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RESEARCH ARTICLE

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Modular design of adaptive robust controller for strict-feedback stochastic nonlinear systems

Abstract A modular approach of the estimation-based design in adaptive linear control systems has been extended to the adaptive robust control of strict-feedback stochastic nonlinear systems with additive standard Wiener noises and constant unknown parameters. By using Itô's differentiation rule, nonlinear damping and adaptive Backstepping procedure, the input-to-state stable controller of global stabilization in probability is developed, which guarantees that system states are bounded and the system has a robust stabilization. According to Swapping technique, we develop two filters and convert dynamic parametric models into static ones to which the gradient update law is designed. Transient performance of the system is estimated by the norm of error. Results of simulation show the effectiveness of the control algorithms. The modular design, which has a concise hierarchy, is more flexible and versatile than a Lyapunov-based algorithm.

Keywords Wiener noises, Itô's differentiation rule, Stabilization in probability, Input-to-state stability, Swapping technique

1 Introduction

Control of nonlinear uncertain system is an exciting research area because of its widespread applications [1–5]. There are two popular adaptive robust control design approaches; one is a Lyapunov-based design, and the other is an estimation-based design. The former involves mainly the tuning function design. In Ref. [6], the over-parameterization problem was eliminated. The Lyapunov-based design has three distinguishing features. First, because a single Lyapunov function encompasses the complete states of the closed-loop system, the adaptive controller and the parameter update law can be obtained simultaneously. Second, the tuning function controller is somewhat complicated because it cancels the effect of parameter estimation variety in the error-system. And third, the choice of a parameter update law is limited to a Lyapunov-type algorithm. The latter exploits the certain equivalence property in the traditional adaptive linear control and achieves a modularity of the controller–identifier pair. Any stable controller can be combined with any identifier. The controller module is capable of stabilizing the plant when all the parameters are known. The identifier module, in turn, guarantees certain boundedness property that is independent of the controller module.

In this paper we extend the idea of the modular approach of the estimation-based design in the adaptive linear control to the adaptive robust control of strict-feedback stochastic nonlinear systems with additive standard Wiener noises and constant unknown parameters. A main obstacle to earlier attempts to apply the estimation-based design to nonlinear systems was the weakness of their certain equivalence controllers. Such controllers cannot achieve any controller–identifier separation without severely restricting the system nonlinearities, such as the linear growth constraints, matching conditions and so on. To overcome the weakness of certain equivalence controllers, we develop a new controller, known as input-to-state stable (ISS) controller [7-8], which has a strong parametric robustness property. In the presence of unknown parameters, this nonlinear controller achieves boundedness without adaptation. It guarantees boundedness not only in the presence of constant parameter errors, but also in the presence of time-varying parameter estimation. The ISS controller is suitable for the modular design of adaptive control scheme for nonlinear systems. It can be combined with any standard identifier. In this paper we extend the well-known Swapping technique to stochastic nonlinear systems. With this technique, we design two filters and convert dynamic parametric models into static ones to which the standard gradient update law is

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applicable.

2 System and problem description

2.1 System description

Consider the following strict-feedback stochastic nonlinear system

$$\begin{aligned} \mathrm{d}x_i &= x_{i+1}\mathrm{d}t + \boldsymbol{\varphi}_i^{\mathrm{T}}(\bar{\boldsymbol{x}}_i)\boldsymbol{\theta}\mathrm{d}t + \boldsymbol{\eta}_i^{\mathrm{T}}(\bar{\boldsymbol{x}}_i)\mathrm{d}\boldsymbol{w} \quad 1 \le i \le n-1 \\ \mathrm{d}x_n &= u\mathrm{d}t + \boldsymbol{\varphi}_n^{\mathrm{T}}(\bar{\boldsymbol{x}}_n)\boldsymbol{\theta}\mathrm{d}t + \boldsymbol{\eta}_n^{\mathrm{T}}(\bar{\boldsymbol{x}}_n)\mathrm{d}\boldsymbol{w} \end{aligned} \tag{1}$$

where x_i is the system state, $\bar{\boldsymbol{x}}_i = (x_1, x_2, \cdots, x_i)^{\mathrm{T}}$, $i = 1, 2, \cdots, n$; $u \in \mathrm{R}$ is the control input; $\boldsymbol{\theta} \in \mathrm{R}^p$ is an uncertain constant parameter vector, and $\varphi_i(\bar{\boldsymbol{x}}_i) \in \mathrm{R}^p$ is its gain function; $\boldsymbol{w} \in \mathrm{R}^r$ is an independent vector-valued standard Wiener process defined on probability space $(\boldsymbol{\Omega}, F, P)$, with $\boldsymbol{\Omega}$ being the sample space, F being a $\sigma-$algebra, and P being the probability measure. Thus $E\{\boldsymbol{w}(t)\} = 0$, $E\{\mathrm{d}\boldsymbol{w}\mathrm{d}\boldsymbol{w}^{\mathrm{T}}\} = \boldsymbol{I}\mathrm{d}t$, where $\boldsymbol{I}$ is a unit matrix, and $\eta_i(\bar{\boldsymbol{x}}_i) \in \mathrm{R}^r$ is the gain function of the random disturbance.

Throughout this paper, system Eq. (1) satisfies the following assumptions:

Assumption 1 The nonlinear functions $\varphi_i(\bar{\boldsymbol{x}}_i), \eta_i(\bar{\boldsymbol{x}}_i)$ are smooth and $\varphi_i(0) = 0$, $\eta_i(0) = 0$.

Assumption 2 θ lies in a compact convex set $\Theta \subset \mathrm{R}^p$, and its estimation is $\hat{\theta}(t)$. Then estimation error $\tilde{\theta}(t)$ is defined as $\tilde{\theta}(t) = \theta - \hat{\theta}(t)$, whose maximum is denoted by a constant M given by $M := \max_{\hat{\theta}\in\Theta}\left|\theta - \hat{\theta}(t)\right|$.

