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RESEARCH ARTICLE

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Minimal axiom group of similarity-based rough set model

Abstract Rough set axiomatization is one aspect of rough set study to characterize rough set theory using dependable and minimal axiom groups. Thus, rough set theory can be studied by logic and axiom system methods. The classical rough set theory is based on equivalence relation, but the rough set theory based on similarity relation has wide applications in the real world. To characterize similarity-based rough set theory, an axiom group named S, consisting of 3 axioms, is proposed. The reliability of the axiom group, which shows that characterizing of rough set theory based on similarity relation is rational, is proved. Simultaneously, the minimization of the axiom group, which requests that each axiom is an equation and independent, is proved. The axiom group is helpful to research rough set theory by logic and axiom system methods.

Keywords rough set theory, axioms, minimization

1 Introduction

As a new effective mathematical tool to deal with vagueness and uncertainty, the rough set theory was first proposed by Pawlak. On the basis of classification, rough set theory is considered as knowledge expressed as a partition of data using equivalence relation, which has been successfully applied in many fields such as machine learning, pattern recognition, decision support and data mining [1].

Lower approximation and upper approximation are two basic concepts in rough set theory. Pawlak derived many interesting properties of upper and lower approximations [2], while some researchers studied the reverse problem. Namely, can we characterize the notion of rough sets in terms of those properties? Lin and Liu studied this problem from the viewpoint of topology and proposed an axiom group consisting of six axioms of rough set and presented the concepts of rough set axiom group and axiomatic rough set theory [3]. Zhu and He [4] and Sun et al. [5] discussed the redundancy of rough set axiom group. But these rough set axiom groups are used to characterize the classical rough set, which is defined by equivalence relation. Dai studied the axiomatization problem of quasi-ordering-based rough set model [6]. In fact, we can find that there exist indiscernibility relations which are not transitive. For example, city A is not far from city B, and city B is not far from C, but A can be far from C. This makes it necessary to study similarity-based rough set model.

To characterize rough set based on similarity, an axiom group named S, consisting of 3 axioms, is proposed. The validity of the axiom group, which shows that characterizing of rough set theory based on similarity is rational, is proved. Simultaneously, the minimization of the axiom group, which requires that each axiom is an equation and each equation is independent, is proved. The axiom group is helpful for researching rough set based on similarity by logic and axiom system methods.

2 Preliminaries

2.1 Pawlak rough set model

Let $U=\{u_1, u_2, \cdots, u_n\}$ denote a finite and non-empty set of objects called the universe, and let $R \subseteq U \times U$ denote an equivalence relation on U. The pair (U, R) is called an approximation space. The equivalence classes partitioned by R are called as elementary sets of (U, R).

The equivalence relation and the induced equivalence classes may be regarded as the available information or the knowledge about the objects under consideration. Given an arbitrary set $X \subseteq U$, it may be impossible to describe X precisely using the equivalence classes of R. That is, the available information is not sufficient to give a precise representation of X. In this case, one may characterize X by a

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pair of lower and upper approximations:

$$L\_APP_R(X) = \bigcup\{[x]_R \mid [x]_R \subseteq X\} \quad (1)$$

$$U\_APP_R(X) = \bigcup\{[x]_R \mid [x]_R \cap X \neq \Phi\} \quad (2)$$

where $[x]_R$ is the equivalence class containing x. The lower approximation $L\_APP(X)$ is the union of all the elementary sets that are subsets of X. It is the largest composed set contained in X. The upper approximation $U\_APP(X)$ is the union of all the elementary sets which have a non-empty intersection with X. It is the smallest composed set containing X.

2.2 Similarity based rough set model

Some researchers generalized the classic rough set by replacing equivalent classes with neighborhood [7]. Suppose that $R \subseteq U \times U$ is a binary relation on U, $x \in U$, then R-neighborhood of x is defined as:

$$N_R(x) = \{y \mid (x, y) \in R\} \quad (3)$$

Based on Eq. (3), the lower and upper approximation of the objects set $X \subseteq U$ can be defined as:

$$L\_APP_R(X) = \{x \mid N_R(x) \subseteq X\} \quad (4)$$

$$U\_APP_R(X) = \{x \mid N_R(x) \cap X \neq \Phi\} \quad (5)$$

In order to get similarity based rough set model, the relation R in Eqs. (3)–(5) is supposed to be reflexive and symmetric.

2.3 Minimum of rough set axiom groups

Theorem 1 Rough set axiom formulas are defined as follows:

1) n arbitrary set $X \subseteq U$ is a rough set axiom formula;

2) If α is a rough set axiom formula, then so are ~α, $L(\alpha)$ and $H(\alpha)$;

3) If α, β are rough set axiom formulas, then so are $\alpha \cup \beta$ and $\alpha \cap \beta$;

4) The only rough set axiom formulas are those obtained by finite application of (1)–(3) in the above.

Definition 1 The conception of rough set axiom formula (RSAF) is not used to describe rough set, which is an opposite concept to definable (crisp) set. Normally, a rough set axiom formula is not a rough set. One can find that $L(\alpha)$ and $H(\alpha)$ are surely crisp sets in classic rough set theory. In fact, a rough set α is described by a pair of definable sets $\{L(\alpha), H(\alpha)\}$.

