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RESEARCH ARTICLE

CHENG Yiping

CARMA-model-based *j*-step-ahead prediction for MIMO systems

Abstract The single-input single-output (SISO) j-step-ahead predictor for generalized predictive control (GPC) controllers was traditionally derived using the polynomial approach through the Diophantine equations. An equivalent version of the predictor in a state-space form is available in the literature. In this paper, a z-domain analysis of the multiple input multiple output (MIMO) extension of the state-space predictor is carried out, and then an MIMO j-step-ahead predictor in polynomial form based on the controlled autoregressive moving average model is derived. The predictor enables us to simplify the GPC algorithm design for multivariable systems. In the SISO case the predictor is just the traditional GPC predictor, therefore this paper gives rigorous proof of the equivalence between the traditional GPC predictor and the state-space predictor.

Keywords generalized predictive control, multivariable control, controlled autoregressive moving average model

1 Introduction

The earliest predictive control schemes called the dynamic matrix control (DMC) and the model algorithmic control (MAC) are based on the step and impulse response models of the plant. Therefore, they are only applicable to stable plants. In contrast, the generalized predictive control (GPC) proposed by Clarke et al. [1] uses a j-step-ahead predictor based on the transfer function model of the plant, and is thus applicable to unstable plants as well. Therefore the j-step-ahead predictor is the cornerstone of the GPC.

The GPC j-step-ahead predictor is traditionally designed for the SISO systems, and uses an input-output formulation (also called the polynomial formulation or the transfer function formulation). Some authors have proposed MIMO extensions and/or state-space formulations of the GPC [2–6]. However, among these works, Refs. [3–6] used predictors that are not equivalent to the traditional GPC predictor in the single-input single-output (SISO) case, and the only equivalent state-space formulation given in Ref. [2] is still for the SISO case and came without rigorous proof of its equivalence to the traditional GPC j-step-ahead predictor.

In this paper, we will generalize the SISO state-space j-step-ahead predictor proposed in Ref. [2] to the multiple-input multiple-output (MIMO) case, and then based on the z-domain analysis of this predictor, we will obtain a controlled autoregressive moving average (CARMA)-model-based MIMO j-step-ahead predictor in polynomial form, which leads to a simplified GPC algorithm design for MIMO systems. In the SISO case, the predictor is completely equivalent to the traditional GPC predictor. Finally, we will give a rigorous proof of the equivalence between the traditional GPC predictor and the state-space predictor.

2 The model of the plant

In this paper we adopt the CARMA (also referred to as ARMAX) model as the input-output model of the plant. This model has the following form

$$\boldsymbol{A}[z^{-1}]\boldsymbol{y}(k) = \boldsymbol{B}[z^{-1}]\boldsymbol{u}(k) + \boldsymbol{C}[z^{-1}]\boldsymbol{v}(k) \tag{1}$$

where $\boldsymbol{y}(k)$ is the $q \times 1$ output sequence, $\boldsymbol{u}(k)$ is the $p \times 1$ control input sequence, $\boldsymbol{v}(k)$ is the $q \times 1$ disturbance input sequence, which is assumed to be a discrete-time white noise with zero expectation. z, z^{-1} are the forward-shift and the backward-shift operators, respectively. $\boldsymbol{A}[z^{-1}]$, $\boldsymbol{B}[z^{-1}]$, $\boldsymbol{C}[z^{-1}]$ are $q \times q$, $q \times p$, $q \times q$ polynomial matrices in z^{-1}. They can be written as

$$\boldsymbol{A}[z^{-1}] = \boldsymbol{I} + \boldsymbol{A}_1 z^{-1} + \boldsymbol{A}_2 z^{-1} + \dots + \boldsymbol{A}_n z^{-n} \tag{2}$$

$$\boldsymbol{B}[z^{-1}] = \boldsymbol{B}_1 z^{-1} + \boldsymbol{B}_2 z^{-1} + \dots + \boldsymbol{B}_n z^{-n} \tag{3}$$

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CHENG Yiping (✉)
School of Electronic and Information Engineering, Beijing Jiaotong University, Beijing 100044, China
E-mail: ypcheng@bjtu.edu.cn

$$C[z^{-1}] = I + C_1 z^{-1} + C_2 z^{-1} + \ldots + C_n z^{-n} \tag{4}$$

where I is the identity matrix. Note that $B_0 = 0$, because we assume the system to be strictly proper. When $q = p = 1$, Eq. (1) reduces to the SISO CARMA model.