2.2 Stabilization in probability

To study the system Eq. (1), consider the following stochastic nonlinear system first

$$\mathrm{d}\boldsymbol{x} = f(\boldsymbol{x})\mathrm{d}t + \boldsymbol{g}_1(\boldsymbol{x})\mathrm{d}\boldsymbol{w} \tag{2}$$

where $\boldsymbol{x} \in \mathrm{R}^n$ is a state vector. Functions $f: \mathrm{R}^n \to \mathrm{R}^n$ and $g_1: \mathrm{R}^n \to \mathrm{R}^{n\times r}$ are smooth, and $f(0) = 0, \boldsymbol{g}_1(0) = 0$; $\boldsymbol{w}$ is defined as Eq. (1).

Definition 1 [3] The equilibrium $x = 0$ for Eq. (2) is globally stable in probability if for $\forall \varepsilon > 0$, there exists a class K function $\gamma(\cdot)$ such that

$$P\{|x(t)| < \gamma(|x_0|)\} \ge 1 - \varepsilon \qquad \forall t \ge 0, \forall x_0 \in \mathrm{R}^n \setminus \{0\}$$

Definition 2 [7] Consider the nonlinear system

$$\dot{x} = f(t, x, u) \tag{3}$$

where f is piecewise continuous in t and locally Lipschitz in x and u. The system is ISS if there exists a function β of class KL and a function γ of class K such that for any bounded input $u(\cdot)$ and each initial state $x(t_0)$ The solution exists for each $t \ge 0$ and satisfies

$$|x(t)| \le \beta(|x(t_0)|, t - t_0) + \gamma(\sup_{t_0 \le \tau \le t}|u(\tau)|) \quad 0 \le t_0 \le t$$

Lemma 1[3] *(Stochastic LaSalle-Yoshizawa Theorem) Consider system Eq. (2) and suppose there exists a C^2 function $\boldsymbol{V}(\boldsymbol{x}): \mathrm{R}^n \to \mathrm{R}_+$ and class K_∞ functions α_1 and α_2, such that for all $\boldsymbol{x} \in \mathrm{R}^n$, $t \ge 0$*

$$\alpha_1(|\boldsymbol{x}|) \le V(\boldsymbol{x}) \le \alpha_2(|\boldsymbol{x}|)$$

$$\boldsymbol{LV}(\boldsymbol{x}) := \frac{\partial V}{\partial \boldsymbol{x}}f(\boldsymbol{x}) + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{g}_1^{\mathrm{T}}(\boldsymbol{x})\frac{\partial^2 V}{\partial \boldsymbol{x}^2}\boldsymbol{g}_1(\boldsymbol{x})\right) \le -\boldsymbol{W}(\boldsymbol{x})$$

where $W: \mathrm{R}^n \to \mathrm{R}$ is continuous and nonnegative. Then, the equilibrium $\boldsymbol{x} = 0$ is globally stable in probability and

$$P\{\lim_{t\to\infty} W(\boldsymbol{x}(t)) = 0\} = 1, \qquad \forall \boldsymbol{x}_0 \in \mathrm{R}^n$$

Lemma 2[9] *Consider system Eq. (2), where $f(\boldsymbol{x}), \boldsymbol{g}_1(\boldsymbol{x})$ are locally Lipschitz functions, and suppose there exists a positive definite, radically unbounded and C^2 function $V(\boldsymbol{x}): \mathrm{R}^n \to \mathrm{R}_+$, such that for constants $c > 0$, and $c_0 \ge 0$*

$$\boldsymbol{LV}(\boldsymbol{x}) \le -cV(\boldsymbol{x}) + c_0$$

then system Eq. (2) is globally uniformly stable in probability. If $c_0 = 0$, the system Eq. (2) is globally asymptotically stable in probability.

Lemma 3 *Consider the following stochastic system*

$$\mathrm{d}\boldsymbol{x} = f(\boldsymbol{x})\mathrm{d}t + F(\boldsymbol{x})\boldsymbol{\theta}\mathrm{d}t + \boldsymbol{g}_1(\boldsymbol{x})\boldsymbol{u}\mathrm{d}t + \boldsymbol{g}_2(\boldsymbol{x})\mathrm{d}\boldsymbol{w} \tag{4}$$

Suppose that there exists a C^2 function $\boldsymbol{V}(\boldsymbol{x})$, class K_∞ functions γ_1, γ_2, ρ, and class K functions γ_3, γ_4, such that

$$\gamma_1(|\boldsymbol{x}|) \le \boldsymbol{V}(\boldsymbol{x}) \le \gamma_2(|\boldsymbol{x}|)$$

$$|\boldsymbol{x}| \ge \rho\left(\left\|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right\|\right)$$

$$\Downarrow$$

$$\begin{aligned} &\frac{\partial V}{\partial \boldsymbol{x}}\left[f(\boldsymbol{x}) + F(\boldsymbol{x})\hat{\boldsymbol{\theta}} + \boldsymbol{g}_1(\boldsymbol{x})u\right] + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{g}_2^{\mathrm{T}}\frac{\partial^2 V}{\partial \boldsymbol{x}^2}\boldsymbol{g}_2\right) \\ &+ \frac{\partial V}{\partial \boldsymbol{x}}F(\boldsymbol{x})\tilde{\boldsymbol{\theta}} + \frac{\partial V}{\partial \hat{\theta}}\dot{\hat{\theta}} \le -\gamma_3(|\boldsymbol{x}|) + \gamma_4\left(\left\|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right\|\right) \end{aligned} \tag{5}$$

Then the system Eq. (4) is ISS.

