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RESEARCH ARTICLE

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# Robust adaptive output feedback control of nonlinearly parameterized systems



**Abstract** The ideas of adaptive nonlinear damping and changing supply functions were used to counteract the effects of parameter and nonlinear uncertainties, unmodeled dynamics and unknown bounded disturbances. The high-gain observer was used to estimate the state of the system. A robust adaptive output feedback control scheme was proposed for nonlinearly parameterized systems represented by input-output models. The scheme does not need to estimate the unknown parameters nor add a dynamical signal to dominate the effects of unmodeled dynamics. It is proven that the proposed control scheme guarantees that all the variables in the closed-loop system are bounded and the mean-square tracking error can be made arbitrarily small by choosing some design parameters appropriately. Simulation results have illustrated the effectiveness of the proposed robust adaptive control scheme.

**Keywords** nonlinearly parameterized systems, adaptive control, output feedback, unmodeled dynamics, robustness, stability

## 1 Introduction

Because many physical systems are nonlinear and uncertain, the adaptive control of nonlinear systems has been receiving a great attention. Much progress has been made in this field recently. In Ref. [1], an adaptive output feedback control scheme is presented for a class of nonlinear systems represented by input-output models. However, the scheme applies only to nonlinear systems while linear parameterization. The scheme does not consider disturbances and unmodeled dynamics, i.e., it is not robust to disturbances and unmodeled dynamics. Although disturbances and unmodeled dynamics are considered in Refs. [2–4], they can be used only in nonlinear systems with linear parameterization.

Although adaptive control schemes have been presented for nonlinearly parameterized systems in Refs. [5,6], the schemes use state feedback.

The scheme in Ref. [7] uses an output feedback, but it needs to generate an additional dynamical signal to dominate the effects caused by unmodeled dynamics and leads to a complicated controller. Therefore, it is important both theoretically and practically to study the adaptive output feedback control scheme, which does not need to generate an additional dynamical signal.

Based on the ideas of adaptive nonlinear damping [7] and changing supply functions [8], this paper presents a robust adaptive output feedback control scheme for nonlinearly parameterized systems represented by input-output models. The scheme can be used in the case of nonlinear systems with parameter and nonlinear uncertainties, unmodeled dynamics and unknown bounded disturbances. It does not need to estimate the unknown parameters or to add a dynamical signal to dominate the effects of unmodeled dynamics. It is shown that the proposed control scheme guarantees that all the variables in the closed-loop system are bounded and the mean-square tracking error can be made arbitrarily small by choosing some design parameters appropriately.

## 2 Problem statement

Consider a single-input single-output (SISO) nonlinearly parameterized system

$$\begin{aligned} y^{(n)} &= f\left(y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(m-1)}, \tilde{\boldsymbol{\theta}}\right) \\ &\quad + \theta_u u^{(m)} + d(t) \\ &\quad + \Delta\left(y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(m-1)}, \boldsymbol{\omega}\right) \end{aligned} \tag{1}$$

where $y$ is the output; $u$ is the control; $y^{(i)}$ and $u^{(i)}$ are the $i$th derivative of $y$ and $u$ respectively; $d(t)$ is the unknown bounded time-varying disturbance; $\Delta(\cdot)$ represents the uncertain

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nonlinearity and the uncertainty related to unmodeled dynamics $\boldsymbol{\omega}$; $\tilde{\boldsymbol{\theta}} \in \mathbb{R}^p$ and $\theta_u$ are unknown constant parameters, but the sign of $\theta_u$ is known. Without loss of generality, we assume that $\theta_u > 0$. It is assumed in Eq. (1) that $f$ is an unknown smooth nonlinear function satisfying

$$\left|f\left(y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(m-1)}, \tilde{\boldsymbol{\theta}}\right)\right| \leqslant \theta \bar{f}\left(y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(m-1)}\right)$$

where $\bar{f}(\cdot)$ is a known smooth nonlinear function, $\theta > 0$ is an unknown constant.

Denote $x_1 = y, x_2 = y^{(1)}, \ldots, x_n = y^{(n-1)}$, $z_1 = u, z_2 = u^{(1)}, \ldots, z_m = u^{(m-1)}$. From Eq. (1), we have

$$\begin{cases} \dot{x}_i = x_{i+1}, & 1 \leqslant i \leqslant n-1 \\ \dot{x}_n = f(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\theta}}) + \theta_u v + \Delta(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\omega}) + d(t) \\ \dot{z}_i = z_{i+1}, & 1 \leqslant i \leqslant m-1 \\ \dot{z}_m = v \end{cases} \tag{2}$$

where $v = u^{(m)}$ is the control input for the augmented system described by Eq. (2), $\boldsymbol{x} = [x_1, \ldots, x_n]^{\mathrm{T}}$, $\boldsymbol{z} = [z_1, \ldots, z_m]^{\mathrm{T}}$. In $\Delta(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\omega})$, $\boldsymbol{\omega} \in \mathbb{R}^l$ is the unmodeled dynamics described by

$$\dot{\boldsymbol{\omega}} = \boldsymbol{q}(\boldsymbol{\omega}, \boldsymbol{x}, \boldsymbol{z}) \tag{3}$$