We remark that the original GPC Ref. [1] used the CARIMA model which is of the form

$$A[z^{-1}]y(k) = B[z^{-1}]u(k) + C[z^{-1}]\Delta^{-1}v(k) \tag{5}$$

where $\Delta = 1 - z^{-1}$. So Δ^{-1} is the discrete integration operator and $\Delta^{-1}v(k)$ is the white noise integrated. However, we may operate both sides of Eq. (5) with Δ to turn it into the following CARMA form

$$(1 - z^{-1})A[z^{-1}]y(k) = (1 - z^{-1})B[z^{-1}]u(k) + C[z^{-1}]v(k) \tag{6}$$

Therefore, Eq. (5) is a special case of Eq. (1). In the sequel only the CARMA model will be used.

The CARMA model Eq. (1) is an operator description, and we can write down its transfer function description as

$$Y(z) = G(z)U(z) + T(z)V(z) \tag{7}$$

where $Y(z)$, $U(z)$, $V(z)$ are the z-transforms of $y(k)$, $u(k)$, $v(k)$, respectively.

$$G(z) = A^{-1}[z^{-1}]B[z^{-1}] \tag{8}$$

$$T(z) = A^{-1}[z^{-1}]C[z^{-1}] \tag{9}$$

Note that in Eqs. (8) and (9), $A[z^{-1}]$, $B[z^{-1}]$, and $C[z^{-1}]$ are viewed as functions of z.

The CARMA model, as an input-output description of the plant, has its state-space realization

$$x(k+1) = \Phi x(k) + \Gamma u(k) + \Lambda v(k) \tag{10}$$

$$y(k) = Hx(k) + v(k) \tag{11}$$

where Φ, Γ, Λ, H are $m \times m$, $m \times p$, $m \times q$, $q \times m$ matrices. And they verify

$$H(zI - \Phi)^{-1}\Gamma = G(z) = A^{-1}[z^{-1}]B[z^{-1}] \tag{12}$$

$$I + H(zI - \Phi)^{-1}\Lambda = T(z) = A^{-1}[z^{-1}]C[z^{-1}] \tag{13}$$

3 Derivation of the MIMO state-space *j*-step-ahead predictor

In the SISO case, Ordys and Clarke in Ref. [2] gave a state-space j-step-ahead predictor equivalent to the traditional GPC j-step-ahead predictor, but concerning the equivalence only an informal explanation is given. In this section we will derive an MIMO state-space j-step-ahead predictor based on Ref. [2]. A rigorous proof of the equivalence result will be given in the next section.

The core of the predictor is the following state-observer

$$\xi(k+1) = (\Phi - \Lambda H)\xi(k) + \Gamma u(k) + \Lambda y(k) \tag{14}$$

It follows from Eqs. (10), (11) and (14) that

$$\xi(k+1) - x(k+1) = (\Phi - \Lambda H)[\xi(k) - x(k)] \tag{15}$$

This implies that if the eigenvalues of $\Phi - \Lambda H$ are all inside the unit disk, then Eq. (14) constitutes an asymptotic state observer of the plant. The prediction of $x(k+1)$ at time k can be obtained by extrapolating Eq. (14).

$$\hat{x}(k+1 \mid k) = (\Phi - \Lambda H)\xi(k) + \Gamma u(k) + \Lambda y(k) \tag{16}$$

Then, we predict $x(k+j)$ based on Eq. (10), using the principle that the optimal prediction of a future disturbance is its expectation 0.

$$\hat{x}(k+j \mid k) = \Phi^{j-1}\hat{x}(k+1 \mid k) + \sum_{i=1}^{j-1}\Phi^{i-1}\Gamma u(k+j-i) \tag{17}$$

From Eqs. (11), (16) and (17) we obtain the prediction of $y(k+j)$ at time k.

$$\hat{y}(k+j \mid k) = H\Phi^{j-1}(\Phi - \Lambda H)\xi(k) + H\Phi^{j-1}\Lambda y(k) + \sum_{i=1}^{j} H\Phi^{i-1}\Gamma u(k+j-i) \tag{18}$$

Equation (18) is just the j-step-ahead predictor in the state-space form. Since the state observer is the only part of the predictor that requires explicit online computation, it needs to be stable to ensure the stability of the entire control system. This is an important problem that is not discussed in Ref. [2].