2.3 Problem discription

The problem is to design the ISS controller of system Eq. (1) with $\tilde{\boldsymbol{\theta}}, \dot{\hat{\boldsymbol{\theta}}}$ as the inputs, and then to design the identifier module to guarantee $\tilde{\boldsymbol{\theta}}, \dot{\hat{\boldsymbol{\theta}}} \in L_\infty[0, t_f)$.

3 ISS-controller design

Consider system Eq. (1) with additive standard Wiener noises and constant unknown parameters, by using Itô's differentiation rule, nonlinear Damping and adaptive Backstepping procedure, we design the ISS controller with respect to $(\tilde{\boldsymbol{\theta}},\dot{\hat{\boldsymbol{\theta}}})$.

Define the error variables and the Lyapunov function

$$z_i = \boldsymbol{x}_i - \boldsymbol{a}_{i-1}(\bar{\boldsymbol{x}}_{i-1},\hat{\boldsymbol{\theta}}), \qquad i=1,2,\cdots,n \tag{6}$$

$$V(z)=\frac{1}{4}\sum_{i=1}^{n} z_i^4 \tag{7}$$

where $\alpha_0=0$. From Itô's differentiation rule and system Eq. (1) and considering the Eq. (6), we have

$$\begin{aligned}
\mathrm{d}z_i &= \mathrm{d}(\boldsymbol{x}_i-\alpha_{i-1})\\
&=[\boldsymbol{z}_{i+1}+\alpha_i+\boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{\theta}-\sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\boldsymbol{x}_k}(\boldsymbol{x}_{k+1}+\boldsymbol{\varphi}_k^{\mathrm{T}}\boldsymbol{\theta})\\
&\quad-\frac{1}{2}\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial\boldsymbol{x}_p\partial\boldsymbol{x}_q}\boldsymbol{\eta}_p^{\mathrm{T}}\boldsymbol{\eta}_q-\frac{\partial\alpha_{i-1}}{\partial\hat{\boldsymbol{\theta}}}\dot{\hat{\boldsymbol{\theta}}}]\mathrm{d}t\\
&\quad+(\boldsymbol{\eta}_i^{\mathrm{T}}-\sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\boldsymbol{x}_l}\boldsymbol{\eta}_l^{\mathrm{T}})\mathrm{d}\boldsymbol{w}
\end{aligned} \tag{8}$$

where $\boldsymbol{x}_{n+1}=\boldsymbol{u}=\alpha_n(\bar{\boldsymbol{x}}_n,\hat{\boldsymbol{\theta}})$ and $z_{n+1}=0$. By Lemma 1 and Eqs. (6) and (8), along the solutions of Eq. (1), the infinitesimal generator $\boldsymbol{L}$ of V is

$$\begin{aligned}
\boldsymbol{L}V &\le z_n^3[\boldsymbol{u}-\sum_{k=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\boldsymbol{x}_k}\boldsymbol{x}_{k+1}-\frac{1}{2}\sum_{p,q=1}^{n-1}\frac{\partial^2\alpha_{n-1}}{\partial\boldsymbol{x}_p\partial\boldsymbol{x}_q}\boldsymbol{\eta}_p^{\mathrm{T}}\boldsymbol{\eta}_q+\boldsymbol{\omega}_n^{\mathrm{T}}\boldsymbol{\theta}\\
&\quad-\frac{\partial\alpha_{n-1}}{\partial\hat{\boldsymbol{\theta}}}\dot{\hat{\boldsymbol{\theta}}}+\frac{1}{4}z_n+\frac{3n}{4}\sum_{m=1}^{n}|\xi_{nm}|^4 z_n+\frac{3n}{4}z_n]\\
&\quad+z_1^3[\alpha_1+\boldsymbol{\varphi}_1^{\mathrm{T}}\boldsymbol{\theta}+\frac{3}{4}z_1+\frac{3}{4}|\xi_{11}|^4 z_1+\frac{3n}{8}(n+1)z_1]\\
&\quad+\sum_{i=2}^{n-1}z_i^3[\alpha_i-\sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_k}x_{k+1}-\frac{1}{2}\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\boldsymbol{\eta}_p^{\mathrm{T}}\boldsymbol{\eta}_q\\
&\quad+\boldsymbol{\omega}_i^{\mathrm{T}}\boldsymbol{\theta}-\frac{\partial\alpha_{i-1}}{\partial\hat{\boldsymbol{\theta}}}\dot{\hat{\boldsymbol{\theta}}}+\frac{3}{4}z_i+\frac{1}{4}z_i+\frac{3i}{4}\sum_{m=1}^{i}|\xi_{im}|^4 z_i\\
&\quad+\frac{3}{8}(n+i)(n+1-i)z_i]
\end{aligned} \tag{9}$$

where $\quad \boldsymbol{\omega}_i(\bar{\boldsymbol{x}}_i,\hat{\boldsymbol{\theta}})=\boldsymbol{\varphi}_i-\sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\boldsymbol{x}_k}\boldsymbol{\varphi}_k$

$$\bar{\boldsymbol{\varpi}}_i=\boldsymbol{\eta}_i-\sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\boldsymbol{x}_l}\boldsymbol{\eta}_l=\sum_{m=1}^{i}z_m\xi_{im} \tag{10}$$

$$\xi_{im}=\boldsymbol{\phi}_{im}-\sum_{l=m}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\boldsymbol{x}_l}\boldsymbol{\phi}_{lm}$$