In addition, $\Delta$ and $\boldsymbol{q}$ are assumed to be unknown continuous functions satisfying

$$\left|\Delta(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\omega})\right| \leqslant c_1 \gamma_1(\boldsymbol{x}) + c_2 \|\boldsymbol{z}\| + c_3 \gamma_2(\boldsymbol{\omega}) + c_4 \tag{4}$$

$$\Delta(0, \boldsymbol{z}, 0) = 0 \tag{5}$$

where $\gamma_1(\cdot)$, $\gamma_2(\cdot)$ are nonnegative functions; $c_1$, $c_2$, $c_3$, $c_4 \geqslant 0$ are unknown constants. Let

$$\zeta_i = z_i - \frac{x_{n-m+i}}{\theta_u}, \quad 1 \leqslant i \leqslant m \tag{6}$$

Using Eq. (2) we obtain

$$\begin{cases} \dot{\zeta}_i = \zeta_{i+1}, & 1 \leqslant i \leqslant m-1 \\ \dot{\zeta}_m = -\left.\dfrac{f(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\theta}}) + \Delta(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\omega}) + d(t)}{\theta_u}\right|_{z_i = \zeta_i + \frac{x_{n-m+i}}{\theta_u}} \end{cases} \tag{7}$$

Denote $\boldsymbol{\zeta} = [\zeta_1, \ldots, \zeta_m]^{\mathrm{T}}$, $\bar{\boldsymbol{b}} = [0, \ldots, 0, 1]^{\mathrm{T}}$, $\boldsymbol{h}(\boldsymbol{\zeta}, \boldsymbol{x}) = \left[\zeta_2, \ldots, \zeta_{m-1}, -\dfrac{f(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\theta}})}{\theta_u}\right]^{\mathrm{T}}$, we have

$$\dot{\boldsymbol{\zeta}} = \boldsymbol{h}(\boldsymbol{\zeta}, \boldsymbol{x}) + \bar{\boldsymbol{b}}\left(\frac{-\Delta(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\omega}) - d(t)}{\theta_u}\right) \tag{8}$$

The nominal system of Eq. (2) can be obtained by substituting $\Delta(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\omega}) = 0$, $d(t) = 0$ into the first $n$ equations in Eqs. (2) and (8).

We assume that the reference signal $y_{\mathrm{r}}(t)$ is bounded with bounded derivatives up to the $n$th order and $y_{\mathrm{r}}^{(n)}(t)$ is piecewise continuous. Denote

$$\bar{\boldsymbol{y}}_{\mathrm{r}} = \left[y_{\mathrm{r}}, y_{\mathrm{r}}^{(1)}, \ldots, y_{\mathrm{r}}^{(n-1)}\right]^{\mathrm{T}}$$

$$\bar{\boldsymbol{y}}_{\boldsymbol{R}} = \left[y_{\mathrm{r}}, y_{\mathrm{r}}^{(1)}, \ldots, y_{\mathrm{r}}^{(n)}\right]^{\mathrm{T}}$$

Let $\boldsymbol{Y} \subset \mathbb{R}^n, \boldsymbol{Y}_R \subset \mathbb{R}^{n+1}, \boldsymbol{Z}_0 \subset \mathbb{R}^m, \boldsymbol{W}_0 \subset \mathbb{R}^l$ be any given compact sets. The objective of this paper is to design a robust adaptive output feedback controller such that for any $\boldsymbol{x}(0) \in Y$, $\boldsymbol{z}(0) \in Z_0$, $\boldsymbol{\omega}(0) \in W_0$ and $\bar{\boldsymbol{y}}_{\boldsymbol{R}} \in Y_R$, the output $y(t)$ of the system tracks the reference signal $y_{\mathrm{r}}(t)$ and all the variables of the closed-loop system are bounded in the presence of unmodeled dynamics and bounded disturbances.

**Assumption 1** In the nominal system, the subsystem $\dot{\boldsymbol{\zeta}} = \boldsymbol{h}(\boldsymbol{\zeta}, \boldsymbol{x})$ has a unique steady-state solution $\bar{\zeta}$ [9]. Without loss of generality, we assume $\bar{\zeta} = 0$. Moreover, the subsystem has a function $w(t, \boldsymbol{\zeta})$ satisfying

$$\begin{cases} \pi_1 \|\boldsymbol{\zeta}\|^2 \leqslant w(t, \boldsymbol{\zeta}) \leqslant \pi_2 \|\boldsymbol{\zeta}\|^2, \\ \dfrac{\partial w}{\partial t} + \dfrac{\partial w}{\partial \boldsymbol{\zeta}} \boldsymbol{h}(\boldsymbol{\zeta}, \boldsymbol{x}) \leqslant -\pi_3 \|\boldsymbol{\zeta}\|^2 + \pi_4 \|\boldsymbol{\zeta}\| \|\boldsymbol{x}\|, \\ \left\|\dfrac{\partial w}{\partial \boldsymbol{\zeta}}\right\| \leqslant \pi_5 \|\boldsymbol{\zeta}\| \end{cases} \tag{9}$$

where $\pi_i > 0$, $i = 1, \ldots, 5$ are constants and $\pi_3 > \dfrac{\pi_5 c_2}{\theta_u} + \pi_6$, $\pi_6 > 0$ is a constant.