Theorem 1 If $C[z^{-1}]$ (as a function of z) has all its zeros inside the unit disk, then we can always find a realization (Φ, Γ, Λ, H) of Eq. (1) such that all the eigenvalues of $\Phi - \Lambda H$ are inside the unit disk.

Proof Let

$$\Phi = \begin{bmatrix} -A_1 & I & & \\ \vdots & & \ddots & \\ -A_{n-1} & & & I \\ -A_n & 0 & \cdots & 0 \end{bmatrix}, \Gamma = \begin{bmatrix} B_1 \\ \vdots \\ B_{n-1} \\ B_n \end{bmatrix}, \Lambda = \begin{bmatrix} C_1 - A_1 \\ \vdots \\ C_{n-1} - A_{n-1} \\ C_n - A_n \end{bmatrix}$$

$$H = [I \quad 0 \quad \cdots \quad 0].$$

Then it is not difficult to verify that (Φ, Γ, Λ, H) is a realization of Eq. (1). Furthermore, we have

$$\Phi - \Lambda H = \begin{bmatrix} -C_1 & I & & \\ \vdots & & \ddots & \\ -C_{n-1} & & & I \\ -C_n & 0 & \cdots & 0 \end{bmatrix}$$

It can be proved (the detailed proof is omitted due to limitation in space) that the eigenvalues of $\boldsymbol{\Phi}-\boldsymbol{\Lambda H}$ are the zeros of $\boldsymbol{C}[z^{-1}]$ (as a function of z) and 0. Therefore, according to the hypothesis we have, all the eigenvalues of $\boldsymbol{\Phi}-\boldsymbol{\Lambda H}$ are inside the unit disk.

In the sequel it will be assumed that $\boldsymbol{C}[z^{-1}]$ (as a function of z) has all its zeros inside the unit disk. From Theorem 1 and Eq. (15) we also find that the convergence rate of the state observation error depends on the positions of the zeros.

4 The unified MIMO CARMA model *j*-step-ahead predictor

In this section we will obtain a unified j-step-ahead predictor for the MIMO CARMA model, based on the z-domain analysis of the state-space predictor we just derived. As a by-product, we will show the equivalence between the state-space and the traditional GPC predictors.

The Matrix Inversion Lemma of linear algebra (see Ref. 7 [Appendix B.2]) is instrumental in the derivation in this section.

Lemma 1 (Matrix Inversion Lemma) If $\boldsymbol{A}$, $\boldsymbol{I}+\boldsymbol{DA}^{-1}\boldsymbol{B}$ are nonsingular, then

$$(\boldsymbol{A}+\boldsymbol{BD})^{-1}=\boldsymbol{A}^{-1}-\boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{I}+\boldsymbol{DA}^{-1}\boldsymbol{B})^{-1}\boldsymbol{DA}^{-1} \tag{19}$$

We first use this lemma to show an interesting and deep result

Theorem 2 If Eqs. (12) and (13) hold, then

$$\boldsymbol{H}(z\boldsymbol{I}-\boldsymbol{\Phi}+\boldsymbol{\Lambda H})^{-1}\boldsymbol{\Gamma}=\boldsymbol{C}^{-1}(z^{-1})\boldsymbol{B}(z^{-1}) \tag{20}$$

$$\boldsymbol{H}(z\boldsymbol{I}-\boldsymbol{\Phi}+\boldsymbol{\Lambda H})^{-1}\boldsymbol{\Lambda}=\boldsymbol{I}-\boldsymbol{C}^{-1}(z^{-1})\boldsymbol{A}(z^{-1}) \tag{21}$$

