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RESEARCH ARTICLE

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Theory of constructing closed parametric curves based on manifolds

Abstract A parametric curve based on a manifold is designed for constructing an accurate closed curve. A circle was defined as the parametric space and a non-uniform B-splines defined on the unit circle were used as base functions. A method to construct knot vectors, control points and corresponding parameters were proposed. A method to determine the coordinates for any point on a curve was also proposed. Some non-uniform rational B-splines (NURBS) control techniques, such as curves with an embedded line, a sharp angle, and so on, were used to verify the proposed method's compatibility with NURBS. Some examples were used to compare the arithmetic with that of NURBS. The results show that the method is not only simple, feasible and reliable but also compatible with a CAD system using NURBS.

Keywords closed curve, differential manifold, base function, shape control

1 Introduction

Constructing a closed curve is often encountered in computer aided design. A closed curve theoretically should have no obvious start-point and end-point in 3D space and have no continuity constraints at the start and the end. When the control points of a closed parametric curve are modified, it should still be a closed curve. But normal modeling systems do not construct this kind of closed curve at the moment. There are currently two kinds of methods to deal with the closed curve: one uses two curves with continuity constraints at their ends to represent a closed curve, such as Pro/Engineer, Catia, and so on. Another uses one curve with continuity constraints at its start-point and end-point to represent a closed curve, for example, ImageWareSurfacer. The above-mentioned shows that the NURBS modeling method cannot really represent a curve with closed topology currently. To construct an accurate model of a closed curve, Goshtasby [1–3], Jackowski et al. [4], and so on, designed a closed curve by using a Gaussian function as the base function and iterating the base function value after a circulating parameter. The computing and designing process of this method is simpler compared with NURBS and gets a better curvature, but this method cannot accurately process local control and construct base graphical units, such as a circle. In recent years, people have gradually turned their attention to generating curves and surfaces based on a manifold for structuring curves and surfaces with any type of topology. Grimm and Hughes [5], Navau and Garcia [6, 7], Wang Qing [8], and so on, have researched on constructing and modeling a curve and surface based on manifolds. A set of curve construction theory have been put forward which can represent not only common curves but also closed curves. But all these methods use only basic principles and cannot meet the requirement for actual engineering application. Also, it is not compatible with the current NURBS method and cannot use the advanced calculation of NURBS, such as tangent vector and curvature calculation of points on curve, curve division, degree elevation, and so on. In the rest of this paper, we will present a method to solve the closed curve construction problem which develops the closed curve construction theory based on manifolds. The method can be applied to actual engineering practice. It is simple and completely compatible with the current NURBS modeling system.

2 Basic principle of closed parametric curve construction theory based on a manifold

Suppose that M is the control grid of a closed parametric curve, which is defined as a closed polygon in a 2D Euclidean space. N is a simple differential manifold, namely a unit circle. First of all, a map $f: M \to N$ is set as the parameter

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of the control grid, which defines the respective parameter $u_i = f(V_i)$ for every control point V_i. Then, the region of each control point is defined, namely a set of open covering $\{U_i | i = 1, 2, \cdots, n\}$ on N, and $u_i \in U_i$ for each i. Define base function $G_i : N \to R$ with C^r continuity for each control point on N whose support region $\mathrm{Supp}(G_i) \subset U_i$ and let $\{\mathrm{Supp}(G_i)\}$ be also a covering on N. A unit decomposition $\{g_i\}$ belonging to $\{U_i\}$ on N is defined as follows.

$$g_i(p) = \frac{w_i G_i(p)}{\sum_{i=1}^{n} w_i G_i(p)}, \quad p \in N \tag{1}$$

where w_i is a weighting factor. Finally, the equation is defined by combining g_i and V_i [8]:

$$C(p) = \sum_{i=1}^{n} V_i g_i(p), \quad p \in N \tag{2}$$

Based on the above constructing theory, a set of systemic methods are put forward in this paper that is completely compatible with the current NURBS modeling system and computation. These methods include techniques for choosing and constructing a basic function, calculating the 3D coordinates of the points on a curve, controlling the shape of the curve, and so on.

2.1 Choosing base function

The base function of closed parametric curve based on a manifold can be any kind of function, such as radial base function, a Gauss function, a B-spline base function, and so on. The B-spline function has many significant properties, such as local support (a degree k B-spline is only nonzero at k+1 knot intervals, namely each segment of degree k B-spline curve only involves k+1 base functions and is defined by k+1 control points), convex hall, and so on. Adopting non-uniform B-spline as base function can reflect the distributing characteristic of control points and get a perfect interpolated curve when interpolate points are non-uniformly distributed. So, non-uniform B-spline base function can meet the requirements for designing closed curve based on a manifold entirely. In order to be compatible with the current NURBS modeling method and be able to use the advanced arithmetic and processing techniques of NURBS, the non-uniform B-spline base function is defined as the base function of closed parametric curve.