**Assumption 2** The unmodeled dynamics described by Eq. (3) is exponentially input-to-state practically stable (exp-ISpS); that is, there exists $V_\omega(\boldsymbol{\omega})$ satisfying

$$\alpha_1(\|\boldsymbol{\omega}\|) \leqslant V_\omega(\boldsymbol{\omega}) \leqslant \alpha_2(\|\boldsymbol{\omega}\|) \tag{10}$$

$$\frac{\partial V_\omega(\boldsymbol{\omega})}{\partial \boldsymbol{\omega}} \boldsymbol{q}(\boldsymbol{\omega}, \boldsymbol{x}, \boldsymbol{z}) \leqslant -W(\|\boldsymbol{\omega}\|) + \gamma(x) \tag{11}$$

where $\alpha_1$, $\alpha_2$ and $W$ are functions of class $K_\infty$, $\gamma$ is a nonnegative smooth function.

**Lemma 1** [5] For any continuous function $f(\boldsymbol{x}, \boldsymbol{y})$, where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{y} \in \mathbb{R}^m$, there exist smooth scalar functions $a(\boldsymbol{x}) \geqslant 0$, $b(\boldsymbol{y}) \geqslant 0$, $c(\boldsymbol{x}) \geqslant 1$ and $d(\boldsymbol{y}) \geqslant 1$ satisfying

$$|f(\boldsymbol{x}, \boldsymbol{y})| \leqslant a(\boldsymbol{x}) + b(\boldsymbol{y}) \tag{12}$$

$$|f(\boldsymbol{x}, \boldsymbol{y})| \leqslant c(\boldsymbol{x}) d(\boldsymbol{y}) \tag{13}$$

Lemma 1 provides a parameter separation method in the adaptive control of nonlinearly parameterized systems.

Using the idea of changing supply functions [8], we construct the following function

$$V_0(\boldsymbol{\omega}) = \int_0^{V_\omega(\boldsymbol{\omega})} \eta(s) \mathrm{d}s \tag{14}$$

where $\eta(\cdot) \geqslant 1$ is monotone nondecreasing function. It can be seen that $V_0(\boldsymbol{\omega})$ is positive definite and continuously differentiable. By Assumption 2 and Lemma 1, for any given $\delta(\boldsymbol{\omega})\geqslant 0$, we can always find a $\eta(\cdot)\geqslant 1$ such that

$$\frac{\partial V_0(\boldsymbol{\omega})}{\partial \boldsymbol{\omega}}q(\boldsymbol{\omega},\boldsymbol{x},\boldsymbol{z})\leqslant -\frac{1}{2}\eta\left(V_\omega(\boldsymbol{\omega})\right)W\left(\|\boldsymbol{\omega}\|\right)+\hat{\gamma}\left(\|\boldsymbol{x}\|\right) \quad (15)$$

$$\frac{1}{4\lambda}\eta\left(V_\omega(\boldsymbol{\omega})\right)W\left(\|\boldsymbol{\omega}\|\right)\geqslant \delta(\boldsymbol{\omega}) \quad (16)$$

where $\lambda\geqslant 1$ is a constant, $\hat{\gamma}\left(\|\boldsymbol{x}\|\right)\geqslant 0$ is a smooth function.

## 3 Robust adaptive control scheme

In this section, we first study the adaptive state feedback control scheme on the basis of the assumption that the state $(\boldsymbol{x}, \boldsymbol{z})$ of Eq. (2) is available for feedback. Then, we study the adaptive output feedback control scheme. Denote $e_1 = x_1 - y_r$, $e_2 = x_2 - \dot{y}_r, \ldots, e_n = x_n - y_r^{(n-1)}$, $\boldsymbol{e} = [e_1, e_2, \ldots, e_n]^{\mathrm{T}}$.