The recursive design procedure is given as follows

$$\alpha_1=-c_1z_1-s_1z_1-\boldsymbol{\omega}_1^{\mathrm{T}}\hat{\boldsymbol{\theta}}-\frac{3}{4}z_1-\frac{3}{4}|\xi_{11}|^4 z_1-\frac{3n}{8}(n+1)z_1 \tag{11}$$

$$\begin{aligned}
\alpha_i &= -z_{i-1}-c_i z_i-s_i z_i-\boldsymbol{\omega}_i^{\mathrm{T}}\hat{\boldsymbol{\theta}}+\sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\boldsymbol{x}_k}\boldsymbol{x}_{k+1}\\
&\quad+\frac{1}{2}\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\boldsymbol{\eta}_p^{\mathrm{T}}\boldsymbol{\eta}_q-\frac{3}{4}z_i-\frac{1}{4}z_i\\
&\quad-\frac{3i}{4}\sum_{m=1}^{i}|\xi_{im}|^4 z_i-\frac{3}{8}(n+i)(n+1-i)z_i
\end{aligned} \tag{12}$$

$$\begin{aligned}
\boldsymbol{u} &= -z_{n-1}-c_n z_n-s_n z_n-\boldsymbol{\omega}_n^{\mathrm{T}}\hat{\boldsymbol{\theta}}+\sum_{k=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_k}\boldsymbol{x}_{k+1}\\
&\quad+\frac{1}{2}\sum_{p,q=1}^{n-1}\frac{\partial^2\alpha_{n-1}}{\partial x_p\partial x_q}\boldsymbol{\eta}_p^{\mathrm{T}}\boldsymbol{\eta}_q-\frac{1}{4}z_n-\frac{3n}{4}\sum_{m=1}^{n}|\xi_{nm}|^4 z_n-\frac{3n}{4}z_n
\end{aligned} \tag{13}$$

and the nonlinear damping functions s_i are given by

$$s_i(\bar{\boldsymbol{x}}_i,\hat{\boldsymbol{\theta}})=\kappa_i|\boldsymbol{\omega}_i|^2+\sigma_i\left|\frac{\partial\alpha_{i-1}}{\partial\hat{\boldsymbol{\theta}}}^{\mathrm{T}}\right|^2+d_i|\bar{\boldsymbol{\varpi}}_i|^4 \tag{14}$$

where $\kappa_i>0$, $\sigma_i>0$, $d_i>0$, $c_i>1/8\kappa_i+1/8\sigma_i$. Now substituting Eqs. (11)–(13) into Eq. (9), and using Young's in Eq. (9), we get

$$\begin{aligned}
\boldsymbol{L}V &\le -\sum_{i=1}^{n}c_i z_i^4+\sum_{i=1}^{n}\frac{1}{8\kappa_i}z_i^4+\sum_{i=1}^{n}\frac{1}{8\kappa_i}|\tilde{\boldsymbol{\theta}}|^4+\sum_{i=1}^{n}\frac{1}{8\sigma_i}z_i^4+\sum_{i=1}^{n}\frac{1}{8\sigma_i}|\dot{\hat{\boldsymbol{\theta}}}|^4\\
&\le -c\sum_{i=1}^{n}z_i^4+\sum_{i=1}^{n}\frac{1}{8\kappa_i}|\tilde{\boldsymbol{\theta}}|^4+\sum_{i=1}^{n}\frac{1}{8\sigma_i}|\dot{\hat{\boldsymbol{\theta}}}|^4+\frac{9}{16d}
\end{aligned} \tag{15}$$

where $c=\min_{1\le i\le n}(c_i-\frac{1}{8\kappa_i}-\frac{1}{8\sigma_i})$, $|z|_4=(\sum_i z_i^4)^{1/4}$, $\kappa=(\sum_{i=1}^{n}\frac{1}{\kappa_i})^{-1}$,

$$\sigma=(\sum_{i=2}^{n}\frac{1}{\sigma_i})^{-1},\ d=(\sum_{i=1}^{n}\frac{1}{d_i})^{-1}.$$

The closed-loop error system can be obtained by substituting Eqs. (11) – (13) into Eq. (8)

$$\begin{aligned}
\mathrm{d}\boldsymbol{z} &= \boldsymbol{A}_z(\boldsymbol{z},\hat{\boldsymbol{\theta}},t)\boldsymbol{z}\mathrm{d}t+\boldsymbol{W}^{\mathrm{T}}(\boldsymbol{z},\hat{\boldsymbol{\theta}},t)\tilde{\boldsymbol{\theta}}\mathrm{d}t\\
&\quad+\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{z},\hat{\boldsymbol{\theta}},t)\dot{\hat{\boldsymbol{\theta}}}\mathrm{d}t+\bar{\boldsymbol{W}}^{\mathrm{T}}(\boldsymbol{z},\hat{\boldsymbol{\theta}},t)\mathrm{d}\boldsymbol{w}
\end{aligned} \tag{16}$$

where $\boldsymbol{z}\in\mathrm{R}^n$ and

$$\boldsymbol{A}_z=\begin{bmatrix} -c_1-s_1 & 1 & 0 & \cdots & 0\\ -1 & -c_2-s_2 & 1 & & \vdots\\ 0 & -1 & & & 0\\ \vdots & & & & 1\\ 0 & \cdots & 0 & -1 & -c_n-s_n \end{bmatrix}$$

$$\boldsymbol{W}^{\mathrm{T}}=\begin{bmatrix}\boldsymbol{\omega}_1^{\mathrm{T}}\\ \boldsymbol{\omega}_2^{\mathrm{T}}\\ \vdots\\ \boldsymbol{\omega}_n^{\mathrm{T}}\end{bmatrix}\in\mathrm{R}^{n\times p},\quad \boldsymbol{Q}^{\mathrm{T}}=\begin{bmatrix}0\\ -\frac{\partial\boldsymbol{\alpha}_1}{\partial\hat{\boldsymbol{\theta}}}\\ \vdots\\ -\frac{\partial\boldsymbol{\alpha}_{n-1}}{\partial\hat{\boldsymbol{\theta}}}\end{bmatrix}\in\mathrm{R}^{n\times p}$$

$$\bar{\boldsymbol{W}}^{\mathrm{T}}=\begin{bmatrix}\boldsymbol{\varpi}_1^{\mathrm{T}}\\ \boldsymbol{\varpi}_2^{\mathrm{T}}\\ \vdots\\ \boldsymbol{\varpi}_n^{\mathrm{T}}\end{bmatrix}\in\mathrm{R}^{n\times r}$$