Proof The proofs of Eqs. (20) and (21) are similar, so here, we only give the proof of Eq. (20). Using the Matrix Inversion Lemma, we have

$$(z\boldsymbol{I}-\boldsymbol{\Phi}+\boldsymbol{\Lambda H})^{-1}=(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}-(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\boldsymbol{\Lambda}[\boldsymbol{I}+\boldsymbol{H}(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\boldsymbol{\Lambda}]^{-1}\boldsymbol{H}(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1},$$

therefore,

$$\begin{aligned}\boldsymbol{H}(z\boldsymbol{I}-\boldsymbol{\Phi}+\boldsymbol{\Lambda H})^{-1}\boldsymbol{\Gamma}&=\boldsymbol{H}(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\boldsymbol{\Gamma}\\&\quad-\boldsymbol{H}(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\boldsymbol{\Lambda}[\boldsymbol{I}+\boldsymbol{H}(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\boldsymbol{\Lambda}]^{-1}\boldsymbol{H}(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\boldsymbol{\Gamma}\\&=\boldsymbol{G}(z)-[\boldsymbol{T}(z)-\boldsymbol{I}]\boldsymbol{T}^{-1}(z)\boldsymbol{G}(z)\\&=\boldsymbol{T}^{-1}(z)\boldsymbol{G}(z)=\boldsymbol{C}^{-1}(z^{-1})\boldsymbol{B}(z^{-1}).\end{aligned}$$

Now, denote the z-transforms of $\xi(k)$, $\hat{y}(k+j|k)$ by $\boldsymbol{\Xi}(z)$, $\hat{\boldsymbol{Y}}(j;z)$, respectively. Then it follows from Eq. (14) that

$$\boldsymbol{\Xi}(z)=(z\boldsymbol{I}-\boldsymbol{\Phi}+\boldsymbol{\Lambda H})^{-1}[\boldsymbol{\Gamma U}(z)+\boldsymbol{\Lambda Y}(z)] \tag{22}$$

From Eq. (18) we obtain

$$\begin{aligned}\hat{\boldsymbol{Y}}(j;z)&=z\boldsymbol{H\Phi}^{j-1}(z\boldsymbol{I}-\boldsymbol{\Phi}+\boldsymbol{\Lambda H})^{-1}\boldsymbol{\Lambda Y}(z)\\&\quad+[\boldsymbol{H\Phi}^{j-1}(\boldsymbol{\Phi}-\boldsymbol{\Lambda H})(z\boldsymbol{I}-\boldsymbol{\Phi}+\boldsymbol{\Lambda H})^{-1}\boldsymbol{\Gamma}\\&\quad+\sum_{i=1}^{j}z^{j-i}\boldsymbol{H\Phi}^{i-1}\boldsymbol{\Gamma}]\boldsymbol{U}(z)\end{aligned} \tag{23}$$

Let $\boldsymbol{G}_i$, $\boldsymbol{T}_i$ be the Markov parameters of $\boldsymbol{G}(z)$, $\boldsymbol{T}(z)$, respectively, i.e., $\boldsymbol{G}(z)=\sum_{i=1}^{\infty}\boldsymbol{G}_i z^{-i}$, $\boldsymbol{T}(z)=\boldsymbol{I}+\sum_{i=1}^{\infty}\boldsymbol{T}_i z^{-i}$. This implies that for all i

$$\boldsymbol{G}_i=\boldsymbol{H\Phi}^{i-1}\boldsymbol{\Gamma} \tag{24}$$

$$\boldsymbol{T}_i=\boldsymbol{H\Phi}^{i-1}\boldsymbol{\Lambda} \tag{25}$$

We have the following lemmas:

Lemma 2 If Eqs. (12), (13), (24) and (25) hold, then

$$\boldsymbol{H\Phi}^{j}(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\boldsymbol{\Gamma}=z^{j}[\boldsymbol{G}(z)-\sum_{i=1}^{j}\boldsymbol{G}_i z^{-i}] \tag{26}$$

$$\boldsymbol{H\Phi}^{j}(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\boldsymbol{\Lambda}=z^{j}[\boldsymbol{T}(z)-\boldsymbol{I}-\sum_{i=1}^{j}\boldsymbol{T}_i z^{-i}] \tag{27}$$