Using Eqs. (2) and (3), we have

$$\dot{\boldsymbol{e}} = \boldsymbol{A}\boldsymbol{e}+\boldsymbol{b}\Big[f\left(\boldsymbol{e}+\bar{\boldsymbol{y}}_r, \boldsymbol{z}, \tilde{\boldsymbol{\theta}}\right)+\theta_u v - y_r^{(n)} +\Delta\left(\boldsymbol{e}+\bar{\boldsymbol{y}}_r, \boldsymbol{z}, \boldsymbol{\omega}\right)+d(t)\Big] \quad (17)$$

$$\dot{\boldsymbol{z}} = \bar{\boldsymbol{A}}\boldsymbol{z}+\bar{\boldsymbol{b}}v \quad (18)$$

$$\dot{\boldsymbol{\omega}} = q(\boldsymbol{\omega}, \boldsymbol{e}+\bar{\boldsymbol{y}}_r, \boldsymbol{z}) \quad (19)$$

where

$$\boldsymbol{A}=\begin{bmatrix}0 & 1 & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & \cdots & 1\\ 0 & 0 & \cdots & 0\end{bmatrix}, \quad \boldsymbol{b}=\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}$$

$\bar{\boldsymbol{A}}$, $\bar{\boldsymbol{b}}$ have the same structure as $\boldsymbol{A}$, $\boldsymbol{b}$, but with different sizes. Let $\boldsymbol{A_m} = \boldsymbol{A}-\boldsymbol{bK}$, where $\boldsymbol{K}$ is so chosen such that $\boldsymbol{A_m}$ is Hurwitz. Then

$$\dot{\boldsymbol{e}} = \boldsymbol{A_m}\boldsymbol{e}+\boldsymbol{b}\Big[\boldsymbol{Ke}+\boldsymbol{f}\left(\boldsymbol{e}+\bar{\boldsymbol{y}}_r, \boldsymbol{z}, \tilde{\boldsymbol{\theta}}\right)- y_r^{(n)} +\theta_u v+\Delta\left(\boldsymbol{e}+\bar{\boldsymbol{y}}_r, \boldsymbol{z}, \boldsymbol{\omega}\right)+d(t)\Big] \quad (20)$$

Since $\boldsymbol{x}=\boldsymbol{e}+\bar{\boldsymbol{y}}_r$ and $\bar{\boldsymbol{y}}_r$ is bounded, from Lemma 1 and Assumption 2, there exist smooth function $\tilde{\gamma}\left(\|\boldsymbol{e}\|\right)$ and constants $c_0, \tilde{c}_0$ such that

$$\hat{\gamma}\left(\|\boldsymbol{x}\|\right)\leqslant c_0\|\boldsymbol{e}\|\tilde{\gamma}\left(\|\boldsymbol{e}\|\right)+\tilde{c}_0 \quad (21)$$

$$\frac{\partial V_0(\boldsymbol{\omega})}{\partial \boldsymbol{\omega}}q(\boldsymbol{\omega}, \boldsymbol{x}, \boldsymbol{z})\leqslant -\frac{1}{2}\eta\left(V_\omega(\boldsymbol{\omega})\right)W\left(\|\boldsymbol{\omega}\|\right) +c_0\|\boldsymbol{e}\|\tilde{\gamma}\left(\|\boldsymbol{e}\|\right)+\tilde{c}_0 \quad (22)$$

It is found in constructing the control Lyapunov function for Eqs. (17)–(19) that adaptive nonlinear damping can dominate the effects of parameter and nonlinear uncertainties, unknown bounded disturbances and unmodeled dynamics on the stability of the system. Therefore, this paper presents the following controller consisting of adaptive nonlinear damping terms

$$v=-\beta\boldsymbol{e}^{\mathrm{T}}\boldsymbol{Pb}\Big\{\left[\bar{f}\left(\boldsymbol{e}+\bar{\boldsymbol{y}}_r, \boldsymbol{z}\right)\right]^2+\left[\gamma_1\left(\boldsymbol{e}+\bar{\boldsymbol{y}}_r\right)\right]^2+\|\boldsymbol{z}\|^2 +\left[\tilde{\gamma}\left(\|\boldsymbol{e}\|\right)\right]^2+(\boldsymbol{Ke})^2+1\Big\} \underline{\underline{\Delta}}\, v(\boldsymbol{e}, \boldsymbol{z}, \bar{\boldsymbol{y}}_r, \beta) \quad (23)$$

where $\boldsymbol{P}$ satisfies

$$\boldsymbol{PA_m}+\boldsymbol{A}_m^T\boldsymbol{P}=-\boldsymbol{Q},\ \boldsymbol{Q}=\boldsymbol{Q}^{\mathrm{T}}>0 \quad (24)$$

$\beta$ is the adaptive parameter of the controller with an adaptive law given by

$$\dot{\beta}=\beta_m(\boldsymbol{e}, \boldsymbol{z}, \bar{\boldsymbol{y}}_r)-\Gamma\sigma\beta \quad (25)$$

where

$$\beta_m(\boldsymbol{e}, \boldsymbol{z}, \bar{\boldsymbol{y}}_r)=\Gamma(\boldsymbol{e}^{\mathrm{T}}\boldsymbol{Pb})^2\Big\{\left[\bar{f}(\boldsymbol{e}+\bar{\boldsymbol{y}}_r, \boldsymbol{z})\right]^2 +\left[\gamma_1\left(\boldsymbol{e}+\bar{\boldsymbol{y}}_r\right)\right]^2+\|\boldsymbol{z}\|^2+\left[\tilde{\gamma}\left(\|\boldsymbol{e}\|\right)\right]^2+(\boldsymbol{Ke})^2+1\Big\}$$

$\Gamma>0$, $\sigma>0$ are constants.