Theorem 1 *In the closed-loop error Eq. (16), the following ISS property holds: If* $\tilde{\theta},\dot{\hat{\theta}} \in L_\infty[0,t_f)$ *, then* $z,x \in L_\infty[0,t_f)$*, and*

$$E\{|z(t)|_4^4\} \le \frac{1}{8c\kappa}\|\tilde{\theta}\|_\infty^4 + \frac{1}{8c\sigma}\|\dot{\hat{\theta}}\|_\infty^4 + |z(0)|_4^4 \mathrm{e}^{-4ct} + \frac{9}{16dc} \tag{17}$$

Proof According to Theorem 3.4 in [9], we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\{E(\frac{1}{4}\sum z_i^4)\} = E\{\boldsymbol{LV}\}$$

Upon multiplication of Eq. (15) by exp(4ct) and integrating over $[0,t]$, we arrive at

$$E\{|z(t)|_4^4\} \le \frac{1}{8c\kappa}\|\tilde{\theta}\|_\infty^4 + \frac{1}{8c\sigma}\|\dot{\hat{\theta}}\|_\infty^4 + |z(0)|_4^4 \mathrm{e}^{-4ct} + \frac{9}{16dc}$$

which proves $z \in L_\infty[0,t_f)$ and Eq. (17). And in addition, because the coordinate transformations Eqs. (6), (11)–(13) are smooth and bounded, we also have $z \in L_\infty[0,t_f)$. Their inverse transformations are also smooth and bounded, so $x \in L_\infty[0,t_f)$.

According to Lemma 3, the closed-loop systems Eqs. (1) and (13) are ISS in respect to disturbances $\tilde{\theta},\dot{\hat{\theta}}$. Now we start to design the identifier module. For the state $\boldsymbol{x}$ model, we design two filters by using Swapping technique, which convert a dynamic parametric model into a static one. According to the gradient algorithms, we design the parametric adaptive law which guarantees that $\tilde{\theta}$ and $\dot{\hat{\theta}}$ are bounded. From Theorem 1, the states of the system are bounded.

4 Swapping identifier module design

System Eq. (1) can be arranged as

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})\mathrm{d}t + \boldsymbol{F}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\theta}\mathrm{d}t + \Lambda^{\mathrm{T}}(\boldsymbol{x})\mathrm{d}\boldsymbol{w} \tag{18}$$

where

$$\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) = \begin{bmatrix} \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_n \\ \boldsymbol{u} \end{bmatrix}, \quad \boldsymbol{F}^{\mathrm{T}}(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{\varphi}_1^{\mathrm{T}}(\boldsymbol{x}_1) \\ \vdots \\ \boldsymbol{\varphi}_{n-1}^{\mathrm{T}}(\bar{\boldsymbol{x}}_{n-1}) \\ \boldsymbol{\varphi}_n^{\mathrm{T}}(\bar{\boldsymbol{x}}_n) \end{bmatrix},$$

$$\Lambda^{\mathrm{T}}(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{\eta}_1^{\mathrm{T}}(\boldsymbol{x}_1) \\ \vdots \\ \boldsymbol{\eta}_{n-1}^{\mathrm{T}}(\bar{\boldsymbol{x}}_{n-1}) \\ \boldsymbol{\eta}_n^{\mathrm{T}}(\bar{\boldsymbol{x}}_n) \end{bmatrix}$$

By applying Lemma 1 in Ref. [5], we design the two filters

$$\dot{\boldsymbol{\Omega}}^{\mathrm{T}} = [\boldsymbol{A}_0 - \lambda \boldsymbol{F}^{\mathrm{T}}(\boldsymbol{x})F(\boldsymbol{x})\boldsymbol{P} - \delta(\Lambda^{\mathrm{T}}(\boldsymbol{x})\Lambda(\boldsymbol{x}))^2]\,\boldsymbol{\Omega}^{\mathrm{T}} + \boldsymbol{F}^{\mathrm{T}}(\boldsymbol{x}),$$

$$\boldsymbol{\Omega} \in \mathrm{R}^{p\times n} \tag{19}$$

$$\dot{\boldsymbol{\psi}} = [\boldsymbol{A}_0 - \lambda \boldsymbol{F}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{F}(\boldsymbol{x})\boldsymbol{P} - \delta(\Lambda^{\mathrm{T}}(\boldsymbol{x})\Lambda(\boldsymbol{x}))^2]$$

$$(\boldsymbol{\psi}+\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}), \quad \boldsymbol{\psi} \in \mathrm{R}^n \tag{20}$$

where the constants $\lambda,\delta > 0$, or $\boldsymbol{A}_0$ is any constant matrix and satisfies

$$\boldsymbol{PA}_0 + \boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{P} = -\boldsymbol{I}, \boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}} > 0 \tag{21}$$