Definition 2 If α, β are rough set axiom formulas, then $\alpha \subseteq \beta$ and $\beta \subseteq \alpha$ are rough set inequalities.

Definition 3 The rough set axiom group satisfying the follow conditions is called minimal rough set axiom group:

1) Each axiom in the axiom group is a rough set inequality;

2) Each axiom in the axiom group is independent of others.

We adopt here the concept of minimization of rough set axiom group from Sun et al. from which three definitions above can be found [5].

3 Minimal axiom group of similarity based rough set model

We now propose a minimal axiom group of rough set based on similarity, named rough set axiom group S, as follows:

(S1) $L(\sim X \cup Y) \subseteq \sim L(X) \cup L(Y)$

(S2) $L(X) \subseteq X$

(S3) $X \subseteq L(\sim L(\sim X))$

3.1 Reliability of axiom group S

Lemma 1 For the pair of rough operators L and H, $X \rightarrow (H(X), L(X))$, which satisfy the following axioms:

(L1) $L(U) = U$

(L2) $L(X \cap Y) = LX \cap LY$

(H1) $H(\Phi) = \Phi$

(H2) $H(X \cup Y) = H(X) \cup H(Y)$

(LH) $H(X) = \sim L(\sim X)$

(M1) $X \subseteq H(X)$

(M2) $H(L(X)) \subseteq X$

There exists a reflexive and transitive relation R on U satisfying $H(X) = R^-(X)$ and $L(X) = R\_(X)$.

Proof The lemma is easily proved by Theorem 3 and Theorem 6 of Yao [7].

Lemma 2 [7] Under the condition L(U)=U, we have

$$L(X \cap Y) = L(X) \cap L(Y) \Leftrightarrow$$

$$L(U) = U \,\&\, L(\sim X \cup Y) \subseteq \sim L(X) \cup L(Y)$$

Theorem 2 For the rough operator $X \rightarrow L(X)$, which satisfies the axiom (S1) through (S3), there is a reflexive and symmetric relation on U such that

$$L(X) = R\_(X).$$

Defining the dual operator of L by $H(X) = \sim L(\sim X)$, we have

$$H(X) = R^-(X).$$

Proof From axiom (S3), we have

$$X \subseteq L(\sim L(\sim X)) \quad (6)$$

then let $X = U$, we get

$$U \subseteq L(\sim L(\Phi)) \quad (7)$$

From axiom (S2), we have

$$L(X) \subseteq X \quad (8)$$

Then let $X = \Phi$, we get

$$L(\Phi) \subseteq \Phi \quad (9)$$

Equation (9) means that

$$L(\Phi) = \Phi \quad (10)$$

From Eqs. (7) and (9), we know

$$L(U) = U \quad (11)$$

Namely, we get (L1) in Lemma 1.

With Eqs. (11) and (S1), we have the following by Lemma 2

$$L(X\bigcap Y)=L(X)\bigcap L(Y) \quad (12)$$

Namely, we have L2 in Lemma 1.

From Eqs. (11) and (LH), we can get (H1). And we can have (H2) from Eqs. (12) and (LH).

From (S2), we know

$$X\subseteq \sim L(\sim X) \quad (13)$$

Then from (LH), we get

$$X\subseteq H(X) \quad (14)$$

Namely, we get (M1).

Similarly, we can get (M2) from (S3) and (LH).

Thus, {(S1), (S2), (S3), (LH)} $\Rightarrow$ {(L1), (L2), (H1), (H2), (LH), (M1), (M2)}

In fact, axiom (LH) is the mutual defining method between lower approximation operator and upper approximation operator. From Lemma 1, we get Theorem 2.

3.2 Minimum of axiom group S

Theorem 3 Rough set axiom group S is a minimal similarity based rough set axiom group.

Proof It is obvious that each axiom in S is a rough set inequality, so axiom group S satisfies the first condition in Definition 3.

We then prove that S also satisfies the second condition in Definition 3 as follows:

1) Suppose that U={a, b, c} is the universe. $L(U)$={a, b}, L({a, b})={a, b}, L({a, c})={c}, L({b, c})={b, c}, L({a})={a}, L({b})=Φ, L({c})={c}, $L(\Phi)=\Phi$. Then we get the following validation Table 1 showing that axiom (S1) is independent of (S2) and (S3).

In Table 1, the items in bold and underline style show contradiction of (S1). For example, when X={a, b}, Y={b}, we can find $L(\sim X\bigcup Y)=\{b,c\}$ and $\sim L(X)\bigcup L(Y)=\{c\}$. Hence, we know $L(\sim X\bigcup Y)\not\subset\sim L(X)\bigcup L(Y)$ which implies that axiom (S1) is not satisfied, but axioms (S2) and (S3) are satisfied. So we know that (S1) is independent of (S2) and (S3). The meaning and function of the validation tables followed are the same as it, and we just give the validation tables followed are the same, and we just give the validation tables without further interpretation in the rest of the paper.