**Theorem 1** Under Assumptions 1 and 2, with the proposed adaptive state feedback controller, for any given $\boldsymbol{x}(0)\in Y$, $\boldsymbol{z}(0)\in Z_0$, $\boldsymbol{\omega}(0)\in W_0$ and $\bar{\boldsymbol{y}}_R\in Y_R$, all the variables of the closed-loop system consisting of Eqs. (18)–(21) are bounded even in the presence of an unmodeled dynamics and bounded disturbances. Furthermore, the mean-square tracking error can be made arbitrarily small by choosing the appropriate design parameters $\Gamma$, $\sigma$ and $\boldsymbol{Q}$.

**Proof** Choose the Lyapunov function candidate

$$V=\frac{1}{\lambda}V_0(\boldsymbol{\omega})+\boldsymbol{e}^{\mathrm{T}}\boldsymbol{Pe}+\theta_u\Gamma^{-1}\left(\beta-\beta^*\right)^2 \quad (26)$$

where $\beta^*>0$ is a constant, which is the desired value of $\beta$, i.e., when $\beta=\beta^*$, the controlled system has a desired performance. Taking the time derivative of $V$ along the solutions of Eqs. (19), (20) and (25) yields

$$\begin{aligned}\dot{V}\leqslant & -\frac{1}{2\lambda}\eta\left(V_\omega(\boldsymbol{\omega})\right)W\left(\|\boldsymbol{\omega}\|\right)+\frac{c_0}{\lambda}\|\boldsymbol{e}\|\tilde{\gamma}\left(\|\boldsymbol{e}\|\right)+\frac{\tilde{c}_0}{\lambda}\\ & -\boldsymbol{e}^{\mathrm{T}}\boldsymbol{Qe}+2\left|\boldsymbol{e}^{\mathrm{T}}\boldsymbol{Pb}\right|\cdot\left|\boldsymbol{Ke}\right|+2\left|\boldsymbol{e}^{\mathrm{T}}\boldsymbol{Pb}\right|\\ & \cdot\left[\theta\bar{f}\left(\boldsymbol{e}+\bar{\boldsymbol{y}}_r, \boldsymbol{z}\right)+c_1\gamma_1\left(\boldsymbol{e}+\bar{\boldsymbol{y}}_r\right)+c_2\|\boldsymbol{z}\|\right.\\ & \left.+c_3\gamma_2(\boldsymbol{\omega})+c_4+\left|y_r^{(n)}(t)\right|+\left|d(t)\right|\right]\\ & -2\theta_u\beta^*\left(\boldsymbol{e}^{\mathrm{T}}\boldsymbol{Pb}\right)^2\Big\{\left[\bar{f}\left(\boldsymbol{e}+\bar{\boldsymbol{y}}_r, \boldsymbol{z}\right)\right]^2\\ & +\left[\gamma_1\left(\boldsymbol{e}+\bar{\boldsymbol{y}}_r\right)\right]^2+\|\boldsymbol{z}\|^2+\left[\tilde{\gamma}\left(\|\boldsymbol{e}\|\right)\right]^2+(\boldsymbol{Ke})^2\\ & +1\Big\}-\theta_u\sigma\beta^2+\theta_u\sigma\beta^{*2}-\theta_u\sigma(\beta-\beta^*)^2\end{aligned} \quad (27)$$

Let

$$\bar{c}_4=\sup\left\{\left|y_r^{(n)}(t)\right|+\left|d(t)\right|\right\}+c_4 \quad (28)$$

$$\delta(\boldsymbol{\omega})=\frac{c_3^2\left[\gamma_2(\boldsymbol{\omega})\right]^2}{\beta^*\theta_u} \tag{29}$$

Using Eqs. (16), (27) and (28) we obtain

$$\dot{V}\leqslant-\frac{1}{4\lambda}\eta\left(V_\omega(\boldsymbol{\omega})\right)W\left(\|\boldsymbol{\omega}\|\right)-\boldsymbol{e}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{e}-\sigma\theta_u\left(\beta-\beta^*\right)^2+M_1 \tag{30}$$

where

$$M_1=\frac{1}{2\beta^*\theta_u}(c_1^2+c_2^2+2c_4^2+\theta^2+1)+\frac{\tilde{c}_0}{\lambda}+\frac{c_0^2}{2\beta^*\theta_u\lambda^2\|\boldsymbol{Pb}\|^2}+\sigma\theta_u\beta^{*2} \tag{31}$$