According to Eqs. (18) and (20), we define a variable $\boldsymbol{Y} = \boldsymbol{x}+\boldsymbol{\psi}$. The prediction of $\boldsymbol{Y}$ is $\hat{\mathrm{Y}} = \boldsymbol{\Omega}^{\mathrm{T}}\hat{\boldsymbol{\theta}}$. Then the prediction error $\boldsymbol{\varepsilon} := \mathrm{Y} - \hat{\mathrm{Y}} = \boldsymbol{x}+\boldsymbol{\psi} - \boldsymbol{\Omega}^{\mathrm{T}}\hat{\boldsymbol{\theta}}$ can be written in a form

$$\boldsymbol{\varepsilon} = \boldsymbol{x}+\boldsymbol{\psi} - \boldsymbol{\Omega}^{\mathrm{T}}\hat{\boldsymbol{\theta}} = \boldsymbol{\Omega}^{\mathrm{T}}\tilde{\boldsymbol{\theta}} + \tilde{\boldsymbol{\varepsilon}} \tag{22}$$

where $\tilde{\varepsilon}$ is the filtered disturbance such that

$$\begin{aligned} \mathrm{d}\tilde{\varepsilon} &= [\boldsymbol{A}_0 - \lambda \boldsymbol{F}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{F}(\boldsymbol{x}) - \delta(\Lambda^{\mathrm{T}}(\boldsymbol{x})\Lambda(\boldsymbol{x}))^2]\tilde{\varepsilon}\mathrm{d}t \\ &\quad + \Lambda^{\mathrm{T}}(\boldsymbol{x})\mathrm{d}\boldsymbol{w} \end{aligned} \tag{23}$$

In order to keep the parameter estimation $\hat{\theta}$ be inside Θ, by using Parameter Projection Theorem [5] the update law for $\hat{\theta}$ is

$$\dot{\hat{\theta}} = \mathrm{Proj}\{\boldsymbol{\Gamma}\frac{\boldsymbol{\Omega\varepsilon}}{1+\nu|\boldsymbol{\Omega}|_{\boldsymbol{F}}^2}\}, \quad \boldsymbol{\Gamma} = \boldsymbol{\Gamma}^{\mathrm{T}} > 0, \quad \nu \ge 0 \tag{24}$$

where the detail qualities of the projection operator Proj{*}can be referred to Ref. [5]. If $\nu = 0$, it is called nonstandard parametric adaptive law ; if $\nu > 0$, it is called standard parametric adaptive law . The Swapping identifier consists of Eqs. (19), (22) and (24) and it has the following property .

Theorem 2 *Suppose the solutions of Eqs. (18)–(19), (22) and (24) are defined on* $[0,t_f)$*. Then for* $\nu \ge 0$*, the following identifier properties hold:* $\tilde{\theta},\varepsilon,\dot{\hat{\theta}} \in L_\infty[0,t_f)$.

Proof First we prove that the filter state $\boldsymbol{\Omega}$ is uniformly bounded on $[0,t_f)$, which is irrespective to the boundedness of its input $\boldsymbol{F}$. Along the solutions of Eq. (19) we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{\Omega P \Omega}^{\mathrm{T}}\right) &\le \boldsymbol{\Omega}\left(\boldsymbol{PA}_0 + \boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{P}\right)\boldsymbol{\Omega}^{\mathrm{T}} \\ &\quad - 2\lambda\boldsymbol{\Omega PF}^{\mathrm{T}}\boldsymbol{F}\boldsymbol{P}\boldsymbol{\Omega}^{\mathrm{T}} + \boldsymbol{\Omega PF}^{\mathrm{T}} \\ &\quad + \boldsymbol{FP}\boldsymbol{\Omega}^{\mathrm{T}} = -\boldsymbol{\Omega\Omega}^{\mathrm{T}} \\ &\quad - 2\lambda(\boldsymbol{FP}\boldsymbol{\Omega}^{\mathrm{T}} - \frac{1}{2\lambda}\boldsymbol{I}_p)^{\mathrm{T}} \\ &\quad (\boldsymbol{FP}\boldsymbol{\Omega}^{\mathrm{T}} - \frac{1}{2\lambda}\boldsymbol{I}_p) + \frac{1}{2\lambda}\boldsymbol{I}_p \end{aligned}$$

From the definitions of Froberius norm and trace operator, the linear quality and $\underline{\lambda}(\boldsymbol{P})|\boldsymbol{\Omega}|_{\boldsymbol{F}}^2 \le \mathrm{tr}\{\boldsymbol{\Omega}\boldsymbol{P}\boldsymbol{\Omega}^{\mathrm{T}}\}$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}\{\boldsymbol{\Omega}\boldsymbol{P}\boldsymbol{\Omega}^{\mathrm{T}}\} &= -|\boldsymbol{\Omega}|_{\boldsymbol{F}}^2 - 2\lambda\left|\boldsymbol{FP}\boldsymbol{\Omega}^{\mathrm{T}} - \frac{1}{2\lambda}\boldsymbol{I}_p\right|_{\boldsymbol{F}}^2 \\ &\quad + \frac{1}{2\lambda}\mathrm{tr}\{\boldsymbol{I}_p\} \le -|\boldsymbol{\Omega}|_{\boldsymbol{F}}^2 + \frac{\boldsymbol{p}}{2\lambda} \end{aligned} \tag{25}$$

This proves that $\boldsymbol{\Omega}$ is uniformly bounded on $[0,t_f)$, namely $\boldsymbol{\Omega} \in L_\infty[0,t_f)$. Let us consider the positive definite function $V=\frac{1}{4}|\tilde{\varepsilon}|^4$, in view of Eqs. (21), (23) and Itô's differentiation rule, the infinitesimal generator L of V is

$$\begin{aligned}
LV &\le -\sum_{i=1}^{n}\tilde{\varepsilon}_i^4-\delta\sum_{i=1}^{n}\tilde{\varepsilon}_i^4(\eta_i^{\mathrm T}\eta_i)^2+\frac{3}{2}\mathrm{tr}\{|\tilde{\varepsilon}|^2\Lambda^{\mathrm T}\Lambda\}\\
&=-\sum_{i=1}^{n}\tilde{\varepsilon}_i^4-\delta\sum_{i=1}^{n}|\eta_i|^4\tilde{\varepsilon}_i^4+\frac{3}{2}\sum_{i=1}^{n}|\eta_i|^2\tilde{\varepsilon}_i^2\\
&=-\sum_{i=1}^{n}\tilde{\varepsilon}_i^4-\delta\sum_{i=1}^{n}(|\varpi_i|^2\tilde{\varepsilon}_i^2-\frac{3}{4\delta})^2+\frac{9}{16\delta}\\
&\le -\sum_{i=1}^{n}\tilde{\varepsilon}_i^4+\frac{9}{16\delta}
\end{aligned}\tag{26}$$