Proof The proofs of Eqs. (26) and (27) are almost identical, so we only need to give the proof of Eq. (26).

$$\begin{aligned}\boldsymbol{H\Phi}^{j}(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\boldsymbol{\Gamma}&=\sum_{i=1}^{\infty}\boldsymbol{H\Phi}^{i+j-1}\boldsymbol{\Gamma}z^{-i}\\&=z^{j}\sum_{i=1}^{\infty}\boldsymbol{H\Phi}^{i+j-1}\boldsymbol{\Gamma}z^{-i-j}\\&=z^{j}\sum_{i=j+1}^{\infty}\boldsymbol{H\Phi}^{i-1}\boldsymbol{\Gamma}z^{-i}=z^{j}[\boldsymbol{G}(z)-\sum_{i=1}^{j}\boldsymbol{G}_i z^{-i}].\end{aligned}$$

Lemma 3 If Eqs. (12), (13), (24) and (25) hold and $j\geqslant 1$, then

$$\begin{aligned}&z\boldsymbol{H\Phi}^{j-1}(z\boldsymbol{I}-\boldsymbol{\Phi}+\boldsymbol{\Lambda H})^{-1}\boldsymbol{\Lambda}\\&\quad=z^{j}[\boldsymbol{T}(z)-\boldsymbol{I}-\sum_{i=1}^{j-1}\boldsymbol{T}_i z^{-i}]\boldsymbol{T}^{-1}(z)\end{aligned} \tag{28}$$

Proof Using the Matrix Inversion Lemma,

$$(z\boldsymbol{I}-\boldsymbol{\Phi}+\boldsymbol{\Lambda H})^{-1}=(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}-(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\boldsymbol{\Lambda}[\boldsymbol{I}+\boldsymbol{H}(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\boldsymbol{\Lambda}]^{-1}\boldsymbol{H}(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}.$$

Therefore,

$$\begin{aligned}z\boldsymbol{H\Phi}^{j-1}(z\boldsymbol{I}-\boldsymbol{\Phi}+\boldsymbol{\Lambda H})^{-1}\boldsymbol{\Lambda}&=z\boldsymbol{H\Phi}^{j-1}(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\boldsymbol{\Lambda}\\&\quad-z\boldsymbol{H\Phi}^{j-1}(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\boldsymbol{\Lambda}[\boldsymbol{I}+\boldsymbol{H}(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\boldsymbol{\Lambda}]^{-1}\boldsymbol{H}(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\boldsymbol{\Lambda}\\&=z\boldsymbol{H\Phi}^{j-1}(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\boldsymbol{\Lambda}[\boldsymbol{I}+\boldsymbol{H}(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\boldsymbol{\Lambda}]^{-1}\\&=z^{j}[\boldsymbol{T}(z)-\boldsymbol{I}-\sum_{i=1}^{j-1}\boldsymbol{T}_i z^{-i}]\boldsymbol{T}^{-1}(z).\end{aligned}$$

Lemma 4 If Eqs. (12), (13), (24) and (25) hold and $j \geqslant 1$, then

$$\boldsymbol{H\Phi}^{j-1}(\boldsymbol{\Phi}-\boldsymbol{\Lambda H})(z\boldsymbol{I}-\boldsymbol{\Phi}+\boldsymbol{\Lambda H})^{-1}\boldsymbol{\Gamma} + \sum_{i=1}^{j} z^{j-i}\boldsymbol{H\Phi}^{i-1}\boldsymbol{\Gamma} = z^j(\boldsymbol{I}+\sum_{i=1}^{j-1}\boldsymbol{T}_i z^{-i})\boldsymbol{T}^{-1}(z)\boldsymbol{G}(z) \quad (29)$$