Thus,

$$\dot{V}\leqslant-\mu V+M_1 \tag{32}$$

$$\mu=\min\left\{\frac{1}{4},\frac{\lambda_{\min}(\boldsymbol{Q})}{\lambda_{\max}(\boldsymbol{P})},\ \sigma\Gamma\right\} \tag{33}$$

Therefore, $V$ decreases monotonically until ($\boldsymbol{e}$, $\boldsymbol{\omega}$, $\beta-\beta^*$) reaches the compact set

$$C_s=\left\{\left(\boldsymbol{e},\boldsymbol{\omega},\beta-\beta^*\right)\in\mathbb{R}^n\times\mathbb{R}^l\times\mathbb{R}:V\blacklozenge\mu^{-1}M_1\right\}$$

This means ($\boldsymbol{e}$, $\boldsymbol{\omega}$, $\beta$) are bounded. Because $\overline{\boldsymbol{y}}_{\mathrm{r}}$ is bounded, $\boldsymbol{x}$ is bounded.

Differentiating $w(t,\boldsymbol{\zeta})$ along the solution of Eq. (8) gives

$$\dot{w}(t,\boldsymbol{\zeta})\leqslant\dot{w}(t,\boldsymbol{\zeta})\leqslant-\frac{1}{2}\pi_6\|\boldsymbol{\zeta}\|^2+M_2 \tag{34}$$

where

$$M_2=\sup\left\{\frac{\pi_4^2}{\pi_6}\|\boldsymbol{x}\|^2+\frac{\pi_5^2}{\pi_6\theta_u^2}\left[c_1\gamma_1(x)+\frac{c_2}{\theta_u}\|\boldsymbol{x}\|+c_3\gamma_2(\boldsymbol{\omega})+c_4+|d(t)|\right]^2\right\}$$

Thus, $\boldsymbol{\zeta}$ is bounded, and $\boldsymbol{z}$ is bounded. We conclude that all the variables of the closed-loop system are bounded.

Furthermore, it can be seen from Eqs. (31) and (33) that choosing $\lambda$, $\boldsymbol{Q}$ and the design constants $\sigma$, $\Gamma$ will appropriately reduce the residual error bound $\mu^{-1}M_1$, and make the tracking error arbitrarily small.

To implement the robust adaptive controller presented above by output feedback, we need a state observer. Because the high-gain observers have the properties of rejecting disturbances and allowing for uncertainties in modeling the systems [1], we adopt the following high-gain observer to estimate $\boldsymbol{e}$.

$$\begin{cases}\dot{\hat{e}}_i=\hat{e}_{i+1}+\dfrac{\sigma_i}{\varepsilon^i}(e_1-\hat{e}_1),\ 1\leqslant i\leqslant n-1\\ \dot{\hat{e}}_n=\dfrac{\sigma_n}{\varepsilon^n}(e_1-\hat{e}_1)\end{cases} \tag{35}$$

where $\varepsilon>0$ is a small constant; $\sigma_i>0$, $i=1,\ldots,n$ are chosen so that $\boldsymbol{A}_n=\boldsymbol{A}-\boldsymbol{K}_{\boldsymbol{\sigma}}\boldsymbol{C}$ is a Hurwitz matrix, where $\boldsymbol{K}_{\boldsymbol{\sigma}}=[\sigma_1,\sigma_2,\ldots,\sigma_n]^{\mathrm{T}}$, $\boldsymbol{C}=[1,0,\ldots,0]$

To eliminate peaking in the implementation of the observer, we define

$$\hat{e}_i=\frac{q_i}{\varepsilon^{i-1}},\ 1\leqslant i\leqslant n \tag{36}$$

Then, Eq. (35) becomes

$$\begin{cases}\varepsilon\dot{q}_i=q_{i+1}+\sigma_i(e_1-q_1),\ 1\leqslant i\leqslant n-1\\ \varepsilon\dot{q}_n=\sigma_n(e_1-q_1)\end{cases} \tag{37}$$

In addition, to prevent the peaking from entering the control system, we saturate $r_m$, $\beta_m$ and $v$.

**Assumption 3** $\overline{\boldsymbol{\theta}}=[\theta,\theta_u]^{\mathrm{T}}\in\Omega$, $d(t)\in\Phi$, where $\Omega$ and $\Phi$ are known compact sets.

