the ideal N-step phase-shifting function. The third section is the background error introduced by background error factor. The fourth is the conjugated error related to the conjugate of the phase to be measured. Because Eq. (14) is a complex variable function, the error function and the phase error of the ideal N-step phase-shifting function are mainly decided by the modular magnitude of the superposition of all error functions and the phase angle difference of the measuring phase. It is obvious that by employing $N+1$-step phase-shifting algorithm, we can convert the phase error in N-step phase-shifting algorithm to the same form as the errors of amplitude and intensity, which simplify the further analysis of the phase error measurement.

Equation (14) is a complicated function and it is difficult to calculate the influence of each error factor on the measuring result accurately, from which we could analyze the weighting of each influencing factor on the error amplitude roughly, such as:

1) if each error factor is small enough, then the influence of the synchronous error can be omitted and the phase errors are mainly introduced by background error and conjugated error;

2) if each error factor has the same magnitude, the comparative variety of the interferential lights amplitude is the main factor affecting the magnitude of error amplitude, the accuracy control of the phase error has fewer requirements than the accuracy control of the relative difference on amplitude;

3) when the ratio of the reference light to the object light is 1:1, it has the least error introduced by the comparative variety of the interferential lights amplitude, etc.

Optical phase measurement is done in the 2-D condition, not many amounts of the fringe patterns are recorded. Additionally, in the periodicity and nonlinearity of the phase expression function, the error distribution characteristics may not be normal. Therefore, we should study the new error estimation method according to the characteristic of the phase measurement. According to the aspect of measured error analysis, gaining maximum measuring error estimation has an important practical significance. In order to estimate the maximum phase error, we should first estimate the phase angle difference between the maximum module of the error function and the waiting measured phase. When all error factors are small enough, the influence of the synchronous error could be omitted. If we replace the magnitude of all error factors with the maximum absolute value between the background error and the conjugated error, the module of all error functions sum will be less than the sum of the absolute value of all modules; the phase angle difference is about $\pi/2$ while it has the maximum measured phase error. Thus, under the condition that produces the maximum measured phase error, Eq. (14) can be written as

$$\begin{aligned}
\bar{A}_{\mathrm{NSPS}}(x,y)\approx NA_{\mathrm{O}}(x,y)A_{\mathrm{R}}(x,y)\Bigg\{&\exp\left[\mathrm{j}\varphi_{\mathrm{O}}(x,y)\right]\\
&+7.5\Delta_{\max}\exp\left[\mathrm{j}\varphi_{O}(x,y)+\mathrm{j}\frac{\pi}{2}\right]\Bigg\}
\end{aligned}\tag{15}$$

where $\Delta_{\max}$ represents the maximum absolute value of the error factor in the right side of Eq. (14). According to Fig. 1, which expresses the meaning of the vector superposition in Eq. (15), the maximum measuring error of N-step phase-shifting measurement in an integral period is

$$\Delta\varphi_{\mathrm{O}\max}(x,y)=\tilde{\varphi}_{\mathrm{O}}(x,y)-\varphi_{\mathrm{O}}(x,y)<7.5\Delta_{\max}\tag{16}$$

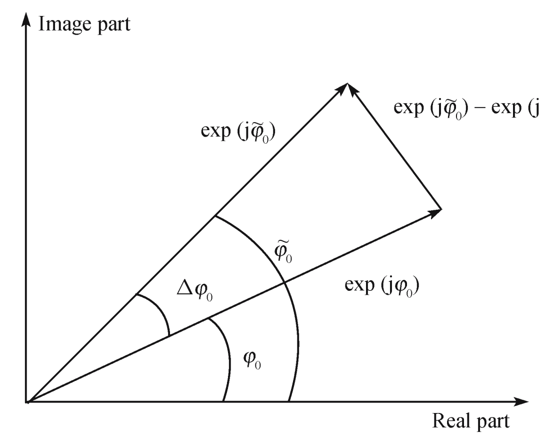

Fig. 1 Relationship of N-steps phase-shifting function, ideal N-steps phase-shifting function, their phase and their phase errors

Once we know the maximum factor generated errors, from Eq. (16), we could easily estimate the control accuracy of the maximum phase error and the other error factors. For example, the phase measurement accuracy could be up to 2 π/56 if we take a laser whose average power output is I_0 and the stability output is 3% as the maximum error control target. Here, the maximum phase-shift error should be controlled within 10° and the total noise power of other error factors should be controlled within 0.007 5 I_0.