2.2 Constructing base function on unit circle

Let p be a point on unit circle. The point p is defined as $p = \varphi(u) = (\cos(u), \sin(u))$, where u is the central angle of p. Then Eq. (2) can be written as:

$$C(\varphi(u)) = \sum_{i=1}^{n} V_i g_i(\varphi(u)), \quad u \in [0, 2\pi] \tag{3}$$

So the base function $G_i(\varphi^{-1}(p))$ on unit circle can be obtained by constructing the base function $G_i(u)$ on a line corresponding to the given closed control manifold vertex.

The method is as follows:

1) Setting up the knot vector for the closed parametric curve.

For the closed parametric curve based on a manifold, whose base function is a degree k B-spline and the number of control points is M ($M \geqslant 3$), it therefore requires M+k+1 knots to construct its base function. The parameter value corresponding to each control point is calculated by the following equation:

$$u_0 = 0$$
$$u_i = u_{i-1} + 2\pi \|V_{i-1}V_i\| / L, \quad i = 1, 2, \cdots, M-1$$
$$L = \sum_{i=1}^{M} \|V_{i-1}V_i\| + \|V_M V_0\|.$$

If k is uneven, then the knot vector of the curve is as follows:

$$\{U_{-i}, \cdots, U_{-1}, u_0, u_1, \cdots, u_{M-1}, U_{M+0}, U_{M+1}, \cdots, U_{M+j}\},$$
$$U_{-i} = -(2\pi - U_{M-i}), \quad i = 1, 2, \cdots, \frac{k+1}{2},$$
$$U_{M+j} = 2\pi + U_j, \quad j = 0, 1, \cdots, \frac{k+1}{2} - 1,$$

where u_i is the parameter value of the corresponding control point.

If k is even, the knot vector of the curve is as follows:

$$(U_{-i}, \cdots, U_{-1}, \frac{u_1}{2}, u_1 + \frac{u_2 - u_1}{2}, u_2 + \frac{u_3 - u_2}{2}, \cdots, u_{M-2} + \frac{u_{M-1} - u_{M-2}}{2}, u_{M-1} + \frac{2\pi - u_{M-1}}{2}, U_{M+0}, \cdots, U_{M+j}),$$
$$U_{-i} = -(2\pi - U_{M-i}), \quad i = 1, 2, \cdots, \frac{k}{2} + 1,$$
$$U_{M+j} = 2\pi + U_j, \quad j = 0, 1, \cdots, \frac{k}{2} - 1,$$

where u_i is the parameter value of corresponding control points.

2) According to the recursive definition of B-spline, the base function $G_{i,k}(u)$ is constructed for each control point beginning from the first knot of the knot vector. Then,

$$C(u) = \frac{\sum_{i=0}^{M-1} G_{i-k+1,k}(u) W_i V_i}{\sum_{i=0}^{M-1} G_{i-k+1,k}(u) W_i} \tag{4}$$

where $0 \leqslant u \leqslant 2\pi$, $M \geqslant 3$, $1 \leqslant k \leqslant (2M-3)$, and W_i is the weighting factor.

Figure 1 is an illustration of the B-spline base function

$G_i(u)$ on a line related to the given closed control manifold vertex. And the corresponding example for base functions $G_i(\varphi^{-1}(p))$ on a unit circle is illustrated in Fig. 2(a) (by polar coordinate). After obtaining the base functions on a unit circle, the closed curve (Fig. 2(b)) can be constructed by the base functions and corresponding control points as Eq. (4).

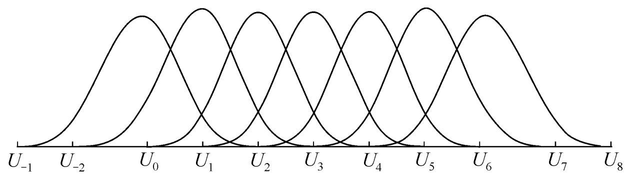
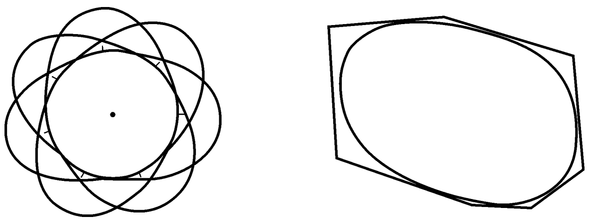


Fig. 1 Degree 3 B-spline base functions on line with 7 control points



(a) (b)

Fig. 2 B-spline base function on circle and closed curve based on a manifold (see text for details). (a) Base function on circle; (b) Closed curve

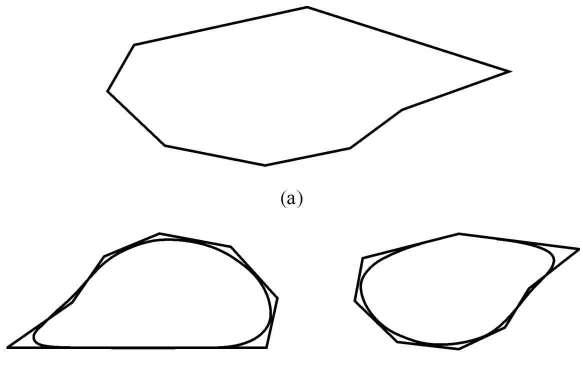
2.3 Calculating 3D coordinates of point on curve corresponding to the given parameter *u*