**Assumption 4** For any $\boldsymbol{e}(0)\in E_0$, $\boldsymbol{z}(0)\in Z_0$, $\boldsymbol{\omega}(0)\in W_0$, $\overline{\boldsymbol{y}}_R(0)\in Y_R$, $\beta(0)\in\mathbb{R}_1^0$, where $E_0$, $Z_0$, $W_0$, $Y_R$ are defined as before and $\mathbb{R}_1^0\subset\mathbb{R}$ is a compact set, using the proposed adaptive state feedback controller, we have $\boldsymbol{e}(t)\in E$, $\boldsymbol{z}(t)\in Z$, $\boldsymbol{\omega}(t)\in W$, $\overline{\boldsymbol{y}}_R(t)\in Y_R$, $\beta(t)\in\mathbb{R}_1$, $\forall t\geqslant0$, where $E\subset\mathbb{R}^n$, $Z\subset\mathbb{R}^m$, $W\subset\mathbb{R}^l$, $Y_R\subset\mathbb{R}^{n+1}$ and $R_1\subset\mathbb{R}$ are known compact sets.

Denote

$$\boldsymbol{R}_{\mathrm{S}}=\left\{\boldsymbol{e}\in E\right\}\times\left\{\boldsymbol{z}\in Z\right\}\times\left\{\boldsymbol{\omega}\in W\right\}\times\left\{\overline{\boldsymbol{y}}_{\boldsymbol{R}}\in Y_R\right\}\times\left\{\beta\in R_1\right\}$$

Choose $M_\beta$ and $M_v$ to be constants that are larger than or equal to the upper bounds of $\beta_m(\boldsymbol{e},\boldsymbol{z},\overline{\boldsymbol{y}}_{\mathrm{r}})$ and $\left|v\left(\boldsymbol{e},\boldsymbol{z},\overline{\boldsymbol{y}}_{\mathrm{r}},\beta\right)\right|$ over $\boldsymbol{R}_{\mathrm{S}}$ respectively. Denote

$$\beta_m^s(\hat{e},\boldsymbol{z},\overline{\boldsymbol{y}}_{\mathrm{r}})=M_\beta\mathrm{sat}\left(\frac{\beta_m(\hat{e},\boldsymbol{z},\overline{\boldsymbol{y}}_{\mathrm{r}})}{M_\beta}\right) \tag{38}$$

$$v^s\left(\hat{e},\boldsymbol{z},\overline{\boldsymbol{y}}_{\mathrm{r}},\beta\right)=M_v\mathrm{sat}\left(\frac{v\left(\hat{e},\boldsymbol{z},\overline{\boldsymbol{y}}_{\mathrm{r}},\beta\right)}{M_v}\right) \tag{39}$$

where sat(·) represents the saturation function [1]. The robust adaptive output controller can be obtained by replacing $\beta_m\left(e,\boldsymbol{z},\overline{\boldsymbol{y}}_{\mathrm{r}}\right)$ and $v\left(e,\boldsymbol{z},\overline{\boldsymbol{y}}_{\mathrm{r}},\beta\right)$ in Eqs. (25) and (23) with $\beta_m^s\left(\hat{e},\boldsymbol{z},\overline{\boldsymbol{y}}_{\mathrm{r}}\right)$ and $v^s\left(\hat{e},\boldsymbol{z},\overline{\boldsymbol{y}}_{\mathrm{r}},\beta\right)$, respectively. We have the following results.

Under Assumptions 1 to 4, with the proposed adaptive output controller, for any $\boldsymbol{e}(0)\in E_0$, $\boldsymbol{z}(0)\in Z_0$, $\boldsymbol{\omega}(0)\in W_0$, $\beta(0)\in\mathbb{R}_1^0$, there exists $\varepsilon_0>0$ and for any $\varepsilon\in(0,\varepsilon_0)$, all the variables of the closed-loop systems Eqs. (18)–(20), (25) and (37) are bounded even in the presence of unmodeled dynamics and bounded disturbances. Furthermore, the mean-square tracking error is of order $O(\varepsilon)$ if the design parameters $\Gamma$, $\sigma$ and $\boldsymbol{Q}$ are chosen appropriately.

The proof of the above results, which is similar to that in Ref. [1], is omitted here due to limited space.

## 4 Simulation illustration

Consider a nonlinear system

$$y^{(3)}=\frac{\theta_0}{2+\sin\theta_1}(u+y-\ddot{y})+2(y\dot{y}+\dot{y}^2+y\ddot{y})\cos\theta_2+\theta_u\dot{u}+\Delta+d(t) \tag{40}$$

where unmodeled dynamics $\boldsymbol{\omega}$ satisfies

$$\dot{\boldsymbol{\omega}}=-\boldsymbol{\omega}+y\sin u \tag{41}$$

$$\Delta=\theta_3\boldsymbol{\omega}+\frac{\theta_4}{2+\sin\theta_5}y^2+0.1u\cos\theta_6$$

It can be seen that Eq. (41) is exp-ISpS and

$$V_\omega(\boldsymbol{\omega})=\boldsymbol{\omega}^2,\ \tilde{\gamma}\left(\|\boldsymbol{e}\|\right)=\|\boldsymbol{e}\|^3+\|\boldsymbol{e}\|^2+\|\boldsymbol{e}\|+1,\ \gamma_1\left(\boldsymbol{x}\right)=y^2.$$