This proves that $\tilde{\varepsilon}$ is uniformly bounded in probability on $[0,t_f)$, namely $\tilde{\varepsilon}\in L_\infty[0,t_f)$. we can further prove $L\{\frac{1}{2}|\tilde{\varepsilon}|^2\}\le -2|\tilde{\varepsilon}|^2+\frac{3}{2\sqrt{\delta}}$

Now let us consider the positive definite function $V=\frac{1}{2}|\tilde{\theta}|^2_{\Gamma^{-1}}+\frac{1}{4}|\tilde{\varepsilon}|^2$. Along the solutions of Eqs. (23) and (24), the derivative of V is

$$\begin{aligned}
\dot V &= -\tilde{\theta}^{\mathrm T}\Gamma^{-1}\dot{\hat{\theta}}-|\tilde{\varepsilon}|^2+\frac{3}{4\sqrt{\delta}}\\
&=-\frac{\tilde{\theta}^{\mathrm T}\Omega\varepsilon}{1+\nu|\Omega|_F^2}-|\tilde{\varepsilon}|^2+\frac{3}{4\sqrt{c_0 d}}\\
&=-\frac{|\varepsilon|^2}{1+\nu|\Omega|_F^2}+\frac{\varepsilon^{\mathrm T}\tilde{\varepsilon}}{1+\nu|\Omega|_F^2}-|\tilde{\varepsilon}|^2+\frac{3}{4\sqrt{\delta}}\\
&\le -\frac{3}{4}\frac{|\varepsilon|^2}{1+\nu|\Omega|_F^2}-\frac{1}{4}\frac{|\varepsilon|^2}{(1+\nu|\Omega|_F^2)^2}\\
&\quad+\frac{\varepsilon^{\mathrm T}\tilde{\varepsilon}}{1+\nu|\Omega|_F^2}-|\tilde{\varepsilon}|^2+\frac{3}{4\sqrt{\delta}}\\
&=-\frac{3}{4}\frac{|\varepsilon|^2}{1+\nu|\Omega|_F^2}-\left|\frac{1}{2}\frac{\varepsilon}{1+\nu|\Omega|_F^2}-\tilde{\varepsilon}\right|^2+\frac{3}{4\sqrt{\delta}}\\
&\le -\frac{3}{4}\frac{|\varepsilon|^2}{1+\nu|\Omega|_F^2}+\frac{3}{4\sqrt{\delta}}
\end{aligned}\tag{27}$$

This proves that $\tilde{\theta}\in L_\infty[0,t_f)$. Recalling Eq. (22), from the boundedness of Ω it follows that $\varepsilon\in L_\infty[0,t_f)$. This in turn shows that $\dot{\hat{\theta}}=\Gamma\frac{\Omega\ \varepsilon}{1+\nu|\Omega|_F^2}\in L_\infty[0,t_f)$.

From the proof we can see that the solution of the identifier equations is bounded and this bound is independent of t_f. So the solution is uniformly bounded. From Theorem 1 and 2 the states of the system are bounded and have the following stability in probability.

Theorem 3 *All the signals in the closed-loop adaptive system consisting of plant Eq. (1), controller Eq. (13), identifiers Eqs. (21) and (22) and the gradient update law Eq. (24) are globally uniformly bounded and the states* z,ε,x *are globally uniformly stable in probability.*

Proof Due to the smoothness of the nonlinearities in Eq. 1, the solution of the closed-loop adaptive system exists and is unique. Let its maximum interval of existence be $[0,t_f)$. From Theorem 2 we obtain $\tilde{\theta},\ \dot{\hat{\theta}},\ \varepsilon\in L_\infty[0,t_f)$. Therefore, by Theorem 1 $z,x\in L_\infty[0,t_f)$. In Theorem 2 we also proved that $\Omega,\tilde{\varepsilon}\in L_\infty[0,t_f)$. Finally, by Eq. (22) $\psi\in L_\infty[0,t_f)$. It is shown that all of the signals of the closed-loop adaptive system are globally bounded on $[0,t_f)$ with the bounds depending only on the initial conditions and design gains but not on t_f. The independence of the bounds proves that $t_f=\infty$. Hence, all signals are globally uniformly bounded on $[0,t_f)$. From Eq. (15), z is globally uniformly stable in probability . From Eqs. (25)–(27) we can see that Ω, $\tilde{\varepsilon}$, $\tilde{\theta}$ are globally uniformly stable in probability . From Eq. (22) ε, ψ are globally uniformly stable in probability . Further we get x is globally uniformly stable in probability.

5 Transient performance

Transient performance of the stochastic system will be estimated by the error norm of state z. To simplify the derivations, without loss of generality, we assume that $\Omega(0)=0$ and $\psi(0)=-z(0)$. Those filter initializations guarantee that $\tilde{\varepsilon}(0)=0$ and should always be performed to eliminate the disturbing effect of the initial estimation error. For simplicity we let $\Gamma=\gamma I$.