For a given parameter *u*, every B-spline base function corresponding to *u* is calculated and the 3D coordinates corresponding to *u* can be computed by Eq. (4). The method for calculating every B-spline base function corresponding to *u* is as follows:

The last knot of the segment *i*–*k*+1 degree *k* B-spline base function $G_{i-k+1,k}(u)$ is labeled as $U_{i-k+1+k+1} = U_{i+2}$. For $G_{i-k+1,k}(u)$, if $U_{i+2} > 2\pi$, the domain of the segment defined as *i*–*k*+1 degree *k* B-spline base function in the range [0, 2π], then the value of this segment B-spline base function corresponding with *u* will be calculated by replacing $G_{i-k+1,k}(u)$ with $G_{i-k+1,k}(u+2\pi)$ when $u < (U_{M+1} - 2\pi)$. If $U_{i-k+1} < 0$, the domain of the segment defined as *i*-*k*+1 degree *k* B-spline base function in the range $[0, 2\pi]$, then the value of this segment B-spline base function corresponding with *u* will be calculated by replacing $G_{i-k+1,k}(u)$ with $G_{i-k+1,k}(u-2\pi)$ when $U_{0-k+1} < (u-2\pi) < 0$. For other cases, the value of the B-spline base function $G_{i-k+1,k}(u)$ will be calculated corresponding with *u* directly.

3 The shape control techniques of a closed parametric curve based on a manifold

The shape control techniques, such as embedding a line in a curve, constructing a curve tangent to its control grid, forming a sharp corner on a curve while going through a control point, and so on, are of great value for product design. The shape control can greatly improve the efficiency and convenience of the design. Because the method of constructing a closed curve is compatible with current NURBS method, those advanced shape control techniques of NURBS may be used to control the shape of a closed parametric curve. Take a degree 3 closed parametric curve for example, the technique of four collinear control points is used to embed a line in closed curve, and the technique of three collinear control points or two concurrent control points is used to construct closed curve tangent to its control grid, and the technique of three concurrent control points is used to let closed curve go through the control point and form a sharp corner at the control point on closed curve. The examples applying these techniques are illustrated in Fig. 3. The results show that the method is compatible with NURBS, and is simple, feasible and reliable. The advanced control techniques and calculation of NURBS can be used to process a closed curve constructed by the method. It can be added to current popular CAD systems easily, avoiding much modification to the CAD systems and repetitive research work.



(a)

(b) (c)

Fig. 3 Techniques of curve control. (a) Curve is tangent to control grid; (b) Line is embedded in curve; (c) Curve with a sharp corner

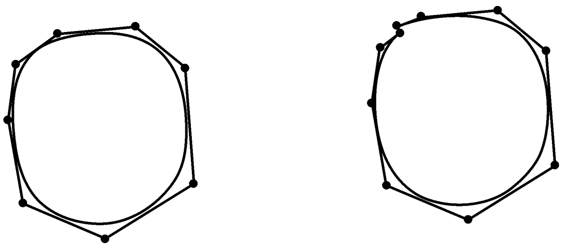
4 Comparing the closed curve based on a manifold with the closed curve defined by NURBS

As shown in Fig. 4 and Fig. 5, the method of constructing a closed curve as proposed holds these merits compared with the current NURBS as follows:

1) There are no obvious start-points and end-points on the constructed closed curve and it needs no disposal for

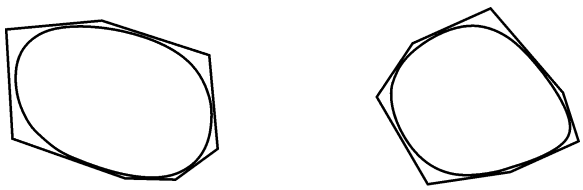
continuity on both ends of curve, the curve does not need to go through both ends of control grid.

2) The constructed closed curve can be modified by changing any control point.



(a) (b)

Fig. 4 Closed curve constructed by imageware surfacer. (a) Closed curve constructed by imageware surfacer; (2) Curve after the first control point is modified (open curve)



(a) (b)

Fig. 5 Closed curve constructed by the method of this paper. (a) Closed curve constructed by method of this paper (without obvious start and end point); (b) Curve after one control point is modified (closed curve)

5 Conclusions

This paper promotes the study on the theory of closed curve construction based on a manifold using the aspects of base function selection and construction, 3D coordinate calculation for points on curve, popular curve control techniques, and so on, which enables it to be applied to engineering practice and resolve the problem that current NURBS modeling method cannot really represent a closed curve. The method is compatible with NURBS modeling system. It make the most of the advanced computations and techniques of NURBS available (e.g., tangent vector and curvature calculation of a point on curve) and avoid much modification to CAD system and repetitive research work. More research can be done on interpolating and fitting of a closed curve based on this method in the future.

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