We assume that

$$f\left(y,\dot{y},\ddot{y},u,\tilde{\boldsymbol{\theta}}\right)=\frac{\theta_0}{2+\sin\theta_1}(u+y-\ddot{y})+2\left(y\dot{y}+\dot{y}^2+y\ddot{y}\right)\cos\theta_2$$

is unknown but satisfies

$$|f(y,\dot{y},\ddot{y},u)|\leqslant\theta\bar{f}(y,\dot{y},\ddot{y},u)$$

where $\tilde{\boldsymbol{\theta}}$, $\theta_u$ and $\theta_i$ are unknown, whereas

$$\bar{f}(y,\dot{y},\ddot{y},u)=[(u+y-\ddot{y})^2+(y\dot{y}+\dot{y}^2+y\ddot{y})^2]^{\frac{1}{2}}$$

is known. The objective is to design a robust adaptive controller such that $y$ tracks $y_r(t)=0.5\sin t$.

Take $\boldsymbol{K}=[2\ 4\ 3]$, and $\boldsymbol{Q}=\boldsymbol{I}$. Solving $\boldsymbol{PA}_m+\boldsymbol{A}_m^{\mathrm{T}}\boldsymbol{P}=-\boldsymbol{I}$, we obtain $\boldsymbol{P}$. In the simulation, we take $\theta_0=\theta_1=\theta_2=\theta_u=1$. $\theta_3=\theta_4=\theta_5=\theta_6=1$, $d(t)=0.5\sin t$. The initial conditions and design parameters are as follows:

$$\begin{aligned}&\boldsymbol{e}(0)=[1,0,0]^{\mathrm{T}},\ \boldsymbol{z}(0)=\boldsymbol{\theta}(0)=0,\\&\beta(0)=20,\ \Gamma=10,\ \sigma=0.00001\end{aligned} \tag{42}$$

By analysis as in Ref. [1] or by simulations of adaptive state feedback control, we can estimate $M_v=10$, $M_\beta=5$. Take $\varepsilon=0.001$. The simulation results of the adaptive output feedback control are given in Fig. 1(a). Choosing $\Gamma=10^6$, $\sigma=10^{-8}$ and the rest of the initial conditions and design parameters to be the same as in Eq. (42), the simulation results are shown in Fig. 1(b), which demonstrates that the mean-square tracking error can be made arbitrarily small by choosing the design parameters $\Gamma$ and $\sigma$ appropriately.

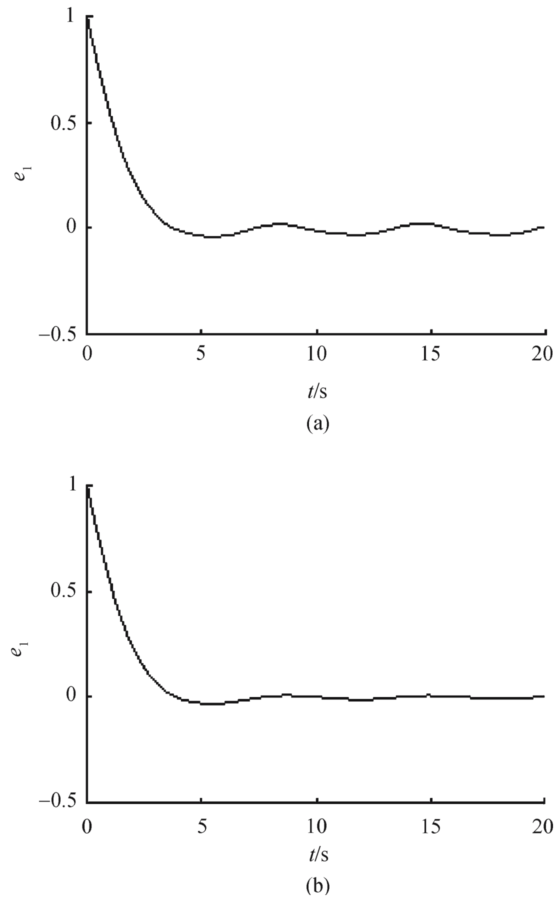


**Fig. 1** Simulation results of the adaptive output feedback controller. (a) Tracking error $e_1$; (b) Tracking error $e_1$ by choosing the design parameters appropriately

## 5 Conclusions

A robust adaptive control scheme is proposed for the nonlinearly parameterized systems represented by input-output models. The scheme can be used in the case of systems with unknown parameters, uncertain nonlinearities, bounded disturbances and unmodeled dynamics. It guarantees the uniform boundedness of all the signals in the closed-loop system. The mean-square tracking error can be made arbitrarily small by choosing some design parameters appropriately. The proposed control scheme does not need to estimate the unknown parameters nor add a dynamical signal to dominate the effects of unmodeled dynamics. No matter how high the order of the system is and how many unknown parameters the system has, there is only one adaptive parameter. Simulation results illustrate the effectiveness of the control scheme.

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