Proof The first term on the left side of Eq. (29) consists of two terms: $\boldsymbol{H\Phi}^j(z\boldsymbol{I}-\boldsymbol{\Phi}+\boldsymbol{\Lambda H})^{-1}\boldsymbol{\Gamma}$ and $-\boldsymbol{H\Phi}^{j-1}\boldsymbol{\Lambda H}(z\boldsymbol{I}-\boldsymbol{\Phi}+\boldsymbol{\Lambda H})^{-1}\boldsymbol{\Gamma}$. By Matrix Inversion Lemma,

$$\begin{aligned}&\boldsymbol{H\Phi}^j(z\boldsymbol{I}-\boldsymbol{\Phi}+\boldsymbol{\Lambda H})^{-1}\boldsymbol{\Gamma} = \boldsymbol{H\Phi}^j(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\boldsymbol{\Gamma} \\ &- \boldsymbol{H\Phi}^j(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\boldsymbol{\Lambda}[\boldsymbol{I}+\boldsymbol{H}(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\boldsymbol{\Lambda}]^{-1}\boldsymbol{H}(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\boldsymbol{\Gamma} \\ &= z^j[\boldsymbol{G}(z)-\sum_{i=1}^{j}\boldsymbol{G}_i z^{-i}] - z^j[\boldsymbol{T}(z)-\boldsymbol{I}-\sum_{i=1}^{j}\boldsymbol{T}_i z^{-i}]\boldsymbol{T}^{-1}(z)\boldsymbol{G}(z).\end{aligned}$$

By Eq. (25) and Theorem 2, we have $-\boldsymbol{H\Phi}^{j-1}\boldsymbol{\Lambda H}(z\boldsymbol{I}-\boldsymbol{\Phi}+\boldsymbol{\Lambda H})^{-1}\boldsymbol{\Gamma} = -\boldsymbol{T}_j\boldsymbol{T}^{-1}(z)\boldsymbol{G}(z)$. By Eq. (24), the second term on the left side of Eq. (29) $\sum_{i=1}^{j} z^{j-i}\boldsymbol{H\Phi}^{i-1}\boldsymbol{\Gamma} = z^j\sum_{i=1}^{j}\boldsymbol{G}_i z^{-i}$. Hence, the left side of Eq. (29) equals

$$\begin{aligned}&z^j\left[\boldsymbol{G}(z)-\sum_{i=1}^{j}\boldsymbol{G}_i z^{-i}\right] + z^j\left[\boldsymbol{T}(z)-\boldsymbol{I}-\sum_{i=1}^{j}\boldsymbol{T}_i z^{-i}\right]\boldsymbol{T}^{-1}(z)\boldsymbol{G}(z) \\ &-\boldsymbol{T}_j\boldsymbol{T}^{-1}(z)\boldsymbol{G}(z)+z^j\sum_{i=1}^{j}\boldsymbol{G}_i z^{-i} \\ &= z^j\boldsymbol{G}(z) - z^j\left[\boldsymbol{T}(z)-\boldsymbol{I}-\sum_{i=1}^{j-1}\boldsymbol{T}_i z^{-i}\right]\boldsymbol{T}^{-1}(z)\boldsymbol{G}(z) \\ &= z^j\left(\boldsymbol{I}+\sum_{i=1}^{j-1}\boldsymbol{T}_i z^{-i}\right)\boldsymbol{T}^{-1}(z)\boldsymbol{G}(z).\end{aligned}$$

To simplify notation, for each $j \geqslant 1$, we define the following polynomial in z^{-1}

$$\boldsymbol{E}_j[z^{-1}] = \boldsymbol{I}+\sum_{i=1}^{j-1}\boldsymbol{T}_i z^{-i} \quad (30)$$

Combining Lemmas 3 and 4, we arrive at the following z-domain characterization of the state-space j-step-ahead predictor

Theorem 3 If Eqs. (12), (13), (23) (25) and (30) hold, then

$$\hat{\boldsymbol{Y}}(j;z) = z^j[\boldsymbol{T}(z)-\boldsymbol{E}_j[z^{-1}]]\boldsymbol{T}^{-1}(z)\boldsymbol{Y}(z) + z^j\boldsymbol{E}_j[z^{-1}]\boldsymbol{T}^{-1}(z)\boldsymbol{G}(z)\boldsymbol{U}(z) \quad (31)$$