2) Suppose that U={a, b, c} is the universe. $L(U)=U$, L({a, b})=U, L({a, c})=U, L({b, c})=U, L({a})=U, L({b})=U, L({c})=U, $L(\Phi)=U$. Then we get the following validation Table 2 showing that axiom (S2) is independent

Table 1 Validation table showing that (S1) is independent of (S2), (S3)

$L(\sim X\bigcup Y)$, $\sim L(X)\bigcup L(Y)$		Y								L(X)	L(~L(~X))
		U	{a, b}	{a, c}	{b, c}	{a}	{b}	{c}	Φ		
X	U	U,U	{a,b},{a,b}	{c},{c}	{b,c},{b,c}	{a},{a}	Φ , Φ	{c},{c}	Φ , Φ	U	U
	{a, b}	U,U	U,U	{c},{c}	{b,c},{b,c}	{c},{a,c}	{b,c},{c}	{c},{c}	{c},{c}	{a,b}	{a,b}
	{a, c}	U,U	{a,b},{a,b}	U,U	{b,c},U	{a,b},{a,b}	Φ ,{a,b}	{b,c},U	Φ ,{a,b}	{c}	U
	{b, c}	U,U	{a,b},{a,b}	{c},{a,c}	U,U	{a},{a}	{a,b},{a}	{c},{a,c}	{a},{a}	{b,c}	{b,c}
	{a}	U,U	U,U	U,{b,c}	{b,c},{b,c}	U,U	{b,c},{b,c}	{b,c},{b,c}	{b,c},{b,c}	{a}	{a}
	{b}	U,U	U,U	{c},U	U,U	{c},U	U,U	{c},U	{c},U	Φ	{a,b}
	{c}	U,U	{a,b},{a,b}	U,U	U,U	{a,b},{a,b}	{a,b},{a,b}	U,U	{a,b},{a,b}	{c}	{c}
	ϕ	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	Φ	Φ

Table 2 Validation table showing that (S2) is independent of (S1), (S3)

$L(\sim X\bigcup Y)$, $\sim L(X)\bigcup L(Y)$		Y								L(X)	L(~L(~X))
		U	{a, b}	{a, c}	{b, c}	{a}	{b}	{c}	Φ		
X	U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U	U
	{a,b}	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U	U
	{a,c}	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U	U
	{b,c}	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U	U
	{a}	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U	U
	{b}	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U	U
	{c}	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U	U
	Φ	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U	U

of (S1) and (S3).

3) Let $U=\{a, b, c\}$ be the universe. Suppose that $L(U)=U$, $L(\{a, b, c\})=\{a, b, c\}$, $L(\{a, b\})=\{a\}$, $L(\{a, c\})=\{a\}$, $L(\{b, c\})=\Phi$, $L(\{a\})=\{a\}$, $L(\{b\})=\Phi$, $L(\{c\})=\Phi$, $L(\Phi)=\Phi$, then we get the following validation Table 3 showing that axiom (S3) is independent of (S1) and (S2).

Table 3 Validation table showing that (S3) is independent of (S1), (S2)

$L(\sim X \cup Y)$, $\sim L(X) \cup L(Y)$		Y								$L(X)$	$L(\sim L(\sim X))$
		U	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	$\{a\}$	$\{b\}$	$\{c\}$	Φ		
X	U	U,U	{a},{a}	{a},{a}	Φ , Φ	{a},{a}	Φ , Φ	Φ , Φ	Φ , Φ	U	U
	$\{a, b\}$	U,U	U,U	{a},U	Φ ,{b,c}	{a},U	Φ ,{b,c}	Φ ,{b,c}	Φ ,{b,c}	{a}	U
	$\{a, c\}$	U,U	{a},U	U,U	Φ ,{b,c}	{a},U	Φ ,{b,c}	Φ ,{b,c}	Φ ,{b,c}	{a}	U
	$\{b,c\}$	U,U	{a},U	{a},U	U,U	{a},U	{a},U	{a},U	{a},U	Φ	Φ
	$\{a\}$	U,U	U,U	U,U	Φ ,{b,c}	U,U	Φ ,{b,c}	Φ ,{b,c}	Φ ,{b,c}	{a}	U
	$\{b\}$	U,U	U,U	{a},U	U,U	{a},U	U,U	{a},U	{a},U	Φ	Φ
	$\{c\}$	U,U	{a},U	U,U	U,U	{a},U	{a},U	U,U	{a},U	Φ	Φ
	Φ	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	Φ	U

4 Conclusions

Rough set axiomatization is an aspect of rough set study aimed at characterizing rough set theory using dependable and minimal axiom groups. Thus, rough set theory can be studied by logic and axiom system methods. To characterize the rough set based on similarity, an axiom group named S, consisting of 3 axioms, is proposed. The validity of the axiom group, which shows that characterizing of rough set theory is reasonable, is proved. Simultaneously, the minimization of the axiom group, which requires that each axiom is an equation and each is independent, is proved. The axiom group is helpful for researching on rough set theory by logic and axiom system methods.

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