5 Conclusions

By employing the optical holographic principle and method, the *N*-step phase-shifting phase measurement algorithm in an integral period has been analyzed in this paper. It has been proven that, in ideal conditions, the proposed method is a kind of synchronous superposition algorithm in which the complex amplitude is separated, measured and superposed. Then, a function of synchronous superposition of object complex amplitude reconstructed from *N*-step phase-shifting in an integral period is proposed, which is also called *N*-step phase-shifting function. When there is an error in measurement, the result of the *N*-step phase-shifting function is the optimal expected value of the least-squares method. Although the analyzed result is the same as that already exists, in principle, the analyzed process here should have more physical property. The $N+1$-step phase-shifting function can be obtained from the *N*-step phase-shifting function, thus an *N*-step phase-shifting phase error estimate method is proposed. It shows that the *N*-step phase-shifting function can be separated into two parts: the ideal *N*-steps phase-shifting function and its errors, the latter contain the errors of amplitude, intensity and phase. According to the significance of $N+1$-step phase-shifting function, the phase-shifting errors can be treated the same as the relative errors of amplitude and intensity and the difficulties of the error estimation in phase-shifting phase measurement are restricted by this error estimation method. Meanwhile, the maximum error estimation method of phase-shifting phase measurement and its formula are proposed.

Acknowledgements This work was supported by the National Natural Science Foundation of China (Grant No. 60467003 and 60277032).

References

1. Crane R. Interference phase measurement. Appl Opt, 1969, 8(3): 538–542
2. Saif B N, Millerd J, Keski-Kuha R, et al. Instantaneous phase-shifted speckle interferometer for measurement of large optical structures. Pro SPIE, 2004, 5494: 152–162
3. Joannes L, Dubios F, Legros J. Phase-shifting schlieren: High-resolution quantitative schlieren that uses the phase-shifting technique principle. Appl Opt, 2003, 42(25): 5 046–5 053
4. Lu Xiaoxu, Zhang Yimo, Zhong Liyun, et al. Analysis on phase-shifting in-line lensless Fourier digital holography. Acta Optica Sinica, 2004, 24(11): 1 511–1 515 (in Chinese)
5. Schwider J, Burow R, Elssner K E, et al. Digital wavefront measuring interferometry: Some systematic error sources. Appl Opt,1983, 22(21): 3 421–3 432
6. Kinnstaetter K, Lohmann A W, Schwider J, et al. Accuracy of phase shifting interferometry. Appl Opt, 1988, 27(24): 5 082–5 089
7. Gutmann B, Weber H. Phase-shifter calibracation and error detection in phase-shifting applications: A new method. Appl Opt, 1998, 37(32): 7 624–7 631
8. Larkin K G, Oreb B F. Design and assessment of symmetrical phase-shifting algorithms. J Opt Soc Am, 1992, 9(10): 1 740–1 748
9. Groot P, Deck L. Numerical simulations of vibration in phase-shifting interferometry. Appl Opt, 1996, 35(13): 2 172–2 178
10. Schmit J, Creath K. Extended averaging technique for derivation of error-compensating algorithms in phase-shifting interferometry. Appl Op, 1995, 34(19): 3 610–3 619
11. Luo Yinlong, Lu Xiaoxu, Zhu Yue, et al. A real-time high accuracy phase-shifting calibrating method based on one-dimensional spatial digital correlation. Chin J Lasers, 2005, 24(5): 1 117–1 122 (in Chinese)
12. Wang Liwu, Su xianyu, Zhou Libing. Correction algorithm of random phase shifting error in phase measuring profilometry. Acta Optica Sinica, 2004, 24(5): 614–618 (in Chinese)