In fact, Theorem 3 gives a unified MIMO j-step-ahead predictor using the frequency (input-output) domain language. Although in this section this predictor is derived via the state-space predictor, it can also be derived using the pure input-output approach, which is omitted here due to the space limitation. To facilitate the use of this predictor, let us now express the right side of Eq. (31) in terms of $\boldsymbol{A}[z^{-1}]$, $\boldsymbol{B}[z^{-1}]$, $\boldsymbol{C}[z^{-1}]$

$$\hat{\boldsymbol{Y}}(j;z) = z^j(\boldsymbol{I}-\boldsymbol{E}_j[z^{-1}]\boldsymbol{C}^{-1}[z^{-1}]\boldsymbol{A}[z^{-1}])\boldsymbol{Y}(z) + z^j\boldsymbol{E}_j[z^{-1}]\boldsymbol{C}^{-1}(z^{-1})\boldsymbol{B}(z^{-1})\boldsymbol{U}(z) \quad (32)$$

The polynomial-operator form of Eq. (32) is

$$\hat{\boldsymbol{y}}(k+j\,|\,k) = z^j(\boldsymbol{I}-\boldsymbol{E}_j[z^{-1}]\boldsymbol{C}^{-1}[z^{-1}]\boldsymbol{A}[z^{-1}])\boldsymbol{y}(k) + \boldsymbol{E}_j[z^{-1}]\boldsymbol{C}^{-1}[z^{-1}]\boldsymbol{B}[z^{-1}]\boldsymbol{u}(k+j) \quad (33)$$

Likewise, we can express $\boldsymbol{E}_j[z^{-1}]$ in terms of $\boldsymbol{A}[z^{-1}]$, $\boldsymbol{C}[z^{-1}]$. Note that since $\boldsymbol{E}_j[z^{-1}]$ is the partial sum of the power-series expansion of $\boldsymbol{T}(z)$, to compute it we need only to compute each $\boldsymbol{T}_i$. From $\boldsymbol{T}(z) = \boldsymbol{A}^{-1}[z^{-1}]\boldsymbol{C}[z^{-1}]$ it follows that for $i \geqslant 1$

$$\boldsymbol{T}_i = \boldsymbol{C}_i - \sum_{s=0}^{i-1}\boldsymbol{A}_{i-s}\boldsymbol{T}_s \quad (34)$$

Thus, Eq. (33) and its auxiliary Eqs. (30) and (34) constitute a unified input-output j-step-ahead predictor for the CARMA model. This predictor is in explicit form, whereas the predictor given in Ref. [8, Lemma 7.4.3] for the same purpose is in implicit form, hence the predictor given here is more convenient to be used.

In the SISO case, Eq. (33) reduces to

$$\hat{\boldsymbol{y}}(k+j\,|\,k) = z^j\,\frac{\boldsymbol{C}[z^{-1}]-\boldsymbol{E}_j[z^{-1}]\boldsymbol{A}[z^{-1}]}{\boldsymbol{C}[z^{-1}]}\boldsymbol{y}(k) + \frac{\boldsymbol{E}_j[z^{-1}]\boldsymbol{B}[z^{-1}]}{\boldsymbol{C}[z^{-1}]}\boldsymbol{u}(k+j) \quad (35)$$

It is easy to see that Eq. (35) is exactly the traditional GPC j-step-ahead predictor derived using Diophantine equations, and we have thus completed a rigorous proof of the equivalence between the state-space, CARMA-model-based, and traditional j-step-ahead predictors.

5 Conclusions

Based on the state-space formulation given in Ref. [2] of the SISO GPC predictor, an MIMO extension is proposed. Then we carry out a frequency domain analysis of the predictor using the Matrix Inversion Lemma, and finally we obtain a complete CARMA-model-based j-step-ahead predictor. This predictor is important for the MIMO systems in that it allows us to express the prediction directly in terms of the CARMA model and thus helps to simplify the GPC algorithm design.

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