Theorem 4 *In the closed-loop stochastic system consisting of plant Eq. (1), controller Eq. (13), identifiers Eqs. (19) and (22) and the update law Eq. (24), the following inequality holds:*

$$E\{|z(t)|_4^4\}\le |z(0)|_4^4\mathrm{e}^{-4ct}+\frac{1}{8c\kappa}M^4+\frac{\gamma^4p^2}{32c\sigma\lambda^2}\left(\sqrt{\frac{p}{2\lambda}}M+\frac{\sqrt{3}}{2\sqrt[4]{\delta}}\right)^4+\frac{9}{16dc}\tag{28}$$

Proof Since $\varepsilon(0)=0$, by using Eqs. (25) and (26), we derive

$$\|\tilde{\varepsilon}\|_\infty^4=\frac{9}{16\delta},\quad \||\Omega|_F\|_\infty=\sqrt{\frac{p}{2\lambda}}$$

From Eq. (22) we obtain

$$\|\varepsilon\|_\infty=\|\Omega^{\mathrm T}\tilde{\theta}+\tilde{\varepsilon}\|_\infty\le\|\Omega^{\mathrm T}\tilde{\theta}\|_\infty+\|\tilde{\varepsilon}\|_\infty\le\sqrt{\frac{p}{2\lambda}}M+\frac{\sqrt{3}}{2\sqrt[4]{\delta}}$$

By applying Eq. (24), we arrive at

$$|\dot{\hat{\theta}}|=\gamma\left|\frac{\Omega\varepsilon}{1+\nu|\Omega|_F^2}\right|\le\gamma|\Omega\varepsilon|$$

Then,

$$\|\dot{\hat{\theta}}\|_{\infty} \le \gamma\|\boldsymbol{\Omega\varepsilon}\|_{\infty} \le \gamma\left(\sqrt{\frac{p}{2\lambda}}M+\frac{\sqrt{3}}{2\sqrt[4]{\delta}}\right)\times\sqrt{\frac{p}{2\lambda}} \tag{29}$$

Substituting the inequality Eq. (29) into Eq. (17), we get

$$E\{|\boldsymbol{z}(\boldsymbol{t})|_4^4\} \le \frac{1}{8c\kappa}M^4+\frac{\gamma^4p^2}{32c\sigma\lambda^2}\left(\sqrt{\frac{p}{2\lambda}}M+\frac{\sqrt{3}}{2\sqrt[4]{\delta}}\right)^4 + \frac{9}{16dc}+|z(0)|_4^4\,\mathrm{e}^{-4ct}$$

6 Simulation results

Consider the second order stochastic system

$$\mathrm{d}x_1 = x_2\mathrm{d}\,t + x_1^3\boldsymbol{\theta}\mathrm{d}\,t+\frac{1}{2}x_1^2\mathrm{d}\,\boldsymbol{w}$$

$$\mathrm{d}x_2 = u\mathrm{d}\,t+(x_1^2+\sin x_2)\boldsymbol{\theta}\mathrm{d}\,t$$

The virtual control α_1 and control $\boldsymbol{u}$ can be calculated according to the above algorithm. We choose $\kappa_1=\kappa_2=\sigma_2=d_1=d_2=1$, $c_1=1.375$, $c_2=1.675$, $\gamma=50$, $\nu=\delta=\lambda=1$, $\boldsymbol{A}_0=\begin{bmatrix}-0.5 & -0.5\\ 0.5 & -0.5\end{bmatrix}$ and set the initial conditions at $x_1(0)=-0.8$, $x_2(0)=0.5$, $\hat{\theta}(0)=0$, $\boldsymbol{\varepsilon}(0)=0$. Suppose $\boldsymbol{\theta}=1$.The states and control of the system are shown in Fig. 1. We can see that the states converge to zero. The estimation of parameter is shown in Fig. 2. We can see that it converges to certain value.

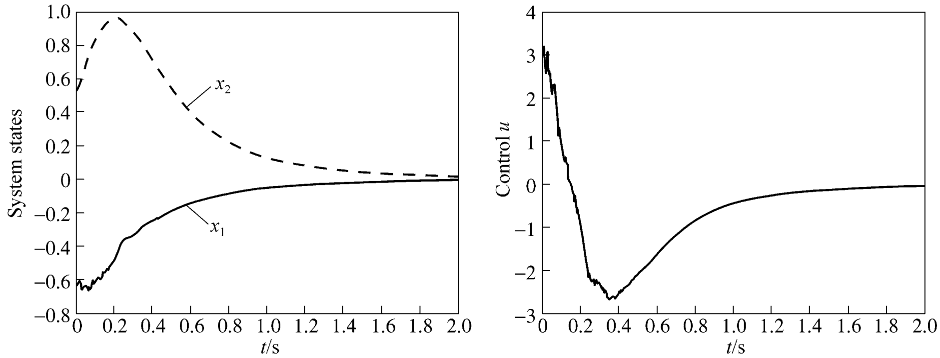

Fig. 1 The system states and control.

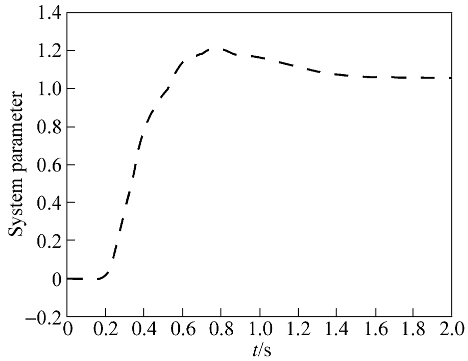

Fig. 2 The system parameter estimation.

7 Conclusions

In this paper, we present an adaptive robust control design for strict-feedback stochastic nonlinear system with additive standard Wiener noises and constant unknown parameters on the basis of a modular approach of the estimation-based design. The controller designed here can guarantee that the system states are bounded and have a robust stabilization. According to Swapping technique, we develop two filters and convert dynamic parametric models into static ones to which the gradient update laws is designed. Transient performance of the system is estimated by the error norm. Results of simulation show the effectiveness of the control algorithms. The questions requiring further study are: 1) how to realize the optimal control of the whole system; 2) when the covariance of the Wiener noises is uncertain, the robust control and optimization problem of this kind of system is used.

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